Triple Math

https://asdia.dev/projects/triplemath

Notes taken by
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Preface

About this Book

This book is a collection of notes and exercises based on the mathematics courses offered at Dunman High School[1]. The scope of this book follows that of the 2025 H2 Mathematics (9758), H2 Further Mathematics (9649) and H3 Mathematics (9820) syllabi for the Singapore-Cambridge A-Level examinations.

Notation

All definitions, results, recipes (methods) and examples are colour-coded green, blue, purple and red respectively.

Challenging exercises are marked with a "🌶" symbol.

The area of a polygon $A_1A_2 \ldots A_n$ is notated $[A_1A_2 \ldots A_n]$. In particular the area of a triangle ABC is notated $[\triangle ABC]$.

For formatting reasons, an inline column vector is notated as $\langle x, y, z\rangle$.

Contributing

The source code for this book is available on GitHub at `asdia0/TripleMath`. Contributions are more than welcome.

[1] It must be stated that these notes are unofficial and are obviously not endorsed by the school.

Contents

NOTES

Part I

Functions and Graphs

1 Equations and Inequalities

1.1 Quadratic Equations

In this section, we will look at the properties of quadratic equations as well as their roots.

Proposition 1.1.1 (Quadratic Formula). The roots α and β of a quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$ can be found using the quadratic formula:

$$\alpha,\ \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof. Completing the square, we get

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0,$$

which rearranges as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking roots and simplifying,

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

□

Definition 1.1.2. The expression under the radical, $b^2 - 4ac$, is known as the **discriminant** and is denoted Δ.

Proposition 1.1.3 (Nature of Roots).

- If $\Delta > 0$, the roots are real and distinct.
- If $\Delta = 0$, the roots are equal.
- If $\Delta < 0$, the roots are complex.

Proof. Let the roots to the quadratic equation $ax^2 + bx + c = 0$ be α and β. By the quadratic formula,

$$\alpha,\ \beta = \frac{-b}{2a} \pm \frac{\sqrt{\Delta}}{2a}.$$

Clearly, if $\Delta > 0$, then $\sqrt{D} > 0$, whence the two roots are different. If $\Delta = 0$, then $\sqrt{D} = 0$, whence $\alpha = \beta = -b/2a$. If $\Delta < 0$, then $\sqrt{D}$ is not real, whence α and β are complex. □

Remark. Not only are α and β complex, but they are also *complex conjugates*. We will cover this later in §11.

Proposition 1.1.4 (Vieta's Formula for Quadratics)**.** Let α and β be the roots of the quadratic $ax^2 + bx + c = 0$, where $a \neq 0$. Then

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

Proof. Since α and β are roots, we can rewrite the quadratic as

$$ax^2 + bx + c = a(x - \alpha)(x - \beta) = a\left[x^2 - (\alpha + \beta)x + \alpha\beta\right].$$

Comparing coefficients yields

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

□

1.2 System of Linear Equations

Definition 1.2.1. A set of two or more equations to be solved simultaneously is called a **system of equations**. If the system has only equations that contain unknowns of the *first degree*, it is a **system of linear equations**.

Definition 1.2.2. A system of equations is said to be **consistent** if it admits solutions. Conversely, if there are no solutions to the system, it is said to be **inconsistent**.

Example 1.2.3. The system

$$\begin{cases} 3x + 6y = 3 \\ 3x + 8y = 9 \end{cases}$$

is consistent, since $x = -5$, $y = 3$ is a solution. On the other hand, the system

$$\begin{cases} 3x + 6y = 3 \\ 6x + 12y = 7 \end{cases}$$

is inconsistent, as it does not admit any solutions (why?).

Proposition 1.2.4. If a system of linear equations is consistent, it either has a unique solution or infinitely many solutions.

Proof. Geometrically, if a collection of lines has more than one common point, they must all be equivalent. □

1.3 Inequalities

Fact 1.3.1 (Properties of Inequalities)**.** Let $a, b, c, \in \mathbb{R}$.

- (transitivity) If $a > b$ and $b > c$, then $a > c$.
- (addition) If $a > b$, then $a + c > b + c$.
- (multiplication) If $a > b$ and $c > 0$, then $ac > bc$; if $c < 0$, then $ac < bc$.

1.3.1 Solving Inequalities

In this section, we introduce two main methods of solving inequalities.

Recipe 1.3.2 (Graphical Method). Plot the function and observe which x-values satisfy the inequality.

Recipe 1.3.3 (Test-Value Method).

1. Indicate the root(s) of the function on a number line (i.e. where $f(x) = 0$).
2. Choose an x-value within each interval as your test-value.
3. Using the test-value, evaluate whether the function is positive/negative within that interval.

Note that the test-value method is only useful for inequalities where one side is 0, e.g. $f(x) > 0$.

Sample Problem 1.3.4 (Test-Value Method). Solve the inequality $2x - x^2 \geq -3$.

Solution. In order to apply the test-value method, we must first make one side of the inequality 0:

$$2x - x^2 \geq -3 \implies x^2 - 2x - 3 \leq 0.$$

Since $x^2 - 2x - 3 = (x+1)(x-3)$, the critical values are $x = -1$ and $x = 3$. Picking $x = -2$, $x = 0$ and $x = 4$ as our test-values, we see that $x^2 - 2x - 3$ is only negative on the interval $(-1, 3)$. Hence, the solution is $[-1, 3]$. □

In the case where the function is rational, i.e. $f(x)/g(x)$, there is an additional method we can use.

Recipe 1.3.5 (Clearing Denominators). Multiply the square of the denominator, i.e. $[g(x)]^2$, throughout the inequality.

Note that the square ensure that the sign of the inequality is preserved.

Sample Problem 1.3.6 (Clearing Denominators). Solve the inequality

$$\frac{(x-1)(x+2)}{x-4} > 0.$$

Solution. Multiplying both sides by the square of the denominator, $(x-4)^2$, we get $(x-1)(x+2)(x-4) > 0$, which we can solve easily using either the graphical or test-value methods. □

1.4 Modulus Function

Definition 1.4.1. The modulus function $|x|$, where $x \in \mathbb{R}$, is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The modulus function can be thought of as the "distance" between a number and the origin (the number 0) on the real number line.

Fact 1.4.2 (Properties of Modulus Function). For any $x \in \mathbb{R}$ and $k > 0$,

- $|x| \geq 0$.
- $\left|x^2\right| = |x|^2 = x^2$ and $\sqrt{x^2} = |x|$.
- $|x| < k \iff -k < x < k$.
- $|x| = k \iff x = -k$ or $x = k$.
- $|x| > k \iff x < -k$ or $x > k$.

2 Numerical Methods of Finding Roots

2.1 Bolzano's Theorem

The following theorem forms the basis for finding roots numerically.

Theorem 2.1.1 (Bolzano's Theorem). Let $f(x)$ be a continuous function on the interval $[a, b]$. If $f(a)$ and $f(b)$ have opposite signs, i.e. $f(a)f(b) < 0$, then there exists at least one real root in $[a, b]$.

Proof. Follows immediately from the Intermediate Value Theorem. □

Additionally, if $f(x)$ is strictly monotonic on $[a, b]$, then there is exactly one real root in $[a, b]$.

2.2 Numerical Methods for Finding Roots

A numerical method for finding roots typically consists of two stages:

1. **Estimate the location of the root**

   Obtain an initial approximate value of this root.

2. **Improve on the estimate (via an iterative process)**

   An iterative process is a repetitive procedure designed to produce a sequence of approximations $\{x_n\}$ so that the sequence converges to a root. The process is continued until the required accuracy is reached.

In this chapter, we will look at three numerical methods for finding roots, namely linear interpolation, fixed point iteration and the Newton-Raphson method.

2.3 Linear Interpolation

Linear interpolation is a numerical method based on approximating the curve $y = f(x)$ to a straight line in the vicinity of the root. The approximate root of the equation $f(x) = 0$ is the intersection of this straight line with the x-axis.

2.3.1 Derivation

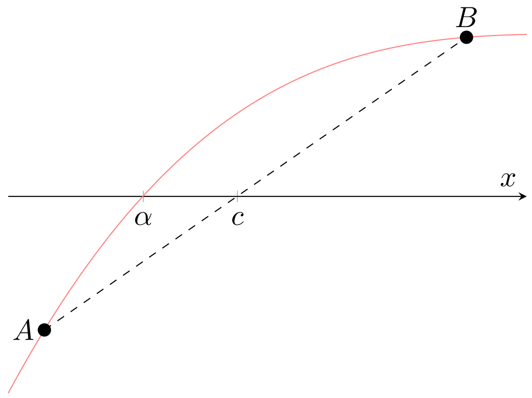


Figure 2.1

Suppose $f(x) = 0$ has exactly one root α in the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs. By the point-slope formula, the line connecting the points $(a, f(a))$ and $(b, f(b))$ is given by

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a).$$

At the point $(c, 0)$,

$$0 - f(a) = \frac{f(b) - f(a)}{b - a}(c - a) \implies c = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

Linear interpolation can be repeatedly applied by replacing either the lower or upper bound of the interval with the previously found approximation.

2.3.2 Convergence

Convergence of the approximations is guaranteed for linear interpolation. However, how good the estimation is depends on how "straight" the graph of $y = f(x)$ is in $[a, b]$, i.e. the rate at which $f'(x)$ is changing in $[a, b]$. This rate also affects the rate of convergence: if $f'(x)$ changes considerably, the rate of convergence is slow; if $f'(x)$ does not change much, the rate of convergence is fast.

2.4 Fixed Point Iteration

Fixed point iteration is used to find a root of an equation $f(x) = 0$ which can be written in the form $x = F(x)$. The roots of the equation are the abscissae of the points of intersection of the line $y = x$ and $y = F(x)$.

2.4.1 Derivation

Let α be a root to $f(x) = 0$. Since $f(x) = 0$ can be written in the form $x = F(x)$, we clearly have $\alpha = F(\alpha)$. Now observe that we can replace the argument α with $F(\alpha)$:

$$\alpha = F(\alpha) = F \circ F(\alpha) = F \circ F \circ F(\alpha) = \dots.$$

Hence,

$$\alpha = F \circ F \circ F \circ \cdots \circ F(x).$$

2.4.2 Geometrical Interpretation

Geometrically, fixed-point iteration can be seen as repeatedly "reflecting" the initial approximation point $(x_1, F(x_1))$ about the line $y = x$, while keeping the resultant point on the curve $y = F(x)$.

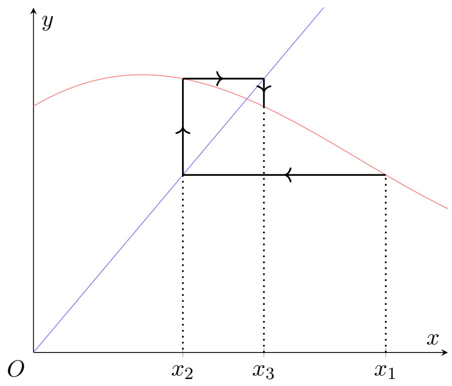


Figure 2.2

2.4.3 Convergence

Convergence is not guaranteed. The rate at which the approximations converge to α depends on the value of $|F'(x)|$ near α. The smaller $|F'(x)|$ is, the faster the convergence. It should be noted that fixed-point iteration fails if $|F'(x)| > 1$ near α.

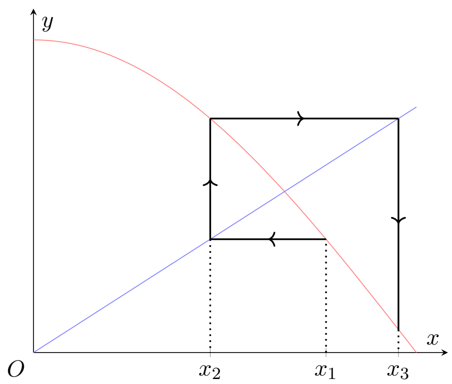


Figure 2.3: Divergence occurs when $|F'(x)| > 1$ near α.

2.5 Newton-Raphson Method

The Newton-Raphson method is a numerical method that improves on linear interpolation by considering the tangent line at the initial approximation to the root.

2.5.1 Derivation

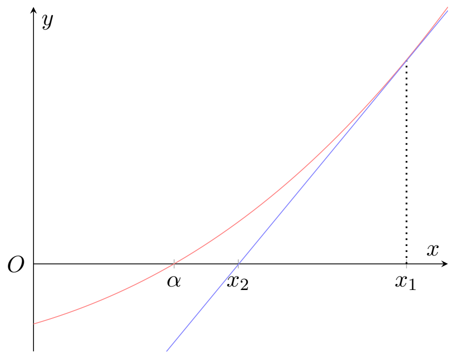


Figure 2.4

Let α be a root to $f(x) = 0$. Consider the tangent to $y = f(x)$ at the point where $x = x_1$. In most circumstances, the point $(x_2, 0)$ where this tangent cuts the x-axis will be nearer to the point $(\alpha, 0)$ than $(x_1, 0)$ was. By the point-slope formula, the equation of the tangent to the curve at $x = x_1$ is

$$y - f(x_1) = f'(x_1)(x - x_1).$$

Since $(x_2, 0)$ lies on the tangent line, we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

By repeating the Newton-Raphson process, we are able to get better approximations to α. In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

2.5.2 Convergence

The rate of convergence when using the Newton-Raphson method depends on the first approximation used and the shape of the curve in the neighbourhood of the root. In extreme cases, these factors may lead to failure (divergence). The three main cases are:

- $|f'(x_1)|$ is too small (extreme case when $f'(x_1) = 0$),
- $f'(x)$ increases/decreases too rapidly ($|f''(x)|$ is too large),
- x_1 is too far away from α.

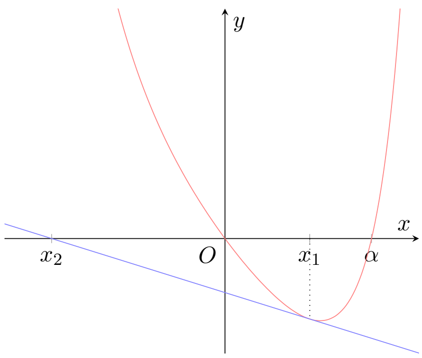


Figure 2.5: Divergence occurs when x_1 is too far away from α.

3 Functions

3.1 Definition and Notation

Definition 3.1.1. A **function** f is a rule or relation that assigns each and every element of $x \in X$ to one and only one element $y \in Y$. We write this as $f : X \to Y$ and read it as "f maps x to Y". X is called the **domain** of f, denoted D_f, while Y is called the **codomain** of f. The elements of y that get mapped to under f is known as the **range** of f, denoted R_f. Mathematically, $R_f = \{f(x) \mid x \in D_f\}$.

To define a function, we must state its rule and specify the domain. There are two ways to represent this:

$$\underbrace{f : x \mapsto x^2 + 1}_{\text{the rule}}, \underbrace{x \in \mathbb{R}}_{D_f} \qquad \text{or} \qquad \underbrace{f(x) = x^2 + 1}_{\text{the rule}}, \underbrace{x \in \mathbb{R}}_{D_f}.$$

Note that two functions are equal if and only if they have the same rule and domain. For instance, the function $g : x \mapsto x^2 + 1$, $x \in \mathbb{Z}$ is not equal to f (as defined above) since their domains are not equal ($\mathbb{R} \neq \mathbb{Z}$).

Note that f is not the same as $f(x)$! f is a *map*, while $f(x)$ is the *value* that f maps x to.

3.2 Graph of a Function

Definition 3.2.1. The **graph** of $f(x)$ is the collection of all points (x, y) in the xy-plane such that the values x and y satisfy $y = f(x)$.

Example 3.2.2 (Graph of a Function)**.**

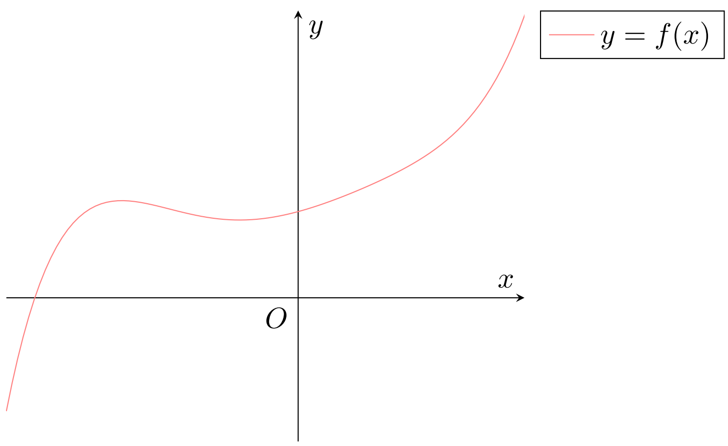


Figure 3.1: An example of a graph of a function.

Proposition 3.2.3 (Vertical Line Test)**.** A relation f is a function if and only if every vertical line $x = k$, $k \in D_f$ cuts the graph of $y = f(x)$ at one and only one point.

Proof. By definition, a function f is a relation which maps each element in the domain to one and only one image. $\square$

3.3 One-One Functions

Definition 3.3.1. A function is said to be **one-one** if no two distinct elements in the given domain have the same image under f. Mathematically,

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

Equivalently, f is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Proposition 3.3.2 (Horizontal Line Test)**.** A function f is one-one if and only if any horizontal line $y = k$, $k \in R_f$ cuts the graph of $y = f(x)$ at one and only one point.

Proof. We only prove the backwards case as the forwards case is trivial. Suppose $y = k$ and $y = f(x)$ intersect more than once. Then there exist two distinct elements x_1 and x_2 in D_f such that $f(x_1) = f(x_2)$, whence f is not one-one. □

Proposition 3.3.3 (Strict Monotonicity Implies One-One)**.** All strictly monotone functions are one-one.

Proof. Seeking a contradiction, assume that there exists a strictly increasing function $f : X \to Y$ which is not one-one. Then there exists $x_1, x_2 \in X$ such that $x_1 \neq x_2 \implies f(x_1) = f(x_2)$. Without loss of generality, assume $x_1 < x_2$, since f is strictly increasing. Then $f(x_1) < f(x_2)$, a contradiction. Therefore, all strictly increasing functions are one-one. Similarly, all strictly decreasing functions are one-one. □

To prove that a function is not one-one, it is sufficient to provide a specific counter-example.

3.4 Inverse Functions

Definition 3.4.1. Let $f : X \to Y$ be a function. Its **inverse function**, $f^{-1} : Y \to X$ is a function that undoes the operation of f. Mathematically, for all $x \in D_f$,

$$f^{-1}(y) = x \iff f(x) = y.$$

Fact 3.4.2 (Properties of Inverse Function)**.**

- f^{-1} exists if and only if f is one-one.
- $D_f = R_{f^{-1}}$ and $R_f = D_{f^{-1}}$.
- The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

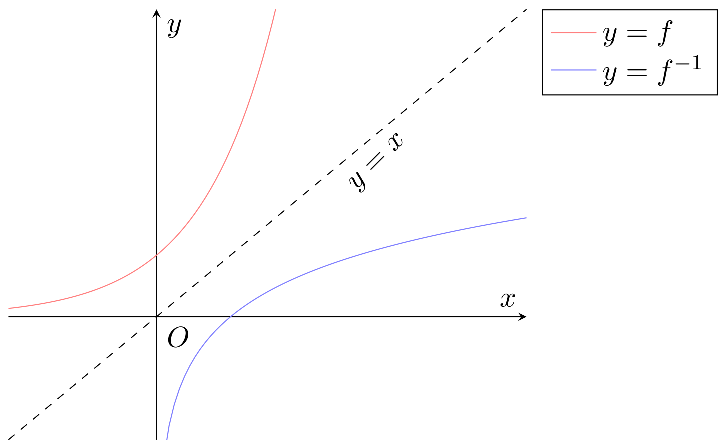


Figure 3.2: The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

3.5 Composite Functions

Definition 3.5.1. Let f and g be functions. Then the **composite function** gf is defined by

$$gf(x) = g(f(x)) = g \circ f(x), \quad x \in D_f.$$

Proposition 3.5.2 (Existence of Composite Function)**.** The composite function gf exists when $R_f \subseteq D_g$.

Proof. Suppose $R_f \nsubseteq D_g$. Then there exists some element y in R_f that is not in D_g. Let the pre-image of y under f be x. Then $gf(x) = g(y)$ is undefined, whence gf is not well-defined and is hence not a function. □

Note that in general, composition of functions is not commutative, i.e. $fg \neq gf$.

We write the composition of f with itself n times as $f^n(x)$. For instance, $ff(x) = f(f(x))$ can be written as $f^2(x)$. This should not be confused with $[f(x)]^n$.

3.5.1 Composition of Inverse Function

Suppose $f : x \mapsto y$ has an inverse $f^{-1} : y \mapsto x$. By the definition of an inverse function.

$$f^{-1} \circ f(x) = f \circ f^{-1}(x) = x.$$

Though $f^{-1}f$ and ff^{-1} have the same rule, they may have different domains. This is because $D_{f^{-1}f} = D_f$, while $D_{ff^{-1}} = D_{f^{-1}}$.

4 Graphs and Transformations

4.1 Characteristics of a Graph

When we sketch a graph, we need to take note of the following characteristics and indicate them on the sketch accordingly:

- **Axial intercepts.** x- and y-intercepts.
- **Stationary points.** Maximum, minimum points and stationary points of inflexion.
- **Asymptotes.** Horizontal, vertical and oblique asymptotes.

When sketching a graph, the shape and any symmetry must be clearly seen.

4.2 Asymptotes

Definition 4.2.1. An **asymptote** is a straight line such that the distance between the curve and the line approaches zero at the extreme end(s) of a graph, i.e. the curve approaches the line but never touches it at these ends.

Definition 4.2.2. Let a and b be constants.

- If $x \to \pm\infty$, $y \to a$, then the line $y = a$ is a **horizontal asymptote**.
- If $x \to a$, $y \to \pm\infty$, then the line $x = a$ is a **vertical asymptote**.
- If $x \to \pm\infty$, $y - (ax + b) \to 0$, then the line $y = ax + b$ is an **oblique asymptote**.

4.3 Even and Odd Functions

Definition 4.3.1. A function $f(x)$ is **even** if and only if $f(-x) = f(x)$ for all x in its domain.

Geometrically, a function is even if and only if the graph $y = f(x)$ is symmetrical about the y-axis.

Definition 4.3.2. A function $f(x)$ is **odd** if and only if $f(-x) = -f(x)$ for all x in its domain.

Geometrically, a function is odd if and only if the graph $y = f(x)$ is symmetrical about the origin.

4.4 Graphs of Rational Functions

A rational function f is a ratio of two polynomials $P(x)$ and $Q(x)$, where $Q(x) \not\equiv 0$.

4.4.1 Rectangular Hyperbola

A rectangular hyperbola is a hyperbola with asymptotes that are perpendicular to each other. The general formula for a rectangular hyperbola is $y = \frac{ax+b}{cx+d}$, where a, b, c and d are constants. Note that the curve $y = \frac{ax+b}{cx+d}$ has a vertical asymptote $x = -d/c$ and a horizontal asymptote $y = a/c$. The two possible shapes of a rectangular hyperbola are shown below.

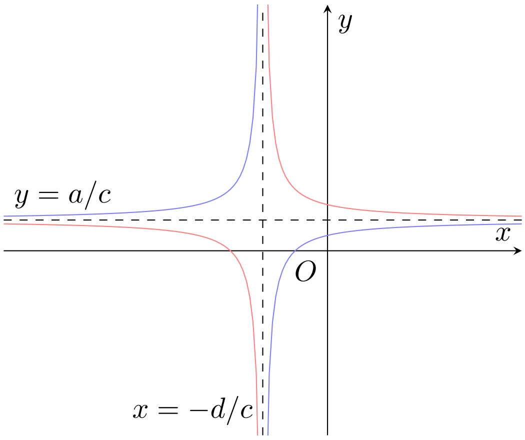


Figure 4.1: Hyperbolas of the form $y = \frac{ax+b}{cx+d}$.

4.4.2 Hyperbolas of the Form $y = \frac{ax^2+bx+c}{dx+e}$

A hyperbola of the form $y = \frac{ax^2+bx+c}{dx+e}$, where a, b, c, d and e are constants, has one vertical and one oblique asymptote. The vertical asymptote has equation $x = -e/d$. To deduce the oblique asymptote, we must first convert the equation to the form $y = px + q + \frac{r}{dx+e}$ (via long division or otherwise). These graphs will generally take one of the two forms below, which can be easily deduced by checking the axial intercepts.

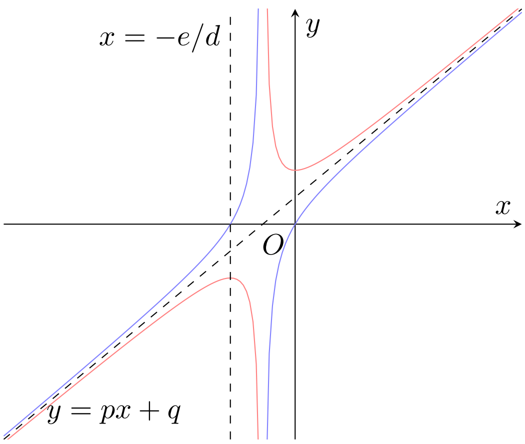


Figure 4.2: Hyperbolas of the form $y = \frac{ax^2+bx+c}{dx+e}$.

4.5 Graphs of Basic Conics

A conic is a curve that can be formed by intersecting a right circular conical surface with a plane. We will examine four types of conics: parabola, circle, ellipse and hyperbola. When sketching graphs of conics, it is important to identify their unique characteristics.

4.5.1 Parabola

Parabolas are curves with equations $y = ax^2$ or $x = by^2$, where a and b are constants.

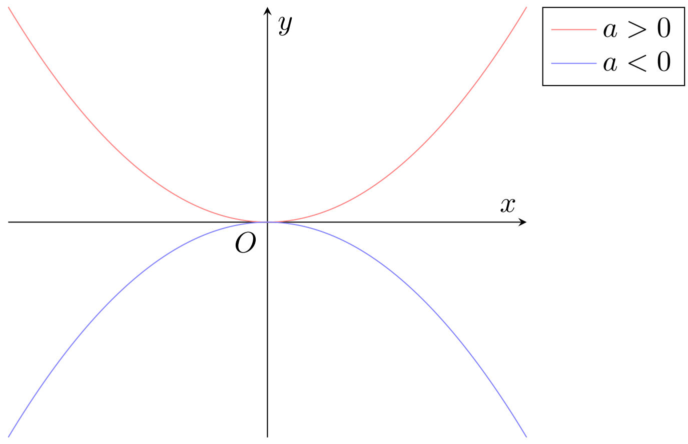


Figure 4.3: Parabolas with equation $y = ax^2$.

Parabolas with equation $y = ax^2$ have a line of symmetry $x = 0$ and a vertex at the origin.

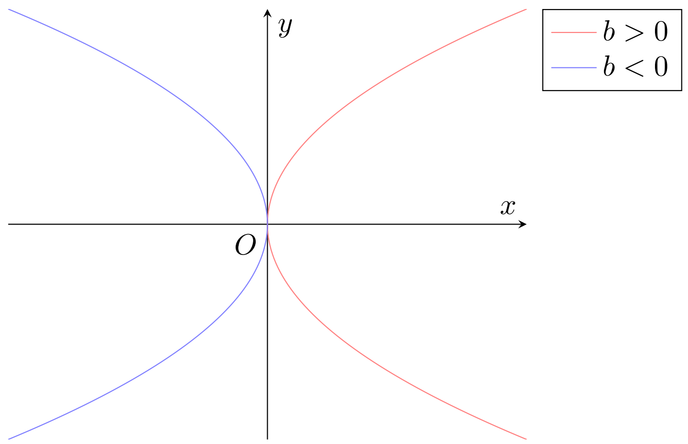


Figure 4.4: Parabolas with equation $x = by^2$.

Parabolas with equation $x = by^2$ have a line of symmetry $y = 0$ and a vertex at the origin.

4.5.2 Circle

A circle is a set of all points in a plane which are the same distance (radius r) from a fixed point (centre). A basic circle with centre at the origin O and radius r is shown below.

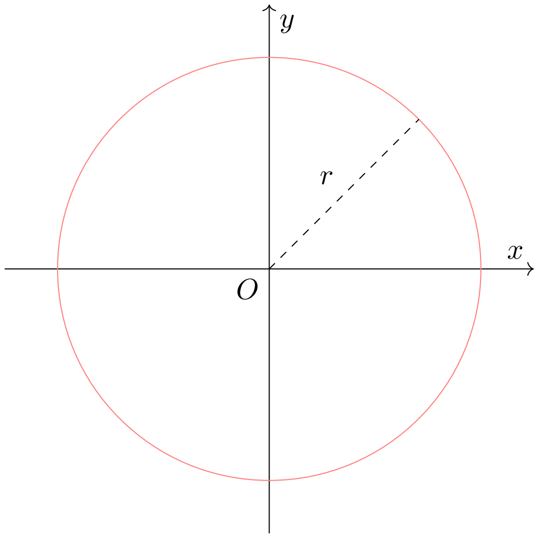


Figure 4.5: Circle with equation $x^2 + y^2 = r^2$.

Any straight line that passes through the centre of the circle is a line of symmetry. The above circle has vertices at $(r, 0)$, $(-r, 0)$, $(0, r)$ and $(0, -r)$.

In general,

- the standard form of the equation of a circle with centre at (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$, where $r > 0$.
- the general form of the equation of a circle is $Ax^2 + Ay^2 + Bx + Cy + D = 0$.

4.5.3 Ellipse

An ellipse is a circle that has been scaled parallel to the x- and/or y-axes. The standard form of the equation of an ellipse centred at $(0, 0)$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b > 0$. a and b are known as the **horizontal** and **vertical radii** respectively.

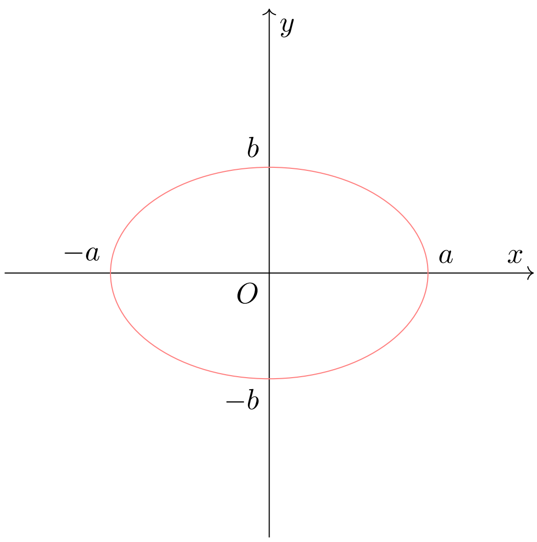


Figure 4.6: Ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The lines of symmetry for the above ellipse are the x- and y-axes, while its vertices are $(a, 0)$, $(-a, 0)$, $(0, b)$ and $(0, -b)$.

In general,

- the standard form of the equation of an ellipse with centre at (h, k) and radius r is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $r > 0$.
- the general form of the equation of an ellipse is $Ax^2 + Bx^2 + Cx + Dy + E = 0$.

4.5.4 Hyperbola

The hyperbola is a conic with two oblique asymptotes. The standard form of a hyperbola centred at the origin O is either $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, where $a, b > 0$, depending on the orientation of the hyperbola.

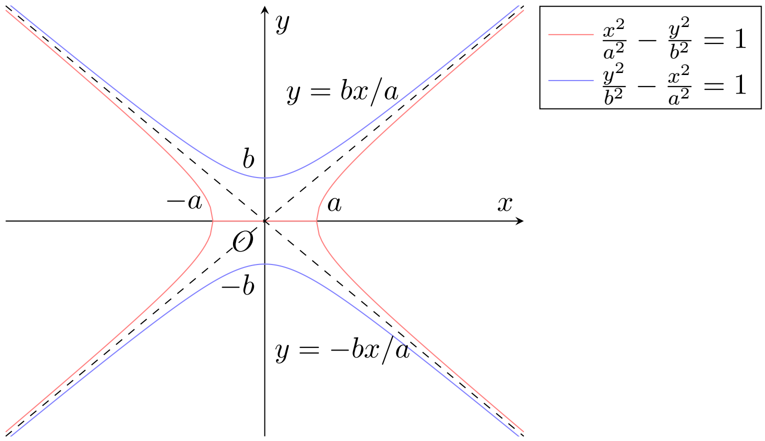


Figure 4.7

Both hyperbolas have the origin as their centres, the x- and y-axes as their lines of symmetry, and their two oblique asymptotes are $y = \pm\frac{b}{a}x$. The hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has vertices $(-a, 0)$ and $(a, 0)$, i.e. a is the horizontal distance from the centre to the vertices. Similarly, the hyperbola with equation $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ has vertices $(0, -b)$ and $(0, b)$, i.e. b is the vertical distance from the centre to the vertices.

In general,

- the standard form of the equation of a hyperbola with centre at (h, k) and radius r is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ where $a, b > 0$.
- the general form of the equation of a hyperbola is $Ax^2 - Bx^2 + Cx + Dy + E = 0$.

4.6 Parametric Equations

Definition 4.6.1. A set of **parametric equations** define a curve by expressing the coordinates (x, y) in terms of an independent variable t (the **parameter**), i.e. $x = f(t)$ and $y = g(t)$.

Example 4.6.2 (Parametric Equations of a Circle)**.** The parametric equations $x = \cos\theta$, $y = \sin\theta$, $\theta \in [0, 2\pi)$ defines a unit circle.

Note that changing the domain of the parameter may change the shape of the curve, even if the same pair of parametric equations are used. Using the above example, if we instead take $\theta \in [0, \pi)$ the resulting curve is that of a semicircle.

To convert a pair of parametric equations to Cartesian form, the parameter must be eliminated. This can be done by either expressing t in terms of x and/or y.

Example 4.6.3 (Parametric to Cartesian via Substitution)**.** Consider the parametric equations $x = t^2 + 2t$, $y = t^2 - 2t$. Observe that $x - y = 4t$, whence $t = (x - y)/4$. Thus, the Cartesian equation of the resulting curve is

$$y = \left(\frac{x-y}{4}\right)^2 + 2\left(\frac{x-y}{4}\right).$$

A similar process is used to convert implicit Cartesian equations into parametric form. Note that explicit Cartesian equations can be trivially converted: simply take $x = t$.

4.7 Basic Linear Transformations

4.7.1 Translation

For $a > 0$,

How $y = f(x)$ was transformed	Graphical effect on $y = f(x)$	Effect on x or y values
y replaced with $y - a$	Translated a units in the positive y-direction.	$(x, y) \mapsto (x, y + a)$
y replaced with $y + a$	Translated a units in the negative y-direction.	$(x, y) \mapsto (x, y - a)$
x replaced with $x - a$	Translated a units in the positive x-direction.	$(x, y) \mapsto (x + a, y)$
x replaced with $x + a$	Translated a units in the negative x-direction.	$(x, y) \mapsto (x - a, y)$

4.7.2 Reflection

For $a > 0$,

How $y = f(x)$ was transformed	Graphical effect on $y = f(x)$	Effect on x or y values
y replaced with $-y$	Reflected in the x-axis.	$(x, y) \mapsto (x, -y)$
x replaced with $-x$	Reflected in the y-axis.	$(x, y) \mapsto (-x, y)$

4.7.3 Scaling

For $a > 0$,

How $y = f(x)$ was transformed	Graphical effect on $y = f(x)$	Effect on x or y values
y replaced with y/a	Scaled by a factor of a parallel to the y-axis.	$(x, y) \mapsto (x, ay)$
x replaced with x/a	Scaled by a factor of a parallel to the x-axis.	$(x, y) \mapsto (ax, y)$

4.8 Relating Graphs to the Graph of $y = f(x)$

4.8.1 Graph of $y = |f(x)|$

Note that

$$y = |f(x)| = \begin{cases} f(x) & f(x) \geq 0, \\ f(-x) & f(x) < 0. \end{cases}$$

Recipe 4.8.1 (Graph of $y = |f(x)|$)**.** To obtain the graph of $y = |f(x)|$ from the graph of $y = f(x)$,

- Retain the portion of $y = f(x)$ above the x-axis.
- Reflect in the x-axis the portion of $y = f(x)$ below the x-axis.

Example 4.8.2 (Graph of $y = |f(x)|$)**.** Consider the following graph of $y = f(x)$.

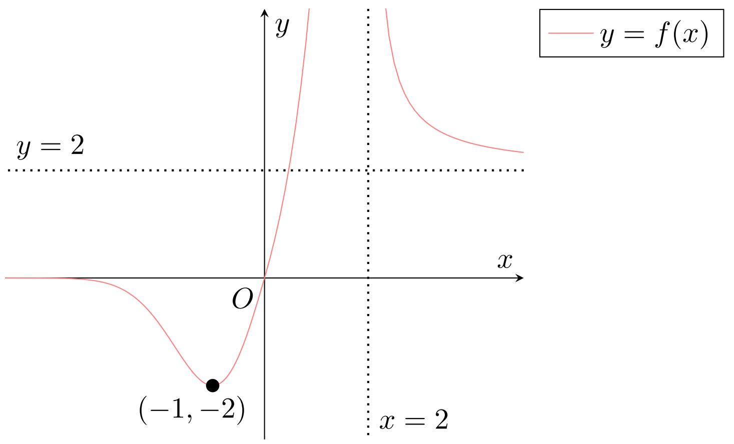


Figure 4.8

Reflecting the portion of the curve below the x-axis, we get the following graph of $y = |f(x)|$.

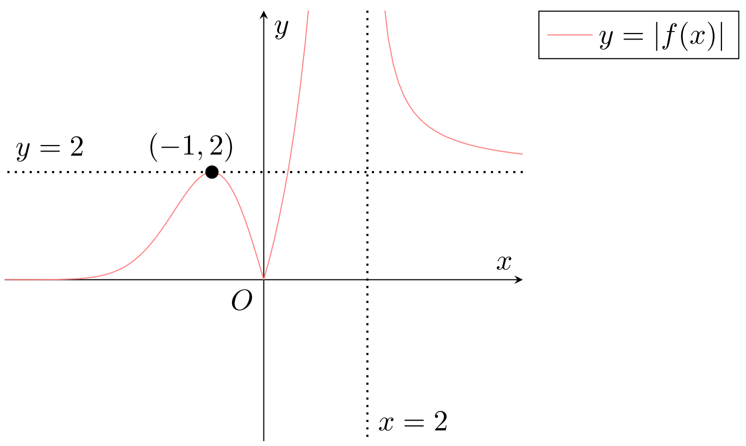


Figure 4.9

4.8.2 Graph of $y = f(|x|)$

Note that

$$y = f(|x|) = \begin{cases} f(x) & x \geq 0, \\ f(-x) & x < 0. \end{cases}$$

Recipe 4.8.3 (Graph of $y = f(|x|)$). To obtain the graph of $y = f(|x|)$ from the graph of $y = f(x)$,

- Retain the portion of $y = f(x)$ where $x \geq 0$.
- Delete the portion of $y = f(x)$ where $x < 0$.
- Copy and reflect in the y-axis the portion of $y = f(x)$ where $x \geq 0$.

Example 4.8.4 (Graph of $y = f(|x|)$). Let the graph of $y = f(x)$ be as in Fig. 4.8. Following the above steps, we see that the graph of $y = f(|x|)$ is

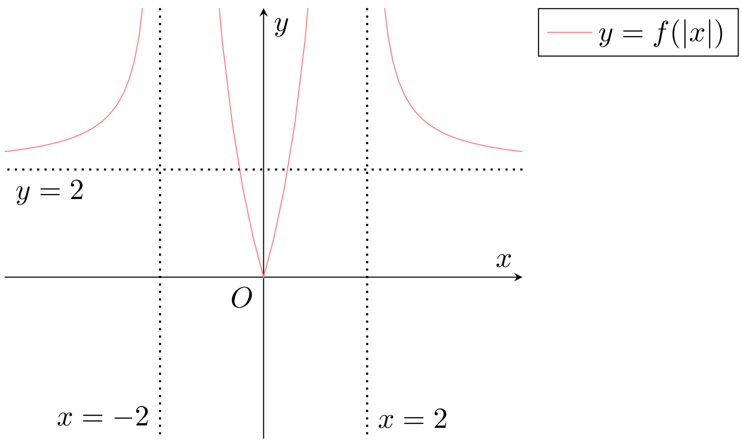


Figure 4.10

4.8.3 Graph of $y = 1/f(x)$

There are several key features and behaviours that we must note when drawing the graph of $y = 1/f(x)$.

- If $y = f(x)$ increases, $1/f(x)$ decreases and vice versa.
- For a minimum point (a, b) where $b \neq 0$ on the graph of $y = f(x)$, it corresponds to a maximum point $(a, 1/b)$ on the graph of $y = 1/f(x)$ and vice versa.
- For an x-intercept $(a, 0)$ on the graph of $y = f(x)$, it corresponds to a vertical asymptote $x = a$ on the graph of $y = 1/f(x)$ and vice versa.
- Oblique asymptotes on the graph of $y = f(x)$ become horizontal asymptotes at $y = 0$ on the graph of $y = 1/f(x)$.

Example 4.8.5 (Graph of $y = 1/f(x)$). Let the graph of $y = f(x)$ be as in Fig. 4.8. Following the above pointers, we see that the graph of $y = 1/f(x)$ is

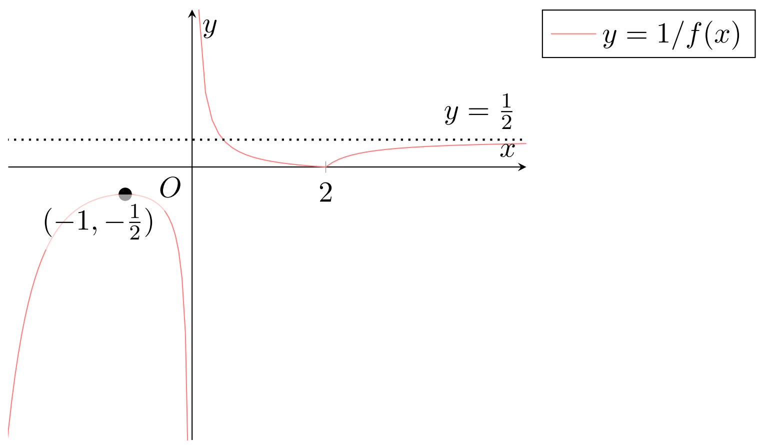


Figure 4.11

5 Polar Coordinates

5.1 Polar Coordinate System

Definition 5.1.1. Let the **pole** (or origin) be a point O in the plane. Let the **initial line** (or polar axis) be a half-line starting at O. Let P be any other point in the plane. Then P has polar coordinates (r, θ), where r is the distance from O to P and θ is the angle between the initial line and the line OP.

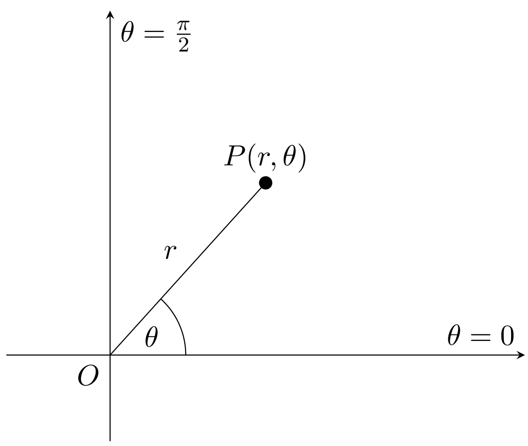


Figure 5.1

There are some conventions regarding the pole and the initial line.

- The initial line is usually drawn horizontally to the right.
- The polar angle θ is positive if measured in the anti-clockwise direction from the initial line and negative in the clockwise direction.
- If $P = 0$, then $r = 0$, and we may use $(0, \theta)$ to represent the pole for any value of θ.

Recall that in the Cartesian coordinate system, each point has a unique representation. This is not the case in the polar coordinate system. For example, the point $(1, \frac{5}{4}\pi)$ could also be written as $(1, \frac{13}{4}\pi)$ or as $(-1, \frac{1}{4}\pi)$. In general, because a complete anti-clockwise rotation is given by the angle 2π, the point (r, θ) can also be represented by $(r, \theta + 2n\pi)$ and $(-r, (2n + 1)\pi)$, where n is any integer.

To avoid this ambiguity, it is common to restrict to $0 \leq \theta < 2\pi$ or $-\pi < \theta \leq \pi$ and to take $r \geq 0$.

5.2 Relationship between the Polar and Cartesian Coordinate Systems

Suppose the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ). From the figure above, we have

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}.$$

Thus,

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Note that while the above were deduced from the case where $r > 0$ and $0 < \theta < \frac{\pi}{2}$, these equations are valid for all values of r and θ.

From the figure, we also have

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x},$$

which allows us to find r and θ when x and y are known.

5.3 Polar Curves

Definition 5.3.1. The **graph of a polar equation** $r = f(\theta)$ consists of all points $P(r, \theta)$ whose coordinates satisfy the equation.

Fact 5.3.2 (Symmetry of Polar Curves)**.**

- If the equation is invariant under $\theta \mapsto -\theta$, the curve is symmetric about the polar axis.
- If the equation is invariant under $r \mapsto -r$, or when $\theta \mapsto \theta + \pi$, the curve is symmetric about the pole (i.e. the curve remains unchanged when rotated by $180°$ about the origin).
- If the equation is invariant when $\theta \mapsto \pi - \theta$, the curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$.
- If r is a function of $\cos n\theta$ only, the curve is symmetric about the horizontal half lines $\theta = \frac{k}{n}\pi$, $k \in \mathbb{Z}$.
- If r is a function of $\sin n\theta$ only, the curve is symmetric about the vertical half-lines $\theta = \frac{2k+1}{2n}\pi$, $k \in \mathbb{Z}$.
- If only even powers of r occur in the equation, the curve is symmetric about the pole.

Proposition 5.3.3 (Tangents to Polar Curves)**.** The gradient of the tangent to a polar curve $r = f(\theta)$ at any point is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}.$$

Proof. Recall that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Differentiating with respect to θ,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = r'\cos\theta - r\sin\theta, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = r'\sin\theta + r\cos\theta.$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}.$$

□

Remark. To find horizontal tangents (i.e. $\mathrm{d}y/\mathrm{d}x = 0$), we can solve $\mathrm{d}y/\mathrm{d}\theta = 0$ (provided $\mathrm{d}x/\mathrm{d}\theta \neq 0$). Likewise, to find vertical tangents (i.e. $\mathrm{d}y/\mathrm{d}x$ undefined), we can solve $\mathrm{d}x/\mathrm{d}\theta = 0$ (provided $\mathrm{d}y/\mathrm{d}\theta \neq 0$). Lastly, if we are looking for tangent lines at the pole, where $r = 0$, the equation simplifies to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta,$$

provided $\mathrm{d}r/\mathrm{d}\theta \neq 0$.

Part II

Sequences and Series

6 Sequences and Series

6.1 Sequences

Definition 6.1.1. A **sequence** or **progression** is a set of numbers $u_1, u_2, u_3, \ldots, u_n, \ldots$ arranged in a defined order according to a certain rule. In general, u_n is called the **nth term**.

Remark. A sequence can be thought of as a function with domain $\mathbb{Z}^+$.

Definition 6.1.2. A sequence is said to be **finite** if it terminates; otherwise it is an **infinite sequence**.

Definition 6.1.3. If an infinite sequence u_n approaches a unique value l as $n \to \infty$, then the sequence is said to **converge** to l. We say that l is the **limit** of u_n. A sequence that does not converge is said to **diverge**.

When describing sequences, one should identify

- Trends (increasing/decreasing, constant, alternating)
- Long-run behaviour of an infinite sequence (convergent or divergent)

6.2 Series

Definition 6.2.1. A **series** is the sum of the terms of a sequence u_n. The sum to n terms is denoted by S_n, i.e.

$$S_n = u_1 + u_2 + \cdots + u_{n-1} + u_n.$$

Similar to sequences, a series can be finite or infinite. If a series is infinite, it can further be categorized as convergent or divergent.

6.3 Arithmetic Progression

Definition 6.3.1. An **arithmetic progression** (AP) is a sequence u_n in which each term differs from the preceding term by a constant called the **common difference**. The first term of an AP is usually denoted by a and the common difference by d. Mathematically,

$$u_n = a + (n-1)d.$$

Definition 6.3.2. An **arithmetic series** is obtained by adding the terms of an arithmetic progression.

Proposition 6.3.3. The nth term S_n of an arithmetic series is given by

$$S_n = \frac{n(a+l)}{2},$$

where l is the last term of the AP, i.e.

$$l = u_n = a + (n-1)d.$$

Proof. Note that for all integers $k \in [1, n]$,

$$u_k + u_{n-k+1} = [a + (k-1)d] + [a + (n-k)d] = a + [a + (n-1)d] = a + l.$$

Hence, by pairing the kth term with the $(n-k+1)$th term, we get

$$2S_n = (u_1+u_n)+(u_2+u_{n-1})+\cdots+(u_{n-1}+u_2)+(u_n+u_1) = n(a+l) \implies S_n = \frac{n(a+l)}{2}.$$

□

6.4 Geometric Progression

Definition 6.4.1. A **geometric progression** (GP) is a sequence u_n in which each term is obtained form the preceding one by multiplying a non-zero constant, called the **common ratio**. The first term of a GP is usually denoted by a and the common ratio by r. Mathematically,

$$u_n = ar^{n-1}.$$

Remark. In the case where $r = 1$, the geometric progression becomes an arithmetic progression.

Definition 6.4.2. A **geometric series** is the sum of the terms of a geometric progression.

Proposition 6.4.3. The nth term S_n of a geometric series is given by

$$S_n = \frac{a(1-r^n)}{1-r},$$

where $r \neq 1$. If the series is infinite, the sum to infinity S_∞ exists only if $|r| < 1$ and is given by

$$S_\infty = \frac{a}{1-r}.$$

Proof. By the definition of a series, we have

$$S_n = a + ar + \cdots + ar^{n-2} + ar^{n-1}. \tag{1}$$

Multiplying both sides by r yields

$$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n. \tag{2}$$

Subtracting (2) from (1), we have

$$(1-r)S_n = a - ar^n \implies S_n = \frac{a(1-r^n)}{1-r}.$$

Suppose $|r| < 1$. In the limit as $n \to \infty$, we have $r^n \to 0$. Hence,

$$S_\infty = \frac{a(1-0)}{1-r} = \frac{a}{1-r}.$$

□

6.5 Sigma Notation

Definition 6.5.1. The series $u_k + u_{k+1} + \cdots + u_m$ can be denoted using Σ (sigma) notation as

$$u_k + u_{k+1} + \cdots + u_m = \sum_{r=k}^{m} u_r.$$

Here, r is called the **index**, and can be replaced with any letter. k is the **lower limit** of r, while m is the **upper limit** of r. There are a total of $m - k + 1$ terms in the sum.

Fact 6.5.2 (Properties of Sigma Notation)**.**

$$\sum_{r=1}^{n} (u_r \pm v_r) = \sum_{r=1}^{n} u_r \pm \sum_{r=1}^{n} v_r.$$

$$\sum_{r=1}^{n} c u_r = c \sum_{r=1}^{n} u_r.$$

$$\sum_{r=m}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{m-1} u_r, \quad n > m > 1.$$

Fact 6.5.3 (Standard Series)**.** The sum of the following standard series can be quoted and applied without proof. Note that $m = q - p + 1$ is the number of terms being summed.

- Series of constants

$$\sum_{r=p}^{q} a = ma.$$

- Arithmetic series

$$\sum_{r=p}^{q} r = \frac{m}{2}(p + q).$$

- Geometric series

$$\sum_{r=p}^{q} a^r = \frac{a^p (a^m - 1)}{a - 1}.$$

7 Recurrence Relations

Definition 7.0.1. A **recurrence relation** is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s).

7.1 First Order Linear Recurrence Relation with Constant Coefficients

Definition 7.1.1. A **first order linear recurrence relation with constant coefficients** is a recurrence relation of the form

$$u_n = au_{n-1} + b,$$

where a and b are constants. If $b = 0$, the recurrence relation is said to be **homogeneous**.

There are two main ways to solve the above recurrence relation: by converting the recurrence relation into a geometric progression, or solving by procedure.

7.1.1 Converting to Geometrical Progression

Recipe 7.1.2 (Converting to Geometrical Progression)**.** Let k be the constant such that

$$u_n + k = a\left(u_{n-1} + k\right).$$

Then we clearly have $k = \frac{b}{a-1}$. We now define a new sequence $v_n = u_n + k$. This turns our recurrence relation into

$$v_n = av_{n-1},$$

whence v_n is in geometric progression. Thus, $v_n = v_1 a^{n-1}$. Writing this back in terms of u_n, we get

$$u_n + k = (u_1 + k)a^{n-1} \implies u_n = (u_1 + k)a^{n-1} - k.$$

Example 7.1.3 (Solving by GP)**.** Consider the recurrence relation

$$u_1 = 0, \quad u_n = \frac{1}{2}u_{n-1} + 10, \quad n > 1.$$

Let k be the constant such that

$$u_n + k = \frac{1}{2}\left(u_{n-1} + k\right).$$

Then

$$k = \frac{10}{1/2 - 1} = -20.$$

We hence have

$$u_n - 20 = \frac{1}{2}\left(u_{n-1} - 20\right),$$

whence the sequence $\{u_n - 20\}$ is in geometric progression with common ratio $1/2$. Thus,

$$u_n - 20 = (u_1 - 20)\left(\frac{1}{2}\right)^{n-1}.$$

Rearranging, we obtain the solution

$$u_n = -20\left(\frac{1}{2}\right)^{n-1} + 20 = -40\left(\frac{1}{2}\right)^{n} + 20.$$

7.1.2 Solving by Procedure

Definition 7.1.4. Given a first order linear recurrence relation with constant coefficients $u_n = au_{n-1} + b$,

- $u_n = au_{n-1}$ is the **associated homogeneous recurrence relation**.
- $u_n^{(c)} = Ca^n$ is the general solution of the associated homogeneous recurrence relation and is called the **complementary solution**.
- $u_n^{(p)} = k$ is the **particular solution** to the recurrence relation.

Fact 7.1.5 (Solving by Procedure)**.** The general solution is given by

$$u_n = u_n^{(c)} + u_n^{(p)} = Ca^n + k.$$

Example 7.1.6 (Solving by Procedure)**.** Consider the recurrence relation

$$u_1 = 0, \quad u_n = \frac{1}{2}u_{n-1} + 10, \quad n > 1.$$

Observe that the associated homogeneous recurrence relation is $u_n = \frac{1}{2}u_{n-1}$. Hence, the complementary solution is

$$u_n^{(c)} = C\left(\frac{1}{2}\right)^n$$

for some arbitrary constant C. Let the particular solution be $u_n^{(p)} = k$. Then

$$k = \frac{1}{2}k + 10 \implies k = 20.$$

Hence, the general solution is

$$u_n = u_n^{(c)} + u_n^{(p)} = C\left(\frac{1}{2}\right)^n + 20.$$

Using the initial condition $u_1 = 0$, we have

$$0 = C\left(\frac{1}{2}\right)^1 + 20 \implies C = -40.$$

Thus,

$$u_n = -40\left(\frac{1}{2}\right)^n + 20.$$

7.2 Second Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Definition 7.2.1. A **second order linear homogeneous recurrence relation with constant coefficients** is a recurrence relation of the form

$$u_n = au_{n-1} + bu_{n-2},$$

where a and b are constants.

Recipe 7.2.2 (Solving by Procedure)**.** To solve the recurrence relation

$$u_n = au_{n-1} + bu_{n-2},$$

1. Form the quadratic equation

$$x^2 - ax - b = 0.$$

   This is called the **characteristic equation**.

2. Find the roots α and β of this characteristic equation.

3. Then u_n has the **general solution**
   - $u_n = A\alpha^n + B\beta^n$, if $\alpha \neq \beta$ (distinct roots, may be real or non-real).
   - $u_n = (A + Bn)\alpha^n$, if $\alpha = \beta$ (real and equal roots).
   - $u_n = Ar^n \cos n\theta + Br^n \sin n\theta$, if $\alpha = re^{i\theta}$ and $\beta = re^{-i\theta}$ (non-real roots).

Proof. For $u_{n+1} = pu_n + qu_{n-1}$ with given initial conditions u_1 and u_2, let the constant k be such that

$$u_{n+1} - ku_n = (p - k)(u_n - ku_{n-1}). \tag{1}$$

Note that this is a GP. Comparing coefficients of u_{n-1}, we get

$$(p - k)k = -q \implies k^2 - pk - q = 0.$$

This is the characteristic equation. Let the roots to the characteristic equation be $k = \alpha$ and $k = \beta$. By Vieta's formulas,

$$\alpha + \beta = -\left(\frac{-p}{1}\right) = p.$$

Now, using the fact that (1) is in GP, we get

$$u_{n+1} - ku_n = (p - k)^{n-1}(u_2 - ku_1). \tag{2}$$

Substituting $k = \alpha$ into (2), we obtain

$$u_{n+1} - \alpha u_n = \beta^{n-1}(u_2 - \alpha u_1). \tag{3a}$$

Substituting $k = \beta$ into (2), we obtain

$$u_{n+1} - \beta u_n = \alpha^{n-1}(u_2 - \beta u_1). \tag{3b}$$

We now analyse the case where $\alpha = \beta$ and $\alpha \neq \beta$ separately.

Case 1: $\alpha = \beta$. Since the two roots are equal, (3a) and (3b) are equivalent. Taking either,

$$u_{n+1} - \alpha u_n = \alpha^{n-1}(u_2 - \alpha u_1) \implies \frac{u_{n+1}}{\alpha^{n-1}} - \frac{u_n}{\alpha^{n-2}} = u_2 - \alpha u_1.$$

The sequence $\left\{\frac{u_n}{\alpha^{n-2}}\right\}$ is hence in AP with common difference $u_2 - \alpha u_1$. Invoking the closed form for AP, we obtain

$$\frac{u_n}{\alpha^{n-2}} = \frac{u_1}{\alpha^{-1}} + (n-1)(u_2 - \alpha u_1) \implies u_n = \alpha^{n-2}\left(\frac{u_1}{\alpha^{-1}} + (n-1)(u_2 - \alpha u_1)\right).$$

Simplifying,

$$u_n = \left[\left(\frac{2u_1}{\alpha} - \frac{u_2}{\alpha^2}\right) + \left(\frac{u_2}{\alpha^2} - \frac{u_1}{\alpha}\right)n\right]\alpha^n = (A + Bn)\alpha^n.$$

Case 2: $\alpha \neq \beta$. Observe that $\frac{(3\text{b})-(3\text{a})}{\alpha-\beta}$ yields

$$u_n = \frac{\alpha^{n-1}(u_2 - \beta u_1) - \beta^{n-1}(u_2 - \alpha u_1)}{\alpha - \beta}.$$

Simplifying, we have

$$u_n = \left[\frac{u_2 - \beta u_1}{\alpha(\alpha - \beta)}\right]\alpha^n + \left[\frac{u_2 - \alpha u_1}{\beta(\beta - \alpha)}\right]\beta^n = A\alpha^n + B\beta^n.$$

We now consider the case where α and β are non-real. By the conjugate root theorem, we can write $\alpha = r\mathrm{e}^{\mathrm{i}\theta}$ and $\beta = r\mathrm{e}^{-\mathrm{i}\theta}$. Substituting this into the above result, we have

$$u_n = A\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^n + B\left(r\mathrm{e}^{-\mathrm{i}\theta}\right)^n = r^n\left(A\mathrm{e}^{\mathrm{i}n\theta} + B\mathrm{e}^{-\mathrm{i}n\theta}\right).$$

By Euler's identity,

$$u_n = r^n\left[(A + B)\cos n\theta + \mathrm{i}(A - B)\sin n\theta\right] = Cr^n\cos n\theta + Dr^n\sin n\theta.$$

□

Part III

Vectors

8 Basic Properties of Vectors

8.1 Basic Definitions and Notations

Definition 8.1.1. A **vector** is an object that has both magnitude and direction. Geometrically, we can represent a vector by a **directed** line segment $\overrightarrow{PQ}$, where the length of the line segment represents the magnitude of the vector, and the direction of the line segment represents the direction of the vector. Vectors are typically denoted by bold print (e.g. $\mathbf{a}$) or by $\overrightarrow{PQ}$.

Definition 8.1.2. The **magnitude** of a vector $\mathbf{a}$ is the length of the line representing $\mathbf{a}$, and is denoted by $|\mathbf{a}|$.

Definition 8.1.3. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be **equal vectors** if they both have the same magnitude and direction. $\mathbf{a}$ and $\mathbf{b}$ are said to be **negative vectors** if they have the same magnitude but opposite directions.

Definition 8.1.4 (Multiplication of a Vector by a Scalar)**.** Let λ be a scalar. If $\lambda > 0$, then $\lambda\mathbf{a}$ is a vector of magnitude $\lambda\,|\mathbf{a}|$ and has the same direction as $\mathbf{a}$. If $\lambda < 0$, then $\lambda\mathbf{a}$ is a vector of magnitude $-\lambda\,|\mathbf{a}|$ and is in the opposite direction of $\mathbf{a}$.

Definition 8.1.5. The **zero vector** is the vector with a magnitude of 0 and is denoted $\mathbf{0}$.

Definition 8.1.6. Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero vectors. Then $\mathbf{a}$ and $\mathbf{b}$ are said to be **parallel** if and only if $\mathbf{b}$ can be expressed as a non-zero scalar multiple of $\mathbf{a}$. Mathematically,

$$\mathbf{a} \parallel \mathbf{b} \iff (\exists \lambda \in \mathbb{R} \setminus \{0\}) : \quad \mathbf{b} = \lambda\mathbf{a}.$$

Definition 8.1.7. A **unit vector** is a vector with a magnitude of 1. Unit vectors are typically denoted with a hat (e.g. $\hat{\mathbf{a}}$).

Observe that for any non-zero vector $\mathbf{a}$, the unit vector parallel to $\mathbf{a}$ is given by

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Definition 8.1.8. The **Triangle Law of Vector Addition** states that

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

Geometrically, we add two vectors $\mathbf{a}$ and $\mathbf{b}$ by placing them head to tail, taking the resultant vector as their sum.

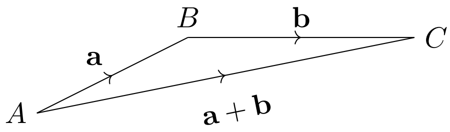


Figure 8.1

We subtract vectors by adding $\mathbf{a} + -(\mathbf{b})$.

Definition 8.1.9. The **angle between two vectors** refers to the angle between their directions when the arrows representing them *both converge* or *both diverge*.

Definition 8.1.10. A **free vector** is a vector that has no specific location in space. The **position vector** of some point A relative to the origin O is unique and is denoted $\overrightarrow{OA}$. A **displacement vector** is a vector that joins its initial position to its final position. For instance, $\overrightarrow{OA}$ is the displacement vector from O to A.

Definition 8.1.11. A set of vectors are said to be **coplanar** if their directions are all parallel to the same plane.

Fact 8.1.12. Any vector $\mathbf{c}$ that is coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be expressed as a **unique linear combination** of $\mathbf{a}$ and $\mathbf{b}$, i.e.

$$(\exists!\, \lambda, \mu \in \mathbb{R}) : \quad \mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}.$$

Theorem 8.1.13 (Ratio Theorem)**.** If P divides AB in the ratio $\lambda : \mu$, then

$$\overrightarrow{OP} = \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}.$$

Proof. Since P divides AB in the ratio $\lambda : \mu$, we have

$$\overrightarrow{AP} = \frac{\lambda}{\lambda + \mu}\overrightarrow{AB} = \frac{\lambda}{\lambda + \mu}(\mathbf{b} - \mathbf{a}).$$

Thus,

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \frac{\lambda}{\lambda + \mu}(\mathbf{b} - \mathbf{a}) = \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}.$$

□

Corollary 8.1.14 (Mid-Point Theorem)**.** If P is the mid-point of AB, then

$$\overrightarrow{OP} = \frac{\mathbf{a} + \mathbf{b}}{2}.$$

8.2 Vector Representation using Cartesian Unit Vectors

8.2.1 2-D Cartesian Unit Vectors

Definition 8.2.1 (2-D Cartesian Unit Vectors). In the 2-D Cartesian plane, $\mathbf{i} = \langle 1,\ 0 \rangle$ is defined to be the unit vector in the positive direction of the x-axis, while $\mathbf{j} = \langle 0,\ 1 \rangle$ is defined to be the unit vector in the positive direction of the y-axis.

Thus, if P is the point with coordinates (a, b), then we can express $\overrightarrow{OP}$ in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. In particular, $\overrightarrow{OP} = a\mathbf{i} + b\mathbf{j}$.

Proposition 8.2.2 (Magnitude in 2-D).

$$\left| \begin{pmatrix} a \\ b \end{pmatrix} \right| = \sqrt{a^2 + b^2}.$$

Proof. Follows immediately from Pythagoras' theorem. □

8.2.2 3-D Cartesian Unit Vectors

Definition 8.2.3 (3-D Cartesian Unit Vectors). In the 3-D Cartesian plane, $\mathbf{i} = \langle 1,\ 0,\ 0 \rangle$, $\mathbf{j} = \langle 0,\ 1,\ 0 \rangle$ and $\mathbf{k} = \langle 0,\ 0,\ 1 \rangle$ denote the unit vectors in the positive direction of the x, y and z-axes respectively.

Proposition 8.2.4 (Magnitude in 3-D).

$$\left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| = \sqrt{a^2 + b^2 + c^2}.$$

Proof. Use Pythagoras' theorem twice. □

Fact 8.2.5 (Operations on Cartesian Vectors). To add vectors given in Cartesian unit vector form, the coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are added separately.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}.$$

Subtraction and scalar multiplication follows immediately.

9 Scalar and Vector Products

9.1 Scalar Product

Definition 9.1.1. The **scalar product** (or dot product) of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ is the angle between the two vectors (note that $0 \leq \theta \leq \pi$).

Remark. $\mathbf{a} \cdot \mathbf{b}$ is called the scalar product as the result is a real number (a scalar). It is also called the dot product because of the notation.

Fact 9.1.2 (Algebraic Properties of Scalar Product)**.** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors and let $\lambda \in \mathbb{R}$. Then

- (commutative) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
- (distributive over addition) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.
- $(\lambda\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda\mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b})$.

Proposition 9.1.3 (Geometric Properties of Scalar Product)**.** Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero vectors, and let θ be the angle between them.

- $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\theta = \frac{\pi}{2}$, i.e. $\mathbf{a} \perp \mathbf{b}$.
- $\mathbf{a} \cdot \mathbf{b} > 0$ if and only if θ is acute.
- $\mathbf{a} \cdot \mathbf{b} < 0$ if and only if θ is obtuse.

Proof. The sign of $\mathbf{a} \cdot \mathbf{b}$ is determined solely by $\cos\theta$. □

Proposition 9.1.4 (Scalar Product in Cartesian Unit Vector Form)**.**

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2.$$

Proof. Since $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are pairwise perpendicular, their pairwise scalar products are 0. That is,

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.$$

Hence, by the distributive property of the scalar product,

$$(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \cdot (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) = x_1x_2\mathbf{i} \cdot \mathbf{i} + y_1y_2\mathbf{j} \cdot \mathbf{j} + z_1z_2\mathbf{k} \cdot \mathbf{k}.$$

Lastly, since $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are all unit vectors,

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1.$$

Thus,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2.$$

□

9.1.1 Applications of Scalar Product

Proposition 9.1.5 (Angle between Two Vectors)**.** Let θ be the angle between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$

Proof. Follows immediately from the definition of the scalar product. □

Definition 9.1.6. Let $\mathbf{a}$ and $\mathbf{b}$ denote the position vectors of A and B respectively, relative to the origin O. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$, and let N be the foot of the perpendicular from the point A to the line passing through O and B.

Then, the length ON is defined to be the **length of projection** of the vector $\mathbf{a}$ onto the vector $\mathbf{b}$. Also, $\overrightarrow{ON}$ is the **vector projection** of $\mathbf{a}$ onto $\mathbf{b}$.

Proposition 9.1.7 (Length of Projection)**.** The length of projection of $\mathbf{a}$ onto $\mathbf{b}$ is $\left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$.

Proof. Consider the case where θ is acute.

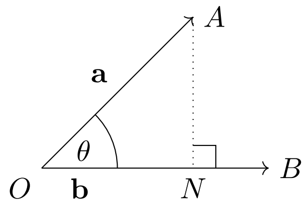


Figure 9.1

From the diagram,

$$ON = OA\cos\theta = |\mathbf{a}|\,\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \mathbf{a}\cdot\hat{\mathbf{b}}.$$

A similar argument shows that when θ is obtuse, $ON = -\mathbf{a}\cdot\hat{\mathbf{b}}$. Hence, in any case, $ON = \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$. □

Proposition 9.1.8 (Vector Projection)**.** The vector projection of $\mathbf{a}$ onto $\mathbf{b}$ is $(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}$.

Proof. Case 1: θ is acute. Then $\overrightarrow{ON}$ is in the same direction as $\mathbf{b}$. Hence,

$$\overrightarrow{ON} = |ON|\,\hat{\mathbf{b}} = (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}.$$

Case 2: θ is obtuse. Then $\overrightarrow{ON}$ is in the opposite direction as $\mathbf{b}$. Hence,

$$\overrightarrow{ON} = |ON|\left(-\hat{\mathbf{b}}\right) = -(\mathbf{a}\cdot\hat{\mathbf{b}})(-\hat{\mathbf{b}}) = (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}.$$

□

9.2 Vector Product

Definition 9.2.1. The **vector product** (or cross product) of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \times \mathbf{b}$ and is defined by

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta \hat{\mathbf{n}},$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, in the direction determined by the right-hand grip rule.

Remark. $\mathbf{a} \times \mathbf{b}$ is called the vector product as the result is a vector. It is also called the cross product due to its notation.

Fact 9.2.2 (Algebraic Properties of Vector Product)**.** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors, and θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

- (anti-commutative) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- (distributive over addition) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$.
- $|\mathbf{a} \times \mathbf{b}| = |a|\,|b| \sin\theta$.
- $(\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$, where $\lambda \in \mathbb{R}$.

Proposition 9.2.3 (Geometric Properties of Vector Product)**.** Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero vectors and θ be the angle between them.

- $|\mathbf{a} \times \mathbf{b}| = 0$ if and only if $\mathbf{a} \parallel \mathbf{b}$.
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$ if and only if $\mathbf{a} \perp \mathbf{b}$.

Proof. Follows from the definition of the vector product (consider $\theta = 0, \frac{\pi}{2}, \pi$). □

Proposition 9.2.4 (Vector Product in Cartesian Unit Vector Form)**.**

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{pmatrix}.$$

Proof. From the geometric properties of the vector product, we have

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

Furthermore, since $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are pairwise perpendicular, by the right-hand grip rule, one has

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Hence, by the distributive property of the vector product,

$$\begin{aligned} &(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \times (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) \\ &\quad = x_1 y_2 \mathbf{k} + x_1 z_2(-\mathbf{j}) + y_1 x_2(-\mathbf{k}) + y_1 z_2 \mathbf{i} + z_1 x_2 \mathbf{j} + z_1 y_2(-\mathbf{i}) \\ &\quad = (y_1 z_2 - z_1 y_2)\mathbf{i} + (z_1 x_2 - x_1 z_2)\mathbf{j} + (x_1 y_2 - y_1 x_2)\mathbf{k}. \end{aligned}$$

□

9.2.1 Applications of Vector Product

Proposition 9.2.5 (Length of Side of Right-Angled Triangle)**.** Let $\mathbf{a}$ and $\mathbf{b}$ denote the position vectors of A and B respectively, relative to the origin O. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$, and let N be the foot of the perpendicular from A to OB. Then

$$AN = \left|\mathbf{a} \times \hat{\mathbf{b}}\right|.$$

Proof. With reference to Fig. 9.1, we have

$$AN = OA \sin\theta = |\mathbf{a}| \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{|\mathbf{a} \times \mathbf{b}|}{\mathbf{b}} = \left|\mathbf{a} \times \hat{\mathbf{b}}\right|.$$

□

Proposition 9.2.6 (Area of Triangles and Parallelogram)**.** Let $ABCD$ be a parallelogram, let $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AC}$, and let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then

$$[\triangle ABC] = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

and

$$[ABCD] = |\mathbf{a} \times \mathbf{b}|.$$

Proof. Recall that the formula for the area of a triangle is

$$[\triangle ABC] = \frac{1}{2}(AB)(AC)\sin\theta = \frac{1}{2}|\mathbf{a}|\,|\mathbf{b}|\sin\theta = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|.$$

Since the area of parallelogram $ABCD$ is twice that of $\triangle ABC$, we immediately have

$$[ABCD] = |\mathbf{a} \times \mathbf{b}|.$$

□

10 Three-Dimensional Vector Geometry

10.1 Lines

10.1.1 Equation of a Line

Definition 10.1.1. The **vector equation** of the line l passing through point A with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ is given by

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \quad \lambda \in \mathbb{R},$$

where $\mathbf{r}$ is the position vector of any point on the line, and λ is a real, scalar parameter. The vector $\mathbf{b}$ is also called the **direction vector** of the line.

Remark. Note that $\mathbf{a}$ can be any position vector on the line and $\mathbf{b}$ can be any vector parallel to the line. Hence, the vector equation of a line is not unique.

Definition 10.1.2. Let $l : \mathbf{r} = \mathbf{a}+\lambda\mathbf{b}$, $\lambda \in \mathbb{R}$. By writing $\mathbf{r} = \langle x, y, z\rangle$, $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, we have

$$\begin{cases} x = a_1 + \lambda b_1 \\ y = a_2 + \lambda b_2, \quad \lambda \in \mathbb{R}. \\ z = a_3 + \lambda b_3 \end{cases}$$

This set of three equations is known as the **parametric equations** of the line l.

Definition 10.1.3. From the parametric form of the line l, by making λ the subject, we have

$$\lambda = \frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}.$$

This equation is known as the **Cartesian equation** of the line l.

Remark. If $b_1 = 0$, we simply have $x = a_1$. A similar result arises when $b_2 = 0$ or $b_3 = 0$.

10.1.2 Point and Line

Proposition 10.1.4 (Relationship between Point and Line)**.** A point C lies on a line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$, if and only if

$$(\exists\lambda \in \mathbb{R}) : \quad \overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{b}.$$

Proof. Trivial. □

Proposition 10.1.5 (Perpendicular Distance between Point and Line)**.** Let C be a point not on the line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$. Let F be the foot of perpendicular from C to l. Then

$$CF = \left|\overrightarrow{AC} \times \hat{\mathbf{b}}\right|.$$

Proof. Trivial (recall the application of the vector product in finding side lengths of right-angled triangles). □

Recipe 10.1.6 (Finding Foot of Perpendicular from Point to Line)**.** Let F be the foot of perpendicular from C to the line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$. To find $\overrightarrow{OF}$, we use the fact that

- F lies on l, i.e. $\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}$.
- $\overrightarrow{CF}$ is perpendicular to l, i.e. $\overrightarrow{CF} \cdot \mathbf{b} = 0$.

10.1.3 Two Lines

Definition 10.1.7. The relationship between two lines in 3-D space can be classified as follows:

- **Parallel lines**: The lines are parallel and non-intersecting;
- **Intersecting lines**: The lines are non-parallel and intersecting;
- **Skew lines**: The lines are non-parallel and non-intersecting.

Remark. Note that parallel and intersecting lines are coplanar, while skew lines are non-coplanar.

Recipe 10.1.8 (Relationship between Two Lines)**.** Consider two distinct lines, $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d}$, $\mu \in \mathbb{R}$.

- l_1 and l_2 are parallel lines if their direction vectors are parallel.
- l_1 and l_2 are intersecting lines if there are unique values of λ and μ such that $\mathbf{a} + \lambda\mathbf{b} = \mathbf{c} + \mu\mathbf{d}$.
- l_2 and l_2 are skew lines if their direction vectors are not parallel and there are no values of λ and μ such that $\mathbf{a} + \lambda\mathbf{b} = \mathbf{c} + \mu\mathbf{d}$.

Proposition 10.1.9 (Acute Angle between Two Lines)**.** Let the acute angle between two lines with direction vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ be θ. Then

$$\cos\theta = \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1|\,|\mathbf{b}_2|}.$$

Proof. Observe that we are essentially finding the angle between the direction vectors of the two lines, which is given by

$$\cos\theta = \frac{\mathbf{b}_1 \cdot \mathbf{b}_2}{|\mathbf{b}_1|\,|\mathbf{b}_2|}.$$

However, to ensure that θ is acute (i.e. $\cos\theta \geq 0$), we introduce a modulus sign in the numerator. Hence,

$$\cos\theta = \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1|\,|\mathbf{b}_2|}.$$

□

10.2 Planes

10.2.1 Equation of a Plane

Definition 10.2.1. Suppose the plane π passes through a fixed point A with position vector $\mathbf{a}$, and π is parallel to two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are not parallel to each other. Then the vector equation (in **parametric form**) of π is given by

$$\pi : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}_1 + \mu \mathbf{b}_2,$$

where $\mathbf{r}$ is the position vector of any point P on π, and λ and μ are real parameters.

Definition 10.2.2. Suppose the plane π passes through a fixed point A with position vector $\mathbf{a}$, and π has normal vector $\mathbf{n}$. Let P be an arbitrary point on π. Then $\overrightarrow{AP}$ is perpendicular to the normal vector $\mathbf{n}$, i.e. $\overrightarrow{AP} \cdot \mathbf{n} = 0$. Since $\overrightarrow{AP} = \mathbf{r} - \mathbf{a}$, by the distributivity of the scalar product, one has

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

This is the **scalar product form** of the vector equation of π, which is more commonly written as

$$\mathbf{r} \cdot \mathbf{n} = d.$$

Definition 10.2.3. Let the plane π have scalar product form

$$\pi : \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

Let $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$. Then

$$\pi : n_1 x + n_2 y + n_3 z = a_1 n_1 + a_2 n_2 + a_3 n_3$$

is the **Cartesian equation** of π, which is more commonly written as

$$\pi : n_1 x + n_2 y + n_3 z = d.$$

Recipe 10.2.4 (Converting between Forms). To convert from parametric form to scalar product form, take $\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2$. To convert from the Cartesian equation to parametric form, express x in terms of y and z, then replace y and z with λ and μ respectively.

Example 10.2.5 (Parametric to Scalar Product Form). Let the plane π have parametric form $\mathbf{r} = \langle 1, 2, 3 \rangle + \lambda \langle 4, 5, 6 \rangle + \mu \langle 7, 8, 9 \rangle$. Then the normal vector to π is given by

$$\mathbf{n} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \times \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} \parallel \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Hence,

$$d = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0,$$

whence π has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0.$$

Example 10.2.6 (Cartesian to Parametric Form)**.** Let the plane π have Cartesian equation

$$x + y + z = 10.$$

Solving for x and replacing y and z with λ and μ respectively, we get

$$x = 10 - \lambda - \mu, \quad y = \lambda, \quad z = \mu.$$

Hence, π has parametric form

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 - \lambda - \mu \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

10.2.2 Point and Plane

Proposition 10.2.7 (Relationship between Point and Plane)**.** A point lies on a plane if and only if its position vector (or its equivalent coordinates) satisfies the equation of the plane.

Proof. Trivial. □

Proposition 10.2.8 (Perpendicular Distance between Point and Plane)**.** Let F be the foot of perpendicular from a point Q to the plane π with vector equation $\pi : \mathbf{r} \cdot \mathbf{n} = d$. Let A be a point on π. Then QF, the perpendicular distance from Q to π, is given by

$$QF = \left| \overrightarrow{QA} \cdot \hat{\mathbf{n}} \right| = \frac{|d - \mathbf{q} \cdot \mathbf{n}|}{|\mathbf{n}|}.$$

Proof. Note that QF is the length of projection of $\overrightarrow{QA}$ onto the normal vector $\mathbf{n}$. Hence,

$$QF = \left| \overrightarrow{QA} \cdot \hat{\mathbf{n}} \right|$$

follows directly from the formula for the length of projection. Now, observe that

$$\overrightarrow{QA} \cdot \mathbf{n} = \overrightarrow{OA} \cdot \mathbf{n} - \overrightarrow{OQ} \cdot \mathbf{n} = d - \mathbf{q} \cdot \mathbf{n}.$$

Hence,

$$QF = \frac{\left| \overrightarrow{QA} \cdot \mathbf{n} \right|}{|\mathbf{n}|} = \frac{|d - \mathbf{q} \cdot \mathbf{n}|}{|\mathbf{n}|}.$$

□

Corollary 10.2.9. OF, the perpendicular distance from the plane π to the origin O, is

$$OF = \frac{|d|}{|\mathbf{n}|}.$$

Proof. Take $\mathbf{q} = \mathbf{0}$. □

Recipe 10.2.10 (Foot of Perpendicular from Point to Plane)**.** Let F be the foot of perpendicular from a point Q to the plane π with vector equation $\pi : \mathbf{r} \cdot \mathbf{n} = d$. To find the position vector $\overrightarrow{OF}$, we use the fact that

- QF is perpendicular to π, i.e. $\overrightarrow{QF} = \lambda\mathbf{n}$ for some $\lambda \in \mathbb{R}$, and
- F lies on π, i.e. $\overrightarrow{OF} \cdot \mathbf{n} = d$.

Example 10.2.11 (Foot of Perpendicular from Point to Plane)**.** Let the plane π have equation $\pi : \mathbf{r} \cdot \langle 1, 2, 3\rangle = 10$. Let $Q(4, 5, 6)$, and let F be the foot of perpendicular from Q to π. We wish to find $\overrightarrow{OF}$.

Since QF is perpendicular to π, we have

$$\overrightarrow{QF} = \lambda \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Hence,

$$\overrightarrow{OF} = \overrightarrow{OQ} + \overrightarrow{QF} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda \begin{pmatrix}1\\2\\3\end{pmatrix}.$$

Taking the scalar product on both sides, we get

$$10 = \overrightarrow{OF} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} = \left[\begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda \begin{pmatrix}1\\2\\3\end{pmatrix}\right] \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} = 32 + 14\lambda.$$

Thus, $\lambda = -11/7$, whence

$$\overrightarrow{OF} = \begin{pmatrix}4\\5\\6\end{pmatrix} - \frac{11}{7}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{7}\begin{pmatrix}17\\13\\9\end{pmatrix}.$$

10.2.3 Line and Plane

Fact 10.2.12 (Relationship between Line and Plane)**.** Given a line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$, and a plane $\pi : \mathbf{r} \cdot \mathbf{n} = d$, there are three possible cases:

- **l and π do not intersect.** l and π are parallel and have no common point.
- **l lies on π.** l and π are parallel and any point on l is also a point on π.
- **l and π intersect once.** l and π are not parallel.

There are two methods to determine the relationship between a line and a plane.

Recipe 10.2.13 (Using Normal Vector)**.**

- If l and π do not intersect, then $\mathbf{b} \cdot \mathbf{n} = 0$ and $\mathbf{a} \cdot vecn \neq d$.
- If l lies on π, then $\mathbf{b} \cdot \mathbf{n} = 0$ and $\mathbf{a} \cdot \mathbf{n} = d$.
- If l and π intersect once, then $\mathbf{b} \cdot \mathbf{n} \neq 0$.

Recipe 10.2.14 (Solving Simultaneous Equations)**.** Solve $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$ and $\pi : \mathbf{r} \cdot \mathbf{n} = d$ simultaneously.

- If there are no solutions, then l and π do not intersect.
- If there are infinitely many solutions, then l lies on π.
- If there is a unique solution, then l and π intersect once.

Proposition 10.2.15 (Acute Angle between Line and Plane)**.** Let θ be the acute angle between the line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$ and the plane $\pi : \mathbf{r} \cdot \mathbf{n} = d$. Then

$$\sin\theta = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}|\,|\mathbf{n}|}.$$

Proof. We first find ϕ, the acute angle between l and the normal. Recall that

$$\cos\phi = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}|\,|\mathbf{n}|}.$$

Since $\phi = \frac{\pi}{2} - \theta$, we have

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}|\,|\mathbf{n}|}.$$

□

10.2.4 Two Planes

Proposition 10.2.16 (Acute Angle between Two Planes)**.** The acute angle θ between two planes $\pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$ is given by

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}.$$

Proof. Consider the following diagram.

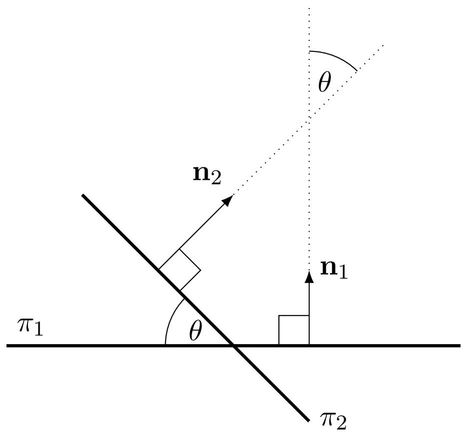


Figure 10.1

It is hence clear that the acute angle between the two planes is equal to the acute angle between the two normal vectors. Thus,

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}.$$

□

Fact 10.2.17 (Relationship between Two Planes). Given two distinct planes $\pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$, there are two possible cases:

- **π_1 and π_2 do not intersect.** The two planes are parallel ($\mathbf{n}_1 \parallel \mathbf{n}_2$).
- **π_1 and π_2 intersect at a line.** The two planes are not parallel ($\mathbf{n}_1 \nparallel \mathbf{n}_2$).

Suppose the two planes are not parallel to each other. There are two methods to obtain the equation of the line of intersection.

Recipe 10.2.18 (Via Cartesian Form). Write the equations of the two planes in Cartesian form and solve the two equations simultaneously.

Recipe 10.2.19 (Via Normal Vectors). Observe that as the line of intersection l lies on both planes, l is perpendicular to both the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Hence, l is parallel to their cross product, $\mathbf{n}_1 \times \mathbf{n}_2$. Thus, if we know a point on the line of intersection l (say point A with position vector $\mathbf{a}$), then the vector equation of l is given by

$$l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}, \quad \lambda \in \mathbb{R},$$

where $\mathbf{b}$ is any scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$.

Part IV

Complex Numbers

11 Introduction to Complex Numbers

Definition 11.0.1. The **imaginary unit** i is a root to the equation

$$x^2 + 1 = 0.$$

11.1 Cartesian Form

Definition 11.1.1. A **complex number** z has **Cartesian form** $x + \mathrm{i}y$, where x and y are real numbers. We call x the **real part** of z, denoted $\operatorname{Re} z$. Likewise, we call y the **imaginary part** of z, denoted $\operatorname{Im} z$.

Definition 11.1.2. The set of complex numbers is denoted $\mathbb{C}$ and is defined as

$$\mathbb{C} = \{z : z = x + \mathrm{i}y, \quad x, y \in \mathbb{R}\}.$$

Remark. The set of real numbers, $\mathbb{R}$, is a proper subset of the set of complex numbers, $\mathbb{C}$. That is, $\mathbb{R} \subset \mathbb{C}$.

Fact 11.1.3 (Algebraic Operations on Complex Numbers). Let $z_1, z_2, z_3 \in \mathbb{C}$.

- Two complex numbers are equal if and only if their corresponding real and imaginary parts are equal.

$$z_1 = z_2 \iff \operatorname{Re} z_1 = \operatorname{Re} z_2 \text{ and } \operatorname{Im} z_1 = \operatorname{Im} z_2.$$

- Addition of complex numbers is commutative, i.e.

$$z_1 + z_2 = z_2 + z_1$$

  and associative, i.e.

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

- Multiplication of complex numbers is commutative, i.e.

$$z_1 z_2 = z_2 z_1,$$

  associative, i.e.

$$z_1(z_2 z_3) = (z_1 z_2) z_3$$

  and distributive, i.e.

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Proposition 11.1.4. Complex numbers cannot be ordered.

Proof. Seeking a contradiction, suppose $\mathrm{i} > 0$. Multiplying both sides by i, we have $\mathrm{i}^2 = -1 > 0$, a contradiction. Hence, we must have $\mathrm{i} < 0$. However, multiplying both sides by i and changing signs (since $\mathrm{i} < 0$), we have $\mathrm{i}^2 = -1 > 0$, another contradiction. Thus, $\mathbb{C}$ cannot be ordered. □

11.2 Argand Diagram

We can represent complex numbers in the complex plane using an Argand diagram.

Definition 11.2.1. The **Argand diagram** is a modified Cartesian plane where the x-axis represents real numbers and the y-axis represents imaginary numbers. The two axes are called the **real axis** and **imaginary axis** correspondingly.

On the Argand diagram, the complex number $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$, can be represented by

- the point $Z(x, y)$ or $Z(z)$; or
- the vector $\overrightarrow{OZ}$.

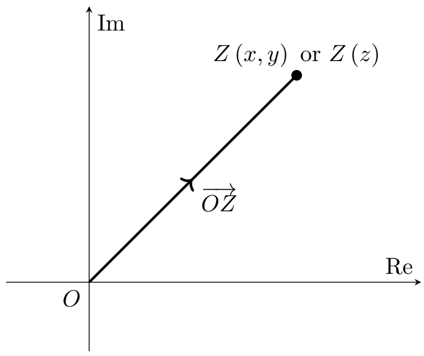


Figure 11.1

In an Argand diagram, let the points Z and W represent the complex numbers z and w respectively. Then $\overrightarrow{OZ}$ and $\overrightarrow{OW}$ are the corresponding vectors representing z and w.

11.2.1 Modulus

Recall in §1, we defined the modulus of a real number x as the "distance" between x and the origin on the real number line. Generalizing this notion to complex numbers, it makes sense to define the modulus of a real number z as the "distance" between z and the origin on the complex plane. This uses Pythagoras' theorem.

Definition 11.2.2. The **modulus** of a complex number z is denoted $|z|$ and is defined as

$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}.$$

11.2.2 Complex Conjugate

Definition 11.2.3. The **conjugate** of the complex number $z = x + \mathrm{i}y$ is denoted z^* with definition

$$z^* = x - \mathrm{i}y.$$

We refer to z and z^* as a **conjugate pair** of complex numbers.

On an Argand diagram, the conjugate z^* is the reflection of z about the real axis.

Fact 11.2.4 (Properties of Complex Conjugates).

- (distributive over addition) $(z + w)^* = z^* + w^*$.
- (distributive over multiplication) $(zw)^* = z^* w^*$.
- (involution) $(z^*)^* = z$.
- $z + z^* = 2\mathrm{Re}(z)$.
- $z - z^* = 2\mathrm{Im}(z)\mathrm{i}$.
- $zz^* = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = |z|^2$.

Because conjugation is distributive over addition and multiplication, we also have the following identities:

$$(kz)^* = kz^*, \qquad (z^n)^* = (z^*)^n,$$

where $k \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Using the conjugate of a complex number z, the reciprocal of z can be computed as

$$\frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}.$$

11.2.3 Argument

Definition 11.2.5. The **argument** of a complex number z is the directed angle θ that $Z(z)$ makes with the positive real axis, and is denoted by $\arg(z)$. Note that $\arg(z) > 0$ when measured in an anticlockwise direction from the positive real axis, and $\arg(z) < 0$ when measured in a clockwise direction from the positive real axis.

Note that $\arg(z)$ is not unique; the position of $Z(z)$ is not affected by adding an integer multiple of 2π to θ. Therefore, if $\arg(z) = \phi$, then $\phi + 2k\pi$, where $k \in \mathbb{Z}$, is also an argument of z. We hence introduce the principal argument of z.

Definition 11.2.6. The value of $\arg(z)$ in the interval $(-\pi, \pi]$ is known as the **principal argument** of z.

The modulus $r = |z|$, complex conjugate z^* and argument $\theta = \arg(z)$ of a complex number z can easily be identified on an Argand diagram:

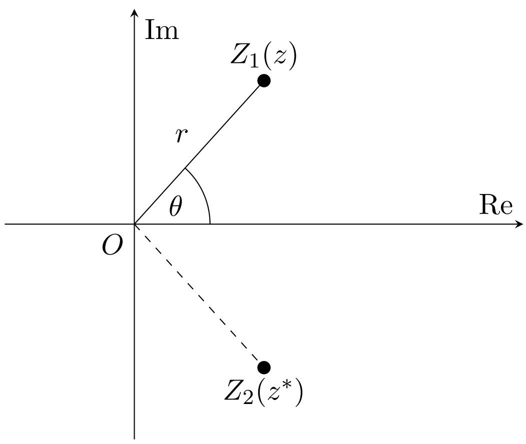


Figure 11.2

11.3 Polar Form

Instead of using Cartesian coordinates on an Argand diagram, we can use polar coordinates, leading to the polar form of a complex number. This polar form can be expressed in two ways: trigonometric form and exponential form.

Definition 11.3.1. The **trigonometric form** of the complex number z is

$$z = r\left(\cos\theta + \mathrm{i}\sin\theta\right),$$

where $r = |z|$ and $\theta = \arg(z)$, $-\pi < \theta \leq \pi$.

Theorem 11.3.2 (Euler's Identity)**.** For all $\theta \in \mathbb{R}$,

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta.$$

Proof 1 (Series Expansion). By the standard series expansion of e^x, we have

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \dots.$$

Simplifying and grouping real and imaginary parts together,

$$\mathrm{e}^{\mathrm{i}\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right),$$

which we recognize to be the standard series expansions of $\cos\theta$ and $\sin\theta$ respectively. Hence,

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta.$$

□

Proof 2 (Differentiation). Let $f(\theta) = \mathrm{e}^{-\mathrm{i}\theta}\left(\cos\theta + \mathrm{i}\sin\theta\right)$. Differentiating with respect to θ,

$$f'(\theta) = \mathrm{e}^{-\mathrm{i}\theta}\left(-\sin\theta + \mathrm{i}\cos\theta\right) - \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\left(\cos\theta + \mathrm{i}\sin\theta\right) = 0.$$

Hence, $f(\theta)$ is constant. Evaluating $f(\theta)$ at $\theta = 0$, we have $f(\theta) = 1$, whence

$$\mathrm{e}^{-\mathrm{i}\theta}\left(\cos\theta + \mathrm{i}\sin\theta\right) = 1 \implies \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta.$$

□

Definition 11.3.3. The **exponential form** of the complex number z is

$$z = r\mathrm{e}^{\mathrm{i}\theta},$$

where $r = |z|$ and $\theta = \arg(z)$, $-\pi < \theta \leq \pi$.

Recall z^* is the reflection of z about the real axis. Hence, we clearly have the following:

Proposition 11.3.4 (Conjugation in Polar Form)**.** If $z = r\mathrm{e}^{\mathrm{i}\theta}$, then $z^* = r\mathrm{e}^{-\mathrm{i}\theta}$. Also,

$$\arg(z^*) = -\theta = -\arg(z), \qquad |z| = r = |z^*|.$$

Using the proposition above, we can convert the results $z + z^* = 2\mathrm{Re}(z)$ and $z - z^* = 2\mathrm{Im}(z)\mathrm{i}$ into polar form:

Proposition 11.3.5.

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta, \qquad e^{i\theta} - e^{-i\theta} = (2\sin\theta)\, i.$$

Lastly, we observe the effect of multiplication and division on the modulus and argument of complex numbers.

Proposition 11.3.6 (Multiplication in Polar Form)**.** Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|, \qquad \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2).$$

Proof. Observe that

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = (r_1 r_2) e^{i(\theta_1+\theta_2)}.$$

The results follow immediately. □

Corollary 11.3.7 (Exponentiation in Polar Form)**.** For $n \in \mathbb{Z}$,

$$|z^n| = r^n = |z|^n, \qquad \arg(z^n) = n\theta = n\arg(z).$$

Proof. Repeatedly apply the above proposition. □

Proposition 11.3.8 (Division in Polar Form)**.** Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}, \qquad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2).$$

Proof. Observe that

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

The results follow immediately. □

11.4 De Moivre's Theorem

Theorem 11.4.1 (De Moivre's Theorem)**.** For $n \in \mathbb{Q}$, if $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, then

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta).$$

Proof. Write z^n in exponential form before converting it into trigonometric form. □

We now discuss some of the applications of de Moivre's theorem.

Recipe 11.4.2 (Finding nth Roots)**.** Suppose we want to find the nth roots of a complex number $w = re^{i\theta}$. We begin by setting up the equation

$$z^n = w = re^{i(\theta+2k\pi)},$$

where $k \in \mathbb{Z}$. Next, we take nth roots on both sides, which yields

$$z = r^{1/n} e^{i(\theta+2k\pi)/n}.$$

Lastly, we pick values of k such that $\arg z = \frac{\theta+2k\pi}{n}$ lies in the principal interval $(-\pi, \pi]$.

Definition 11.4.3. Let $n \in \mathbb{Z}$. The **nth roots of unity** are the n solutions to the equation

$$z^n - 1 = 0.$$

Proposition 11.4.4 (Roots of Unity in Polar Form)**.** The nth roots of unity are given by

$$z = \cos\frac{2k\pi}{n} + \mathrm{i}\sin\frac{2k\pi}{n} = \mathrm{e}^{\mathrm{i}(2k\pi/n)},$$

where $k \in \mathbb{Z}$.

Proof. Use de Moivre's theorem. □

Fact 11.4.5 (Geometric Properties of Roots of Unity)**.** On an Argand diagram, the nth roots of unity

- all lie on a circle of radius 1.
- are equally spaced apart.
- form a regular n-gon.

De Moivre's theorem can also be used to derive trigonometric identities. The trigonometric identities one will be required to prove typically involve reducing "powers" to "multiple angles" (e.g. expressing $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$), or vice versa.

Proposition 11.4.6 (Power to Multiple Angles)**.** Let $z = \cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}$. Then

$$z^n + z^{-n} = 2\cos n\theta, \qquad z^n - z^{-n} = 2\mathrm{i}\sin n\theta.$$

Proof. Use de Moivre's theorem □

Recipe 11.4.7 (Multiple Angles to Powers)**.** Suppose we want to express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. We begin by invoking de Moivre's theorem:

$$\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n.$$

Next, using the binomial theorem,

$$\cos n\theta + \mathrm{i}\sin n\theta = \sum_{k=0}^{n}\binom{n}{k}\cos^k\theta\sin^{n-k}\theta.$$

We then take the real and imaginary parts of both sides to isolate $\cos n\theta$ and $\sin n\theta$:

$$\cos n\theta = \operatorname{Re}\sum_{k=0}^{n}\binom{n}{k}\cos^k\theta\sin^{n-k}\theta, \qquad \sin n\theta = \operatorname{Im}\sum_{k=0}^{n}\binom{n}{k}\cos^k\theta\sin^{n-k}\theta.$$

Example 11.4.8. Suppose we want to write $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$. Using de Moivre's theorem,

$$\cos 2\theta + \mathrm{i}\sin 2\theta = (\cos\theta + \mathrm{i}\sin\theta)^2 = \cos^2\theta + 2\mathrm{i}\cos\theta\sin\theta - \sin^2\theta.$$

Comparing imaginary parts, we obtain

$$\sin 2\theta = 2\cos\theta\sin\theta$$

as expected.

Another way to derive new trigonometric identities is to differentiate known identities.

Example 11.4.9. Using the "power to multiple angle" formula above, one can show that

$$\cos^6\theta = \frac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right).$$

Differentiating, we obtain a new trigonometric identity:

$$\sin\theta\cos^5\theta = \frac{1}{32}\left(\sin 6\theta + 4\sin 4\theta + 5\sin 2\theta\right).$$

11.5 Solving Polynomial Equations over $\mathbb{C}$

Theorem 11.5.1 (Fundamental Theorem of Algebra)**.** A non-zero, single-variable, degree n polynomial with complex coefficients has n roots in $\mathbb{C}$, counted with multiplicity.

Theorem 11.5.2 (Conjugate Root Theorem)**.** For a polynomial equation with all real coefficients, non-real roots must occur in conjugate pairs.

Proof. Suppose z is a non-real root to the polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$. Consider $P(z^*)$.

$$P(z^*) = a_n\left(z^*\right)^n + a_{n-1}\left(z^*\right)^{n-1} + \cdots + a_1\left(z^*\right) + a_0.$$

By conjugation properties, this simplifies to

$$P(z^*) = \left(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)^*,$$

which clearly evaluates to 0, whence z^* is also a root of $P(z)$. □

12 Geometrical Effects of Complex Numbers

12.1 Geometrical Effect of Addition

The following diagram shows the geometrical effect of addition on complex numbers. Here, the point P represents the complex number $z+w$. Observe that $OWPZ$ is a parallelogram (due to the parallelogram law of vector addition).

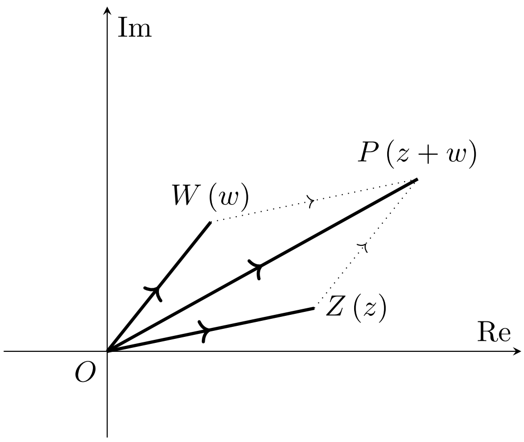


Figure 12.1

12.2 Geometrical Effect of Scalar Multiplication

The following diagram shows the geometrical effect of multiplying a complex number by a real number k. Here, Z_1 represents a point where $k > 1$, Z_2 where $0 < k < 1$, and Z_3 where $k < 0$. Observe that the points lie on the straight line passing through the origin O and the point Z.

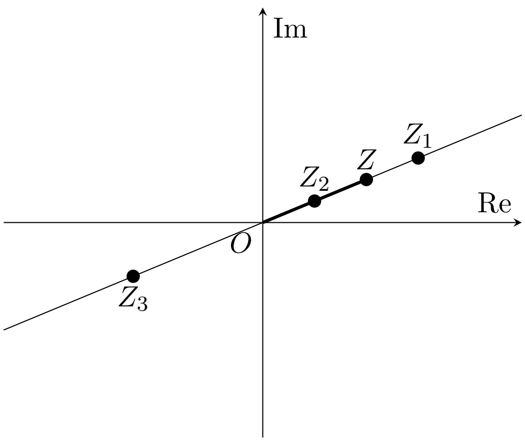


Figure 12.2

12.3 Geometrical Effect of Complex Multiplication

Let points P, Q and R represent the complex numbers z_1, z_2 and z_3 respectively, as illustrated in the Argand diagram below.

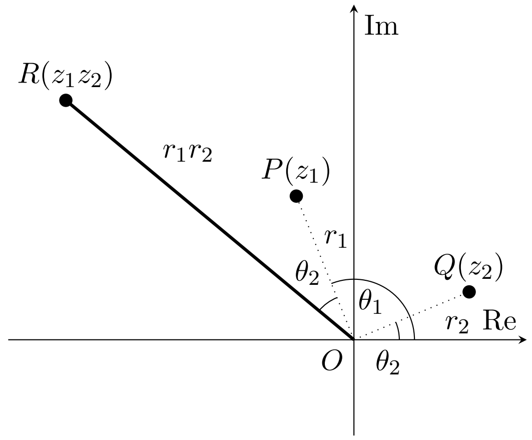


Figure 12.3

Geometrically, the point $R(z_1z_2)$ is obtained by

1. scaling by a factor of r_2 on $\overrightarrow{OP}$ to obtain a new modulus of r_1r_2, followed by
2. rotating $\overrightarrow{OP}$ through an angle θ_2 about O in an anti-clockwise direction if $\theta_2 > 0$ to obtain a new argument $\theta_1 + \theta_2$ (or in a clockwise direction if $\theta_2 < 0$).

12.4 Loci in Argand Diagram

Definition 12.4.1. The **locus** (plural: loci) of a variable point is the path traced out by the point under certain conditions.

12.4.1 Standard Loci

Fact 12.4.2 (Circle). For $|z - a| = r$, with P representing the complex number z and A representing the fixed complex number a and $r > 0$, the locus of P is a circle with centre A and radius r.

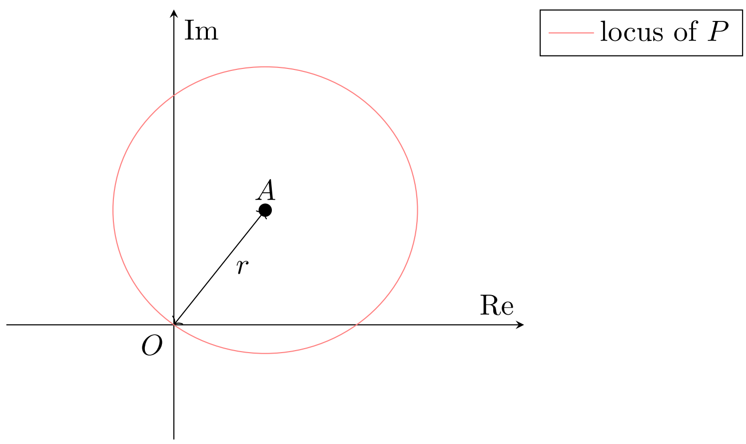


Figure 12.4

Fact 12.4.3 (Perpendicular Bisector). For $|z - a| = |z - b|$, with P representing the complex number z, points A and B representing the fixed complex numbers a and b respectively, the locus of P is the perpendicular bisector of the line segment joining A and B.

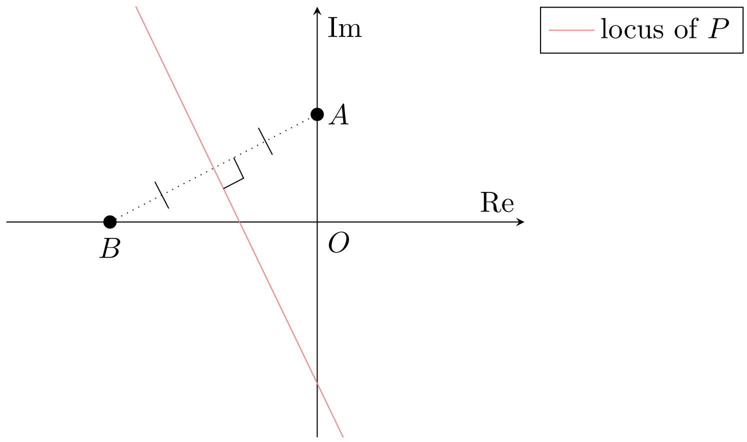


Figure 12.5

Fact 12.4.4 (Half-Line). For $\arg(z - a) = \theta$, with P representing the complex number z and point A representing the fixed complex number a, the locus of P is the half-line starting from A (excluding this point) and inclined at a directed angle θ to the positive real axis.

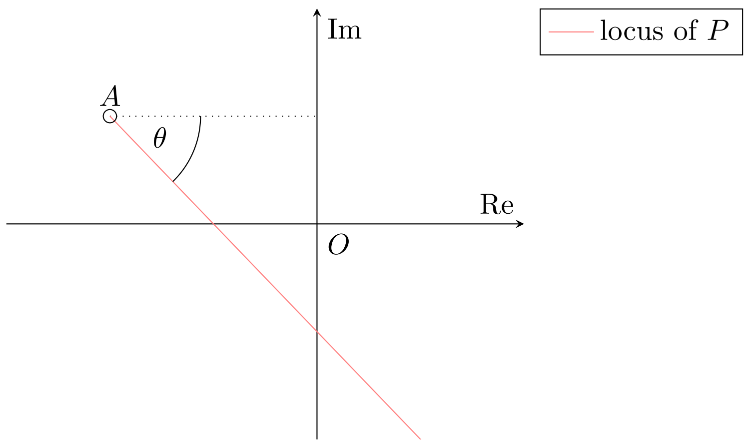


Figure 12.6

12.4.2 Non-Standard Loci

When sketching non-standard loci, one useful technique is to write the equation in Cartesian form, i.e. letting $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$.

Example 12.4.5. Let P be the point representing the complex number z, where z satisfies the equation $\operatorname{Re} z + 2 \operatorname{Im} z = 2$. We begin by writing z in Cartesian form, i.e. $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$. Substituting this into the equation, we have $x + 2y = 2$. Thus, the locus of P is given by the equation $x + 2y = 2$.

12.4.3 Loci and Inequalities

We will use the inequality $|z-(3+4\mathrm{i})|<5$ as an example to illustrate the general procedure of finding the locus of an inequality.

We begin by considering the equality case. As we have seen above, $|z-(3+4\mathrm{i})|=5$ corresponds to a circle centred at $(3,4)$ with radius 5. This is the "boundary" of our locus.

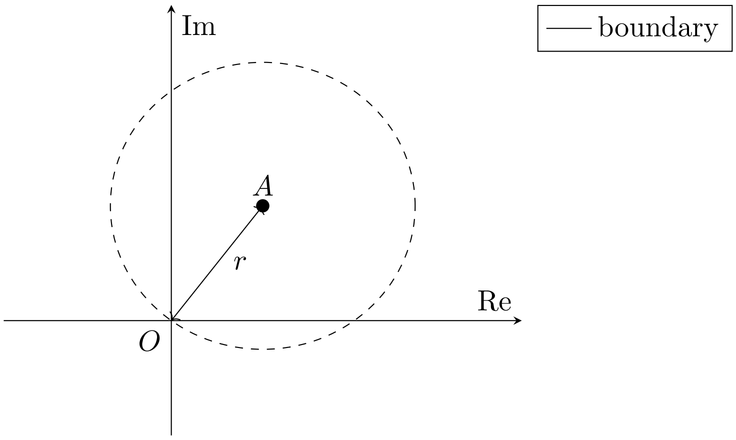


Figure 12.7

Notice that the circle is dashed as the inequality is strict; if the inequality was not strict, i.e. $|z-(3+4\mathrm{i})|\leq 5$, the circle would be drawn with a solid line.

Now, observe that the complex plane has been split into two parts: the interior and exterior of the circle. To determine which region satisfies our inequality, we simply test a complex number in each region.

- Since $3+4\mathrm{i}$ is in the interior of the circle, and $|(3+4\mathrm{i})-(3+4\mathrm{i})|=0<5$, the interior of the circle satisfies the inequality.
- Since $10+4\mathrm{i}$ is in the exterior of the circle, and $|(10+4\mathrm{i})-(3+4\mathrm{i})|=7>5$, the exterior of the circle does not satisfy the inequality.

We thus conclude that the locus of $|z-(3+4\mathrm{i})|<5$ is the interior region of the circle, as shaded below:

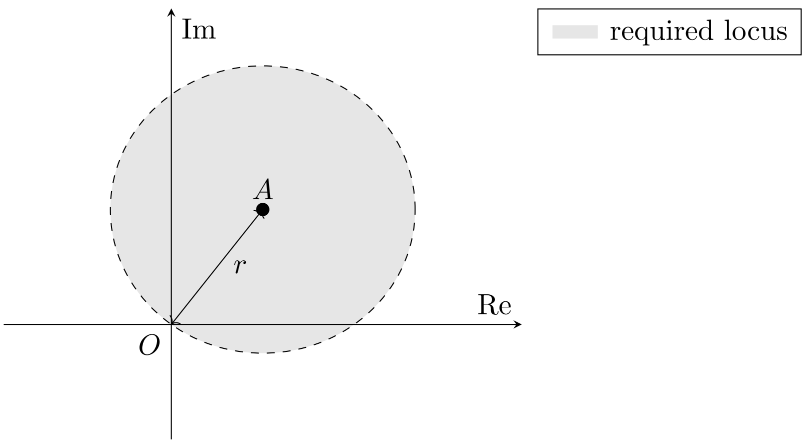


Figure 12.8

12.4.4 Further Use of the Argand Diagram

Many interesting and varied problems involving complex numbers can be solved simply using an Argand diagram. For instance, one may ask what the range of $\arg z$ is, given that z satisfies some other constraint, e.g. $|z - \mathrm{i}| = 1$. Given how diverse these problems may be, there is no general approach to solving them. However, there are several tips that one should keep in mind when doing these problems:

- Think geometrically, not algebraically. Draw out the given constraints on an Argand diagram. Most of the time, the given constraints are simply the three standard loci above (circles, perpendicular bisector and half-lines).
- When working with circles and an external point, drawing tangents and diameters may help. This allows one to use properties of circles (e.g. tangents are perpendicular to the radius).
- Keep an eye out for symmetry or similar figures.

Part V

Calculus

13 Differentiation

13.1 Limits

Let a be a constant.

- $x \to a$ means "x approaches the value a",
- $x \to a^-$ means "x approaches the value a from a value slightly more than a",
- $x \to a^+$ means "x approaches the value a from a value slightly more than a",
- $\lim_{x\to a} f(x)$ means "the limit of $f(x)$ as x approaches a".

Definition 13.1.1. The **limit** of $f(x)$ as x approaches a exists if there exists some $l \in \mathbb{R}$ such that

$$\lim_{x\to a^-} f(x) = l = \lim_{x\to a^+} f(x).$$

We write

$$\lim_{x\to a} f(x) = l.$$

13.2 Derivative

Definition 13.2.1. The **gradient of a straight line** is defined as the ratio of the change in the y-coordinate to that of the x-coordinate between any two points on the line. Mathematically, the gradient m is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

where (x_1, y_1) and (x_2, y_2) are two points on the line.

Definition 13.2.2. The **tangent to the curve at A** is the line *touching* the curve at A.

Definition 13.2.3. The **instantaneous rate of change** or **gradient** of a curve at any point is defined as the gradient of the tangent to the curve at the point.

Definition 13.2.4. The **derivative** of a function $f(x)$, denoted $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ or $f'(a)$, represents the instantaneous rate of change of $f(x)$ with respect to x.

If $y = f(x)$, we write the derivative as $\frac{\mathrm{d}y}{\mathrm{d}x}$ or y'. Note that the symbol $\frac{\mathrm{d}}{\mathrm{d}x}$ means "the derivative with respect to x of" and should be treated as an operation, not a fraction.

Definition 13.2.5. The **nth derivative** of y with respect to x is

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right),$$

where $n \in \mathbb{Z}^+$.

13.2.1 Differentiation from First Principles

Consider a curve $y = f(x)$. Let $A(x, f(x))$ and $B(x + \Delta x, f(x + \Delta x))$ be two points on the curve, where Δx is a small increment in x.

Observe that the gradient of the tangent to the curve at A can be approximated by the gradient of the chord AB, denoted m_{AB}. The closer B is to A, the better the approximation. Therefore, the gradient of the curve at point A is $\lim_{B\to A} m_{AB}$. Now observe that

$$m_{AB} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Additionally, as $B \to A$, $\Delta x \to 0$. Hence,

$$\lim_{B\to A} m_{AB} = \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

For convenience, we replace Δx with h. The derivative is hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}.$$

13.3 Differentiation Rules

Proposition 13.3.1 (Differentiation Rules)**.** Let $k \in \mathbb{R}$ and suppose u and v are functions of x. Then

- (Sum/Difference Rule) If $y = u \pm v$ then $y' = u' \pm v'$.
- (Product Rule) If $y = uv$, then $y' = u'v + uv'$.
- (Quotient Rule) If $y = \frac{u}{v}$, then $y' = \frac{u'v - uv'}{v^2}$.
- (Chain Rule) If $y = f(x)$ and $x = g(t)$, then $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$.

The sum, product and quotient rules are easy to prove from first principles. We hence only prove the chain rule. However, we first need to define differentiability of a function:

Definition 13.3.2. A function $f(x)$ is **differentiable** at a if there exists some function $q(x)$ continuous at a such that

$$[q(x) = \frac{f(x) - f(a)}{x - a}.$$

Note that there is at most one such $q(x)$, and if it exists, then $q(x) = f'(x)$.

We now prove the chain rule.

Proof of Chain Rule. Suppose $y = f(x)$ and $x = g(t)$. Suppose also that $f(x)$ is differentiable at $x = g(a)$, and that $g(t)$ is differentiable at a.

Since $f(x)$ is differentiable at $x = g(a)$, by the above definition, there exists a function $q(x)$ such that

$$q(x) = \frac{f(x) - f(g(a))}{x - g(a)}.$$

Replacing x with $g(t)$, we get

$$q(g(t)) = \frac{f(g(t)) - f(g(a))}{g(t) - g(a)} \implies g(t) - g(a) = \frac{f(g(t)) - f(g(a))}{q(g(t))}. \tag{1}$$

Similarly, since $g(t)$ is differentiable at a, by the above definition, there must exist a function $r(t)$ continuous at a such that

$$r(t) = \frac{g(t) - g(a)}{t - a} \implies g(t) - g(a) = r(t)(t - a). \tag{2}$$

Equating (1) and (2), we have

$$\frac{f(g(t)) - f(g(a))}{q(g(t))} = r(t)(t - a).$$

Rearranging,

$$q(g(t))r(t) = \frac{f(g(t)) - f(g(a))}{t - a} = \frac{(f \circ g)(t) - (f \circ g)(a)}{t - a}.$$

By our assumptions, $q(g(t))r(t)$ is continuous at $t = a$. Hence, by the above definition, $q(g(t))r(t)$ is the derivative of $(f \circ g)'(t)$. Since $q(x) = f'(x)$ and $r(t) = g'(t)$, we arrive at

$$(f \circ g)'(t) = f'(g(t))g'(t).$$

In Liebniz notation, this reads as

$$\frac{\mathrm{d}}{\mathrm{d}t} f(g(t)) = \left[\frac{\mathrm{d}}{\mathrm{d}x} f(g(t))\right] \left[\frac{\mathrm{d}}{\mathrm{d}t} g(t)\right].$$

Since $x = g(t)$ and $y = f(x) = f(g(t))$, this can be written more compactly as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

□

From the chain rule, we can derive the following property:

Proposition 13.3.3. Suppose $\mathrm{d}x/\mathrm{d}y \not\equiv 0$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}.$$

Proof. By the chain rule,

$$1 = \frac{\mathrm{d}y}{\mathrm{d}y} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}.$$

□

Note that this property does not generalize to higher derivatives. For instance, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \frac{1}{\mathrm{d}^2x/\mathrm{d}y^2}$.

13.4 Derivatives of Standard Functions

Let $n, a \in \mathbb{R}$.

y	y'	y	y'	y	y'
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
a^x	$a^x \ln a$	$\sec x$	$\sec x \tan x$	$\csc x$	$-\csc x \cot x$
$\log_a x$	$1/(x \ln a)$	$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$

y	y'
$\arcsin x$	$1/\sqrt{1-x^2}$, $\lvert x\rvert < 1$
$\arccos x$	$-1/\sqrt{1-x^2}$, $\lvert x\rvert < 1$
$\arctan x$	$1/(1+x^2)$

13.5 Implicit Differentiation

Definition 13.5.1. An **explicit function** is one of the form $y = f(x)$, i.e. the dependent variable y is expressed explicitly in terms of the independent variable x, e.g. $y = 2x \sin x + 3$. An **implicit function** is one where the dependent variable y is expressed implicitly in terms of the independent variable x, e.g. $xy + \sin y = 2$.

Recipe 13.5.2 (Implicit Differentiation)**.** y' is found by differentiating every term in the equation with respect to x and with subsequent arrangement, making y' the subject.

Implicit differentiation requires the use of the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} g(y) = \frac{\mathrm{d}}{\mathrm{d}y} g(y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Example 13.5.3 (Implicit Differentiation)**.** Consider the implicit function $3y^3 + x^2 y = 2$. Implicitly differentiating each term with respect to x, we obtain

$$9y^2 y' + \left(x^2 y' + 2xy\right) = 0 \implies y' = \frac{-2xy}{9y^2 + x^2}.$$

Proposition 13.5.4 (Derivative of Inverse Functions)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Proof. Let $y = f^{-1}(x)$. Then $f(y) = x$. Implicitly differentiating,

$$f'(y)\, y' = 1 \implies y' = \frac{1}{f'(y)} = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

□

We can use the above result to derive the derivatives of the inverse trigonometric functions and the logarithm.

Example 13.5.5 (Derivative of $\arcsin x$)**.** Let $f(x) = \sin x$. Then $f'(x) = \cos x$. Using the above result,

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}.$$

Example 13.5.6 (Derivative of $\log_a x$)**.** Let $f(x) = a^x$. Then $f'(x) = a^x \ln a$. Using the above result,

$$\frac{\mathrm{d}}{\mathrm{d}x} \log_a x = \frac{1}{a^{\log_a x} \ln a} = \frac{1}{x \ln a}.$$

13.6 Parametric Differentiation

Sometimes it is difficult to obtain the Cartesian form of a parametric equation, so we are unable to express $\mathrm{d}y/\mathrm{d}x$ in terms of x. However, we are still able to obtain $\mathrm{d}y/\mathrm{d}x$ in terms of the parameter t using the chain rule. If $x = f(t)$ and $g(t)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}.$$

Example 13.6.1 (Parametric Differentiation)**.** Suppose $x = \sin 2\theta$, $y = \cos 4\theta$. Differentiating x and y with respect to θ, we see that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos 2\theta, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = -4\sin 4\theta.$$

Hence, by the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{-2\sin 4\theta}{\cos 2\theta}.$$

14 Applications of Differentiation

14.1 Monotonicity

Definition 14.1.1. Let f be a function, and let $I \subseteq D_f$ be an interval. Let x_1 and x_2 be distinct elements in I.

- f is **strictly increasing** if $x_1 < x_2 \implies f(x_1) < f(x_2)$.
- f is **strictly decreasing** if $x_1 < x_2 \implies f(x_1) > f(x_2)$

Proposition 14.1.2 (Sign of $f'(x)$ Describes Monotonicity)**.** If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I. Similarly, if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

Proof. Suppose $f'(x) > 0$ for all $x \in I$. By the Mean Value Theorem, there exists some $c \in I$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) > 0$ and $x_1 < x_2$, it follows that $f(x_1) < f(x_2)$, whence f is strictly increasing. The proof of the second statement is similar. □

Note that the converse of the above results is not true. Consider the function $f(x) = x^{1/3}$. Clearly, $f(x)$ is increasing on $\mathbb{R}$, yet $f'(x) = x^{-2/3}/3$ is undefined at $x = 0$.

14.2 Concavity

Definition 14.2.1. Let f be a function, and let $I \subseteq D_f$ be an interval.

- f is **concave upwards** on I if the gradient of f increases as x increases.
- f is **concave downwards** on I if the gradient of f decreases as x increases.

Geometrically, f is concave upwards if the graph of $y = f(x)$, $x \in I$ lies above its tangents. Likewise, f is concave downwards if the graph lies below its tangents.

Proposition 14.2.2 (Sign of $f''(x)$ Describes Concavity)**.** If $f''(x) > 0$ for all $x \in I$, then f is concave upwards on I. Similarly, if $f''(x) < 0$ for all $x \in I$, then f is concave downwards on I.

Proof. Suppose $f''(x) > 0$ for all $x \in I$. Then f' is increasing on I. The gradient of f hence increases as x increases, whence f is concave upwards. The proof of the second statement is similar. □

14.3 Stationary Point

Definition 14.3.1. A **stationary point** on a curve $y = f(x)$ is a point where $f'(x) = 0$.

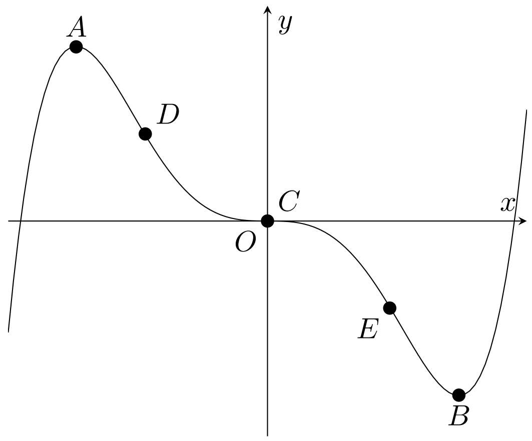


Figure 14.1: Types of stationary points.

There are two types of stationary points:

- turning points: maximum points (A) and minimum points (B)
- stationary points of inflexion: C

Definition 14.3.2. A **point of inflexion** is a point on the curve at which the curve crosses its tangent and the concavity of the curve changes from up to down or vice versa.

Note that a point of inflexion is not necessarily stationary; points D and E in the above figure are **non-stationary points of inflexion**.

14.3.1 Turning Points

In the neighbourhood of turning points, the gradient of the curve, $f'(x)$, changes sign.

Maximum Points

In the neighbourhood of a maximum turning point A, the gradient $f'(x)$ decreases from positive values, through zero at A, to negative values. The y-coordinate of A is known as the **maximum value** of y.

Minimum Points

In the neighbourhood of a minimum turning point B, the gradient $f'(x)$ increases from negative values, through zero at B, to positive values. The y-coordinate of B is known as the **minimum value** of y.

14.3.2 Stationary Points of inflexion

In the neighbourhood of a stationary point of inflexion, the gradient of the curve, $f'(x)$ does not change sign.

14.3.3 Methods to Determine the Nature of Stationary Points

Suppose $y = f(x)$ has stationary point at $x = a$.

Recipe 14.3.3 (First Derivative Test)**.** Check the signs of $f'(x)$ when $x \to a^-$ and $x \to a^+$.

x	a^-	a	a^+	a^-	a	a^+	a^-	a	a^+
$f'(x)$	+ve	0	−ve	−ve	0	+ve	+ve −ve	0 0	+ve −ve
Nature	Maximum point			Minimum point			Stationary point of inflexion		

Example 14.3.4 (First Derivative Test)**.** Let $f(x) = x^2$. Note that $f'(x) = 2x$. Solving for $f'(x) = 0$, we see that $x = 0$ is a stationary point. Checking the signs of y' as $x \to 0^-$ and $x \to 0^+$,

x	0^-	0	0^+
$f'(x)$	−ve	0	+ve

Thus, by the first derivative test, the stationary point at $x = 0$ is a minimum point.

Proposition 14.3.5 (Second Derivative Test)**.** Suppose $f(x)$ has a stationary point at $x = a$.

- If $f''(a) < 0$, then the stationary point is a maximum.
- If $f''(a) > 0$, then the stationary point is a minimum.
- If $f''(a) = 0$, the test is inconclusive.

Proof. At $x = a$, the function $f(x)$ is given by the Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots .$$

When x is arbitrarily close to a, the terms $(x-a)^3$, $(x-a)^4$, ... become negligibly small, whence $f(x)$ is well-approximated by

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Since $x = a$ is a stationary point, $f'(a) = 0$, whence

$$f(x) \approx f(a) + \frac{f''(a)}{2}(x-a)^2.$$

Now observe that $\frac{1}{2}(x-a)^2$ is non-negative. Hence, the sign of $\frac{f''(a)}{2}(x-a)^2$ depends solely on the sign of $f''(a)$: if $f''(a)$ is positive, the entire term is positive and

$$f(x) \approx f(a) + \frac{f''(a)}{2}(x-a)^2 > f(a),$$

whence $f(a)$ is a minimum (since $f(a) < f(x)$ for all x in the neighbourhood of a). Similarly, if $f''(a)$ is negative, the entire term is negative and

$$f(x) \approx f(a) + \frac{f''(a)}{2}(x-a)^2 < f(a),$$

whence $f(a)$ is a maximum. If $f''(a)$ is zero, we cannot say anything about $f(x)$ around $f(a)$ and the test is inconclusive. □

Example 14.3.6 (Second Derivative Test). Let $f(x) = x^2$. From the previous example, we know that $x = 0$ is a stationary point. Since $f''(0) = 2 > 0$, by the second derivative test, it must be a minimum point.

14.4 Graph of $y = f'(x)$

The table below shows the relationships between the graphs of $y = f(x)$ and $y = f'(x)$.

	Graph of $y = f(x)$	Graph of $y = f'(x)$
1a	vertical asymptote $x = a$	vertical asymptote $x = a$
1b	horizontal asymptote $y = b$	horizontal asymptote $y = 0$
1c	oblique asymptote $y = mx + c$	horizontal asymptote $y = b$
2	stationary point at $x = a$	$x = a$ is the x-intercept
3a	f is strictly increasing	curve above the x-axis
3b	f is strictly decreasing	curve below the y-axis
4a	f is concave upward	curve is increasing
4b	f is concave downward	curve is decreasing
5	point of inflexion at $x = a$	maximum or minimum point at $x = a$

For most cases, we can deduce the graph of $y = f'(x)$ by using points (1) to (3) only. Points (4) and (5) are usually for checking.

14.5 Tangents and Normals

Let $P(k, f(k))$ be a point on the graph of $y = f(x)$.

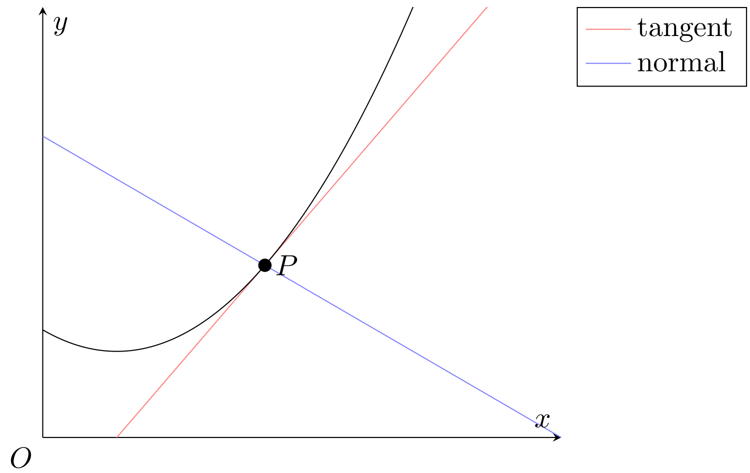


Figure 14.2

The gradient of the tangent to the curve at P is $f'(k)$, while the gradient of the normal to the curve at P is $-1/f'(k)$. This follows from the fact that the tangent and the normal are perpendicular, hence the product of their gradients is -1.

14.6 Optimization Problem

Many real-life situations require that some quantity be minimized (e.g. cost of manufacture) or maximized (e.g. profit on sales). We can use differentiation to solve many of these problems.

Recipe 14.6.1. Suppose we have a dependent variable y that we wish to maximize. We first express y in terms of a single independent variable, say x. We then differentiate y with respect to x and solve for stationary points. Lastly, we determine the nature of the stationary points to obtain the maximum point.

Example 14.6.2. Suppose we wish to enclose the largest rectangular area with only 20 metres of fence. Let x m and y m be the length and width of the rectangular area. The perimeter of the rectangular area is

$$2(x+y) = 20 \implies y = 10 - x.$$

We can hence express the area of the rectangular area A solely in terms of x:

$$A = xy = x(10-x) = -x^2 + 10x.$$

Differentiating A with respect to x, we see that

$$\mathrm{d}A/\mathrm{d}x = -2x + 10.$$

There is hence a stationary point at $x = 5$. By the second derivative test, this is a maximum point. Thus, $x = y = 5$ gives the largest rectangular area.

14.7 Connected Rates of Change

$\mathrm{d}y/\mathrm{d}x$ measures the instantaneous rate of change of y with respect to x. If t represents time, then $\mathrm{d}y/\mathrm{d}t$ represents the rate of change of the variable y with respect to time t. At the same instant, the rates of change can be connected using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Sample Problem 14.7.1. An oil spill spreads on the surface of the ocean, forming a circular shape. The radius of the oil spill r is increasing at a rate of $\mathrm{d}r/\mathrm{d}t = 0.5$ m/min. At what rate is the area of the oil spill increasing when the radius is 10 m?

Solution. Let A be the area of the oil spill. Note that $A = \pi r^2$. Differentiating with respect to r, we get $\mathrm{d}A/\mathrm{d}r = 2\pi r$. Hence, by the chain rule,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = (2\pi r)(0.5) = \pi r.$$

Thus, when the radius is 10 m, the area of the oil spill is increasing at a rate of 10π m/min.
□

15 Maclaurin Series

Definition 15.0.1. A **power series** is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots,$$

where a_n is the constant coefficient of the nth term and c is the **centre** of the power series.

Under certain conditions, a function $f(x)$ can be expressed as a power series. This makes certain operations, such as integration, easier to perform. For instance, the integral $\int x\mathrm{e}^x \, \mathrm{d}x$ is non-elementary. However, we can approximate it by replacing $x\mathrm{e}^x$ with its power series and integrating a polynomial instead.

In this chapter, we will learn how to determine the power series of a given function $f(x)$ with centre $c = 0$ by using differentiation. This particular power series is called the Maclaurin series.

15.1 Deriving the Maclaurin Series

Suppose we can express a function $f(x)$ as a power series with centre $c = 0$. That is, we wish to find constant coefficients such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots. \tag{1}$$

Notice that we can obtain a_0 right away: substituting $x = 0$ into (1) gives

$$f(0) = a_0 + a_1(0) + a_2(0)^2 + \dots = a_0.$$

Now, observe that if we differentiate (1), we get

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots. \tag{2}$$

Once again, we can obtain a_1 using the same trick: substituting $x = 0$ into (2) yields

$$f'(0) = a_1 + 2a_2(0) + 3a_3(0)^2 + \dots = a_1.$$

If we continue this process of differentiating and substituting $x = 0$ into the resulting formula, we can obtain any coefficient we so desire. In general,

$$f^{(n)}(0) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(a_n x^n). \tag{3}$$

However, by repeatedly applying the power rule, we clearly have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^n = \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}nx^{n-1} = \frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}n(n-1)x^{n-2} = \dots = n(n-1)(n-2)\dots(3)(2)(1) = n!.$$

Thus, a simple rearrangement of (3) gives

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

We thus arrive at the formula for the Maclaurin series of $f(x)$:

Definition 15.1.1. The **Maclaurin series** of $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots.$$

There are a few caveats, though:

- The Maclaurin series of $f(x)$ can only be found if $f^{(n)}(0)$ exists for all values of n. For example, $f(x) = \ln x$ cannot be expressed as a Maclaurin series because $f(0) = \ln 0$ is undefined.
- The Maclaurin series may converge to $f(x)$ for only a specific range of values of x. This range is called the **validity range**.

15.2 Binomial Series

Proposition 15.2.1 (Binomial Series Expansion)**.** Let $n \in \mathbb{Q} \setminus \mathbb{Z}^+$. Then

$$(1+x)^n = \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} x^k,$$

with validity range $|x| < 1$.

Proof. Consider $f(x) = (1+x)^n$, where $n \in \mathbb{Q} \setminus \mathbb{Z}^+$. By repeatedly differentiating $f(x)$, it is not too hard to see that

$$f^{(k)}(x) = n(n-1)(n-2)\dots(n-k+1)\,(1+x)^{n-k}.$$

Hence,

$$f^{(k)}(0) = n(n-1)(n-2)\dots(n-k+1).$$

Substituting this into the formula for the Maclaurin series, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} x^k.$$

We now consider the range of validity. If $|x| \geq 1$, then x^k diverges to ∞ as $k \to \infty$. Meanwhile, if $|x| < 1$, then x_k converges to 0 as $k \to \infty$. Hence, the range of validity is $|x| < 1$. □

Note that the binomial theorem is similar to the above result: taking $n \in \mathbb{Z}^+$, we see that

$$\frac{n(n-1)(n-2)\dots(n-k+1)}{k!} = \begin{cases} \binom{n}{k} & k \leq n, \\ 0 & k > n, \end{cases}$$

whence

$$(1+x)^n = \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} x^k = \sum_{k=0}^{n} \binom{n}{k} x^n,$$

which is exactly the binomial theorem. The only difference between the two results is that the range of validity is $\mathbb{R}$ when n is a positive integer. This is because the series is finite (all terms $k > n$ vanish), hence it will always converge.

15.3 Methods to Find Maclaurin Series

15.3.1 Standard Maclaurin Series

Using repeated differentiation, we can derive the following standard Maclaurin series.

$f(x)$	Standard series	Validity range
$(1+x)^n$	$\sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} x^k$	$\lvert x \rvert < 1$
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	all x
$\sin x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	all x (in radians)
$\cos x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	all x (in radians)
$\ln(1+x)$	$\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{r}$	$-1 < x \leq 1$

We can use these standard series to find the Maclaurin series of their composite functions.

Example 15.3.1 (Standard Maclaurin Series). Suppose we wish to find the first three terms of the Maclaurin series of $e^x(1+\sin 2x)$. Using the above standard series, we see that

$$e^x = 1 + x + \frac{x^2}{2} + \cdots, \quad \text{and} \quad 1 + \sin 2x = 1 + 2x + \cdots.$$

Hence,

$$e^x(1+\sin 2x) = \left(1 + x + \frac{x^2}{2} + \cdots\right)(1 + 2x + \cdots)$$
$$= (1+2x) + \left(x + 2x^2\right) + \left(\frac{x^2}{2}\right) + \cdots = 1 + 3x + \frac{5}{2}x^2 + \cdots.$$

15.3.2 Repeated Implicit Differentiation

For complicated functions, it is more efficient to repeatedly implicitly differentiate and substitute $x = 0$ to find the values of $y'(0)$, $y''(0)$, etc.

Example 15.3.2 (Repeated Implicit Differentiation). Suppose we wish to find the first three terms of the Maclaurin series of $y = \ln(1+\cos x)$. Rewriting, we get $e^y = 1 + \cos x$. Implicitly differentiating repeatedly with respect to x,

$$e^y y' = -\sin x \implies e^y\left[\left(y'\right)^2 + y''\right] = -\cos x \implies e^y\left[\left(y'\right)^3 + 3y'y'' + y'''\right] = \sin x$$
$$\implies e^y\left[\left(y'\right)^4 + 3\left(y''\right)^2 + 6\left(y'\right)^2 y'' + 4y'y''' + y^{(4)}\right] = \cos x.$$

Evaluating the above at $x = 0$, we get

$$y(0) = \ln 2, \quad y'(0) = 0, \quad y''(0) = -\frac{1}{2}, \quad y'''(0) = 0, \quad y^{(4)}(0) = -\frac{1}{4}.$$

Thus,

$$\ln(1+\cos x) = \ln 2 + \frac{-1/2}{2!}x^2 + \frac{-1/4}{4!}x^4 + \cdots = \ln 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4 + \cdots.$$

15.4 Approximations using Maclaurin series

Maclaurin series can be used to approximate a function $f(x)$ near $x = 0$.

Example 15.4.1 (Approximating Integrals)**.** Suppose we wish to approximate

$$\int_0^{0.5} \ln(1+\cos x)\,\mathrm{d}x.$$

Doing so analytically is very hard, so we can approximate it using the Maclaurin series of $\ln(1+\cos x)$, which we previously found to be $\ln 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4 + \cdots$. Integrating this expression over the interval $[0, 0.5]$, we get

$$\int_0^{0.5} \ln(1+\cos x)\,\mathrm{d}x \approx \int_0^{0.5} \left(\ln 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4\right)\mathrm{d}x = 0.336092,$$

which is close to the actual value of 0.336091.

Example 15.4.2 (Approximating Constants)**.** For small x,

$$\sin x \approx x - \frac{x^3}{3!}.$$

Since $\sin(\pi/4) = 1/\sqrt{2}$, the numerical value of $1/\sqrt{2}$ can be approximated by substituting $x = \pi/4$ into the above equation:

$$\frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \approx \frac{\pi}{4} - \frac{(\pi/4)^3}{3} = 0.70465.$$

This is close to the actual value of $1/\sqrt{2} \approx 0.70711$.

To improve the approximation, we can

- choose an x-value closer to 0;
- use more terms of the series.

Example 15.4.3 (Improving Approximations)**.** Continuing on from the previous example, we note that $\sin(3\pi/4)$ is also equal to $1/\sqrt{2}$. If we substitute $x = 3\pi/4$ into $\sin x \approx x - x^3/3!$, we get

$$\frac{1}{\sqrt{2}} = \sin\frac{3\pi}{4} \approx \frac{3\pi}{4} - \frac{(3\pi/4)^3}{3} = 0.17607,$$

which is a worse approximation than if we had used $x = \pi/4$. This is because $|\pi/4| < |3\pi/4|$.

15.5 Small Angle Approximation

For x near zero, we can approximate trigonometric functions with just the first few terms of their respective Maclaurin series:

$$\sin x \approx x, \qquad \cos x \approx 1 - \frac{x^2}{2}, \qquad \tan x \approx x.$$

16 Integration

16.1 Indefinite Integration

In the previous chapters, we learnt about differentiation, which can be thought as finding the derivative $f'(x)$ from a function $f(x)$. Reversing this, we define integration as the process of finding the function $f(x)$ from its derivative $f'(x)$. Simply put, integration "undoes" differentiation and vice versa.

16.1.1 Notation and Terminology

Definition 16.1.1. We write the **indefinite integral** with respect to x of a function $f(x)$ as

$$\int f(x)\,\mathrm{d}x.$$

Here, $f(x)$ is called the **integrand**.

Let the derivative of $F(x)$ be $f(x)$, and let c be an arbitrary constant. Since the derivative of a constant is zero, the function $F(x)+C$ will always have the same derivative: $f(x)$. Thus, when we integrate $f(x)$, we don't get back a single function $F(x)$. Instead, we get back a *class* of functions of the form $F(x)+C$. We call $F(x)$ the **primitive** of $f(x)$, and c the **constant of integration**.

With our notation, we can write down the notion of integration "undoing" differentiation mathematically:

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\right]\,\mathrm{d}x = f(x) + C, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\int f(x)\,\mathrm{d}x\right] = f(x).$$

16.1.2 Basic Rules

Fact 16.1.2 (Properties of Indefinite Integrals). Let $f(x)$ and $g(x)$ be any two functions, and let k be a constant.

- (linearity) $\int \left[f(x) + g(x)\right]\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$.
- $\int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x$.

16.2 Definite Integration

Definition 16.2.1. Suppose f is a continuous function defined on the interval $[a, b]$ and $\int f(x)\,\mathrm{d}x = F(x) + C$. Then, the **definite integral** of $f(x)$ from a to b with respect to x is denoted by

$$\int_a^b f(x)\,\mathrm{d}x = \left[F(x)\right]_a^b = F(b) - F(a).$$

We call a the **lower limit** and b the **upper limit** of the integral.

Note that the indefinite integral $\int f(x)\,\mathrm{d}x$ is a function in x, while the definite integral $\int_a^b f(x)\,\mathrm{d}x$ is a numerical value. Also note that x is a **dummy variable** as it does not appear in the final expression of the definite integral; it can be replaced by any symbol.

Fact 16.2.2 (Properties of Definite Integrals). Let $f(x)$ and $g(x)$ be any two functions. Let k and c be constants.

- (linearity) $\int_a^b [f(x)+g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$.
- $\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$.
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.

Note that from the last property, we can deduce the following properties:

$$\int_a^a f(x)\,\mathrm{d}x = 0, \qquad \text{and} \qquad \int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$$

16.3 Integration Techniques

16.3.1 Systematic Integration

Proposition 16.3.1 (Integrals of Standard Functions).

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \qquad (n \neq -1)$$
$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C,$$
$$\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + C.$$

Proposition 16.3.2 (Integrals of Trigonometric Functions).

$$\int \sin x\,\mathrm{d}x = -\cos x + C, \qquad \int \cos x\,\mathrm{d}x = \sin x + C,$$
$$\int \sec x\,\mathrm{d}x = -\ln|\sec x - \tan x| + C, \qquad \int \csc x\,\mathrm{d}x = \ln|\csc x - \cot x| + C,$$
$$\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C, \qquad \int \cot x\,\mathrm{d}x = \ln|\sin x| + C.$$

Equivalently,

$$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| \qquad \text{and} \qquad \int \csc x\,\mathrm{d}x = -\ln|\csc x + \cot x|.$$

Products of trigonometric functions can be easily integrated using the following identities:

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}, \qquad \sin P - \sin Q = 2\sin\frac{P-Q}{2}\cos\frac{P+Q}{2},$$
$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}, \qquad \cos P - \cos Q = 2\sin\frac{P-Q}{2}\sin\frac{P+Q}{2}.$$

Powers of trigonometric functions can also be integrated using the following identities:

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2},$$
$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}, \qquad \cos^3 x = \frac{3\cos x + \cos 3x}{4}.$$

Proposition 16.3.3 (Algebraic Fractions).

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C$$
$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$$
$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

16.3.2 Integration by Substitution

If the given integrand is not in one of the standard forms, it may be possible to reduce it to a standard form by a change of variable. This method is called **integration by substitution**, and it "undoes the chain rule".

Proposition 16.3.4 (Integration by Substitution). Let $F' = f$. Then

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C.$$

Proof. Recall that by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[F(g(x))\right] = F'(g(x))g'(x) = f(g(x))g'(x).$$

Integrating both sides with respect to x,

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C.$$

□

A simpler way to interpret the above formula is as follows:

Recipe 16.3.5 (Integration by Substitution). Given an integral $\int f(x)\,\mathrm{d}x$ and a substitution $x = g(u)$, convert all instances of x in terms of u. This includes replacing $\mathrm{d}x$ with $\mathrm{d}u$, which can be found by "splitting" $\mathrm{d}x/\mathrm{d}u$:

$$\frac{\mathrm{d}x}{\mathrm{d}u} = g'(u) \implies \mathrm{d}x = g'(u)\,\mathrm{d}u.$$

If the integral is definite, the bounds should also be converted to their corresponding u values. Once the integral has been evaluated, all instances of u should be converted back to x.

Example 16.3.6 (Definite Integration by Substitution). Consider the definite integral

$$\int_{2/\sqrt{3}}^{2} \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x.$$

Under the substitution $x = 1/u$, we have

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u^2} \implies \mathrm{d}x = -\frac{1}{u^2}\,\mathrm{d}u.$$

When $x = 2/\sqrt{3}$, $u = \sqrt{3}/2$. When $x = 2$, $u = 1/2$. Thus, the integral becomes

$$\int_{\sqrt{3}/2}^{1/2} \frac{u}{\sqrt{u^{-2}-1}} \frac{1}{u^2}\,\mathrm{d}u = \int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u = [\arcsin u]_{1/2}^{\sqrt{3}/2} = \frac{\pi}{6}.$$

Example 16.3.7 (Indefinite Integration by Substitution). Consider the indefinite integral

$$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x.$$

Following the same substitution as above ($x = 1/u$), we get

$$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u = \arcsin u + C = \arcsin\frac{1}{x} + C.$$

16.3.3 Integration by Parts

Just like integration by substitution "undoes" the chain rule, **integration by parts** "undoes" the product rule.

Proposition 16.3.8 (Integration by Parts). Let u and v be functions of x. Then

$$\int uv'\,\mathrm{d}x = uv - \int vu'\,\mathrm{d}x.$$

For definite integrals,

$$\int_a^b uv'\,\mathrm{d}x = [uv]_a^b - \int_a^b vu'\,\mathrm{d}x.$$

Proof. By the product rule,

$$(uv)' = uv' + u'v.$$

Integrating both sides and rearranging yields the desired result. □

The statement is also sometimes written as

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

As we just learnt in the previous section, the two forms are perfectly equivalent under substitution (simply substitute x for u and v in the integrands).

Care must be exercised in the choice of the factor u. The aim is to ensure that $u'v$ on the RHS is easier to integrate than uv'. To choose u, we can use the following guideline:

Recipe 16.3.9 (LIATE). In decreasing order of suitability, u should be

- **L**ogarithmic
- **I**nverse trigonometric
- **A**lgebraic
- **T**rigonometric
- **E**xponential

Example 16.3.10 (Integration by Parts). Consider the integral $\int \ln x \, \mathrm{d}x$. Picking $u = \ln x$ and $v' = 1$, we get

$$\int \ln x \, \mathrm{d}x = uv - \int u'v \, \mathrm{d}x = (\ln x)(x) - \int \left(\frac{1}{x}\right)(x) \, \mathrm{d}x = x \ln x - x + C.$$

The astute reader would have noticed that we actually dropped an arbitrary constant when integrating v in the above example. We picked $v' = 1$ but only got $v = x$, instead of the expected $v = x + C$. However, including the arbitrary constant does not matter: if we replace v with $v + C$ into the integration by parts formula, we get

$$\int u \, \mathrm{d}v = u(v + C) - \int (v + C) \, \mathrm{d}u = uv + Cu - \left(\int v \, \mathrm{d}u + Cu\right) = uv - \int v \, \mathrm{d}u,$$

which is what we would have got had we not included the arbitrary constant C.

However, this is not to say that we should always drop the arbitrary constant. In certain situations, including it might actually prove more useful, as demonstrated in the following example.

Example 16.3.11 (Including Arbitrary Constant). Consider the integral $\int \ln(x + 1) \, \mathrm{d}x$. Picking $u = \ln(x + 1)$ and $v' = 1$ (which implies $v = x + C$), we get

$$\int \ln(x + 1) \, \mathrm{d}x = uv - \int u'v \, \mathrm{d}x = (x + C) \ln(x + 1) - \int \frac{x + C}{x + 1} \, \mathrm{d}x.$$

Here, a convenient choice for C would be 1, as the integral on the RHS would simplify to $\int 1 \, \mathrm{d}x$, which we can easily integrate. Thus,

$$\int \ln(x + 1) \, \mathrm{d}x = (x + 1) \ln(x + 1) - x + C.$$

If evaluating an integral requires doing multiple integration by parts in succession, the DI method is more convenient.

Recipe 16.3.12 (DI Method). Given the integral $\int uv \, \mathrm{d}x$, construct the following table:

	D	I
$+$	u	v
$-$	u'	$v^{(-1)}$
$+$	u''	$v^{(-2)}$
$\vdots$	$\vdots$	$\vdots$
$\pm$	$u^{(n)}$	$v^{(-n)}$

In other words, keep differentiating the middle column (u) and keep integrating the right column (v), while alternating the sign in the left column. This sign is "attached" to the u terms.

Next, draw diagonal arrows from the middle column to the right column one row below. For instance, u is arrowed to $v^{(-1)}$, while u' is arrowed to $v^{(-2)}$ and so on. Multiply the terms connected by an arrow, keeping in mind the sign of the u terms. Add these terms up, and add the integral of the product of the last row (i.e. $\int u^{(n)} v^{(-n)} \, \mathrm{d}x$).

Essentially, the DI method allows us to easily compute the extended integration by parts formula, which states that

$$\int uv \, \mathrm{d}x = uv^{(-1)} - u'v^{(-2)} + u''v^{(-3)} - u^{(3)}v^{(-4)} + \cdots \pm \int u^{(n)} v^{(-n)} \, \mathrm{d}x,$$

where the sign of the integral depends on the parity of n.

Example 16.3.13 (DI Method). Consider the integral $\int x^3 \sin x \, \mathrm{d}x$. Taking $u = x^3$ and $v = \sin x$, we construct the DI table:

	D	I
$+$	x^3	$\sin x$
$-$	$3x^2$	$-\cos x$
$+$	$6x$	$-\sin x$
$-$	6	$\cos x$

Thus,

$$\begin{aligned}\int x^3 \sin x \, \mathrm{d}x &= x^3(-\cos x) - 3x^2(-\sin x) + 6x(\cos x) - 6 \int \cos x \, \mathrm{d}x \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.\end{aligned}$$

17 Applications of Integration

17.1 Area

17.1.1 The Riemann Sum and Integral

Suppose we wish to find exact area bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, where $a \leq b$ and $f(x) \geq 0$ for $a \leq x \leq b$.

We can approximate this area by drawing n rectangles of equal width, as shown in the diagram below:

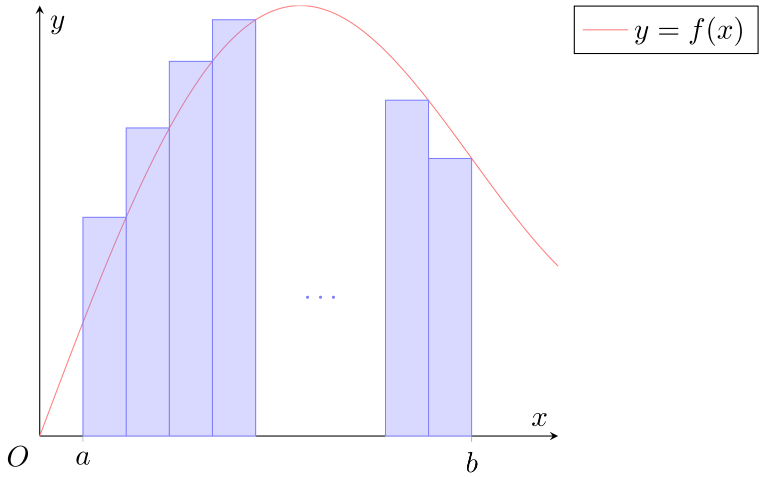


Figure 17.1

Observe that the kth rectangle has width $\Delta x = (b - a)/n$ and height $f(a + k\Delta x)$. The total area of the rectangles is hence

$$\sum_{k=1}^{n} f(a + k\Delta x)\Delta x.$$

This is known as the **Riemann sum** of f over $[a, b]$.

As the number of rectangles approaches ∞, the total area of rectangles approaches the actual area under the curve. In other words,

$$\text{Area} = \lim_{n\to 0} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x = \lim_{\Delta x\to 0} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x.$$

In the limit, the Riemann sum becomes the **Riemann integral**, which is conventionally written as the definite integral

$$\int_a^b f(x)\,\mathrm{d}x.$$

Note that this is where the integral and differential sign comes from: in the limit, $\sum \to \int$ and $\Delta x \to \mathrm{d}x$.

17.1.2 Definite Integral as the Area under a Curve

Proposition 17.1.1 (Area between a Curve and the x-axis)**.** Let A denote the area bounded by the curve of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. Then

$$\text{Area } A = \int_a^b |y| \, \mathrm{d}x = \int_a^b |f(x)| \, \mathrm{d}x.$$

Proposition 17.1.2 (Area between Two Curves)**.** The area A between two curves $y = f(x)$ and $y = g(x)$ is given by

$$\text{Area } A = \int_a^b |f(x) - g(x)| \, \mathrm{d}x.$$

Similar results hold when integrating with respect to the y-axis instead.

Proposition 17.1.3 (Area between a Parametric Curve and the x-axis)**.** Let C be the curve with parametric equations $x = f(t)$ and $y = g(t)$. Then the area A bounded between C and the x-axis is

$$\text{Area } A = \int_a^b |y| \, \mathrm{d}x = \int_{t_1}^{t_2} |g(t)| \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t,$$

where t_1 and t_2 are the values of t when $x = a$ and b respectively.

The formula can be applied similarly when we wish to find the area bounded between C and the y-axis.

Proposition 17.1.4 (Area Enclosed by Polar Curve)**.** Let $r = f(\theta)$ be a polar curve, and let A be the area of the region bounded by a segment of the curve and two half-lines $\theta = \alpha$ and $\theta = \beta$. Then

$$\text{Area } A = \frac{1}{2} \int_\alpha^\beta r^2 \, \mathrm{d}\theta.$$

Proof. Divide the enclosed region A into n sectors with the same interior angle $\Delta\theta$. Consider that a typical sector of A can be approximated by a sector of a circle. Thus, the area of that sector is approximately

$$\Delta A \approx \frac{1}{2} r^2 \Delta\theta.$$

Summing up these approximations, we see that

$$A \approx \sum_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r^2 \Delta\theta.$$

This approximation will improve as the number of sectors increases, i.e. $\Delta\theta \to 0$. Hence,

$$\text{Area } A = \lim_{\Delta\theta \to 0} \sum_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r^2 \Delta\theta = \frac{1}{2} \int_\alpha^\beta r^2 \, \mathrm{d}\theta.$$

□

17.2 Volume

Definition 17.2.1. If an enclosed region is rotated about a straight line, the three-dimensional object formed is called a **solid of revolution**, and its volume is a **volume of revolution**.

The line about which rotation takes place is always an axis of symmetry for the solid of revolution, and any cross-section of the solid which is perpendicular to the axis of rotation is circular.

17.2.1 Disc Method

Consider the solid of revolution formed when the region bounded between $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is rotated about the x-axis.

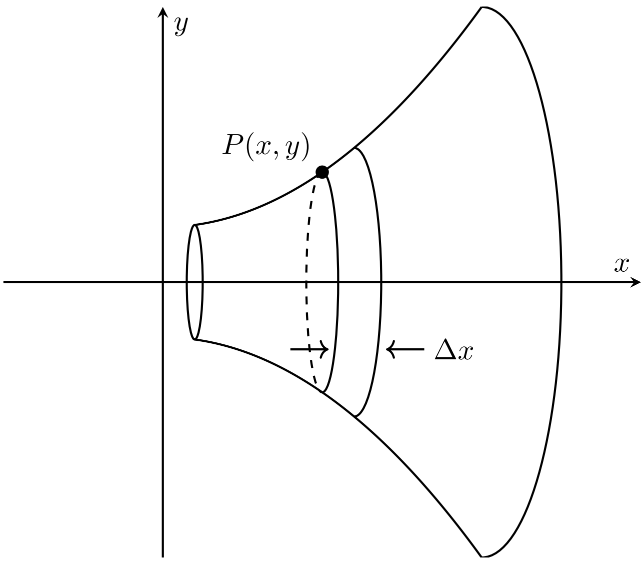


Figure 17.2

To calculate the volume of this solid, we can cut it into thin slices (or discs) of thickness Δx. Each disc is approximately a cylinder and the approximate volume of the solid can be found by summing the volumes of these cylinders. The smaller Δx is, the better the approximation.

Consider a typical disc formed by a one cut through the point $P(x, y)$ and the other cut distant Δx from the first. The volume of this disc is approximately

$$\Delta V \approx \pi y^2 \Delta x.$$

Summing over all discs,

$$V \approx \sum_{x=a}^{b} \pi y^2 \Delta x.$$

As more cuts are made, $\Delta x \to 0$, whence

$$V = \lim_{\Delta x \to 0} \sum_{x=a}^{b} \pi y^2 \Delta x = \pi \int_a^b y^2 \, \mathrm{d}x.$$

Proposition 17.2.2 (Disc Method). When the region bound by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is rotated 2π radians about the x-axis, the volume of the solid of revolution generated is given by

$$V = \pi \int_a^b y^2 \, \mathrm{d}x = \pi \int_a^b [f(x)]^2 \, \mathrm{d}x.$$

Proposition 17.2.3 (Disc Method: Volume Enclosed by Two Curves). When the region enclosed by two curves $y = f(x)$ and $y = g(x)$ is rotated 2π radians about the x-axis, the volume of the solid of revolution generated is given by

$$V = \pi \int_a^b [f(x)]^2 \, \mathrm{d}x - \pi \int_a^b [g(x)]^2 \, \mathrm{d}x = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) \mathrm{d}x.$$

Similar results hold when the axis of rotation is the y-axis.

17.2.2 Shell Method

Suppose a region R is rotated about the y-axis. Consider a typical vertical strip in the region R with height y and thickness Δx. It will form a cylindrical shell with inner radius x, outer radius $x + \Delta x$ and height y when rotated about the y-axis. Hence, it has volume

$$\Delta V = \pi(x + \Delta x)^2 y - \pi x^2 y = 2\pi xy\Delta x + \pi \Delta x^2 y \approx 2\pi xy\Delta x.$$

Hence, the volume of revolution is approximately

$$V \approx \sum_{x=a}^{b} 2\pi xy\Delta x.$$

As more strips are considered, $\Delta x \to 0$, whence

$$V = \lim_{\Delta x \to 0} = 2\pi \int_a^b xy \, \mathrm{d}x.$$

Proposition 17.2.4 (Shell Method). When the region bound by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is rotated 2π radians about the y-axis, the volume of the solid of revolution is given by

$$V = 2\pi \int_a^b xy \, \mathrm{d}x.$$

A similar result holds when the axis of rotation is the x-axis.

17.3 Arc Length

17.3.1 Parametric Form

Proposition 17.3.1 (Arc Length of Parametric Curve). Let $A(t_1)$ and $B(t_2)$ be points the parametric curve with equations $x = f(t)$, $y = g(t)$, $t \in [t_1, t_2]$. Then

$$\widehat{AB} = \int_{t_1}^{t_2} \sqrt{[f'(t)]^2 + [g'(t)]^2} \, \mathrm{d}t = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t.$$

Proof. Let $s = \wideparen{AB}$ be the arc length of AB. Let P and Q be points on AB with parameters t and $t + \Delta t$ respectively. By the Pythagorean theorem, the straight line PQ is given by

$$PQ^2 = [f(t+\Delta t) - f(t)]^2 + [g(t+\Delta t) - g(t)] \, .$$

Dividing both sides by $(\Delta t)^2$,

$$\left(\frac{PQ}{\Delta t}\right)^2 = \left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right]^2 + \left[\frac{g(t+\Delta t) - g(t)}{\Delta t}\right]^2 .$$

As $\Delta t \to 0$, we can write the RHS in terms of $f'(t)$ and $g'(t)$:

$$\lim_{\Delta t \to 0} \left(\frac{PQ}{\Delta t}\right)^2 = \left[f'(t)\right]^2 + \left[g'(t)\right]^2 .$$

Rearranging,

$$\lim_{\Delta t \to 0} PQ = \sqrt{[f'(t)]^2 + [g'(t)]^2} \Delta t.$$

However, observe that as $\Delta t \to 0$, the straight line PQ approximates the arc length PQ (i.e. Δs) better and better. Hence,

$$\Delta s = \wideparen{PQ} = \sqrt{[f'(t)]^2 + [g'(t)]^2} \Delta t.$$

Integrating from A to B, we thus obtain

$$s = \wideparen{AB} = \int_{t_1}^{t_2} \sqrt{[f'(t)]^2 + [g'(t)]^2} \, \mathrm{d}t = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t.$$

□

17.3.2 Cartesian Form

Taking $t = x$ or $t = y$, we get the following formulas involving $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}x/\mathrm{d}y$, which is suitable for Cartesian curves.

Proposition 17.3.2 (Arc Length of Cartesian Curve)**.** Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points on the curve $y = f(x)$. The arc length AB is given by

$$\wideparen{AB} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x = \int_{y_1}^{y_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1} \, \mathrm{d}y.$$

17.3.3 Polar Form

Proposition 17.3.3. Let $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ be points on the polar curve $r = f(\theta)$. Then the arc length AB is given by

$$\wideparen{AB} = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} \, \mathrm{d}\theta.$$

Proof. Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta \frac{\mathrm{d}r}{\mathrm{d}\theta} - r\sin\theta, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \sin\theta \frac{\mathrm{d}r}{\mathrm{d}t} + r\cos\theta.$$

It follows that

$$\left(\frac{\mathrm{d}(r\cos\theta)}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}(r\sin\theta)}{\mathrm{d}\theta}\right)^2=(\cos^2\theta+\sin^2\theta)\left[r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right]=r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2.$$

Taking $t=\theta$,

$$\widehat{AB}=\int_{\theta_1}^{\theta_2}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$

□

17.4 Surface Area of Revolution

Definition 17.4.1. The surface area of a solid of revolution is called the **surface area of revolution**.

Proposition 17.4.2 (Surface Area of Revolution of Parametric Curve)**.** Let $A(t_1)$ and $B(t_2)$ be points the parametric curve with equations $x=f(t)$, $y=g(t)$, $t\in[t_1,t_2]$. Then the surface area of revolution about the x-axis of arc AB is given by

$$A=2\pi\int_{t_1}^{t_2}y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Similarly, the surface area of revolution about the y-axis is given by

$$A=2\pi\int_{t_1}^{t_2}x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Proof. Let $s=\widehat{AB}$ be the arc length of AB. Let P and Q be points on AB with parameters t and $t+\Delta t$ respectively. Recall that

$$\Delta s=\widehat{PQ}=\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\Delta t.$$

Now consider the surface area of revolution about the x-axis of arc PQ. For small Δs, the solid of revolution is approximately a disc wish radius y and width Δs. The surface area of this disc can be calculated as

$$\Delta A=2\pi y\Delta s=2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\Delta t.$$

Integrating from A to B, we see that

$$A=2\pi\int_{t_1}^{t_2}y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

A similar argument is used when the axis of rotation is the y-axis. □

17.5 Approximating Definite Integrals

In §17.1, we saw how Riemann sums could approximate definite integrals using rectangles. This is a blunt tool which utilizes very little information from the curve and thus will often not give a good estimate of the definite integral for a fixed number of rectangles.

In this chapter, we will be exploring two other methods: the trapezium rule and Simpson's rule, for finding the approximate value of an area under a curve. These methods often give better approximations to the actual area as compared to using Riemann sums. Similar to Riemann sums, these methods can be extended to estimate the value of a definite integral.

17.5.1 Trapezium Rule

Consider the curve $y = f(x)$ which is non-negative over the interval $[a, b]$.

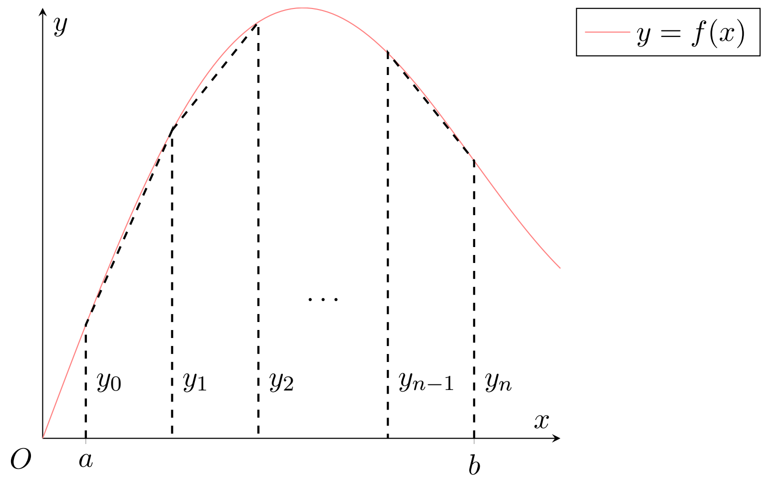


Figure 17.3

Divide the interval $[a, b]$ into n equal intervals (strips) with each having width $h = (b - a)/n$. Then the area of the n trapeziums is given by

$$\text{Area} = \sum_{k=0}^{n} \frac{h}{2} (y_k + y_{k+1}) = \frac{h}{2} [y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n] .$$

Recipe 17.5.1 (Trapezium Rule). The trapezium rule with $(n + 1)$ ordinates (or n intervals) gives the approximation

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{k=0}^{n} \frac{h}{2} (y_k + y_{k+1}) = \frac{h}{2} [y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n] ,$$

where $h = (b - a)/n$.

Sample Problem 17.5.2. Use the trapezium rule with 4 strips to find an approximation for

$$\int_0^2 \ln(x + 2)\,\mathrm{d}x.$$

Find the percentage error of the approximation.

Solution. Let $f(x) = \ln(x+2)$. By the trapezium rule,

$$\begin{aligned}\int_0^2 \ln(x+2)\,\mathrm{d}x &\approx \frac{1}{2}\cdot\frac{2-0}{4}\Big(f(0)+2\left[f(0.5)+f(1)+f(1.5)\right]+f(2)\Big)\\ &= 2.15369 \text{ (5 d.p.)}.\end{aligned}$$

One can easily verify that the integral evaluates to 2.15888 (5 d.p.). Hence, the percentage error is

$$\left|\frac{2.15888-2.15369}{2.15888}\right| = 0.240\%.$$

□

Error in Trapezium Rule Approximation

If the curve is concave upward, the secant lines lie above the curve. Hence, the trapezium rule will give an overestimate.

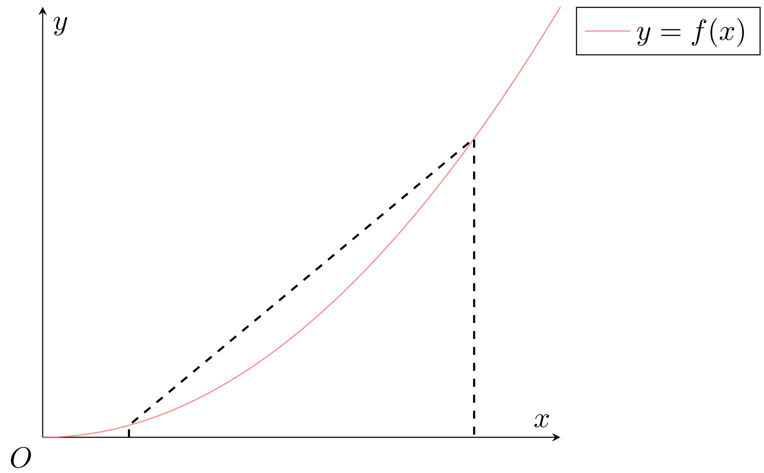


Figure 17.4

If the curve is concave downward, the secant lines lie below the curve. Hence, the trapezium rule will give an underestimate.

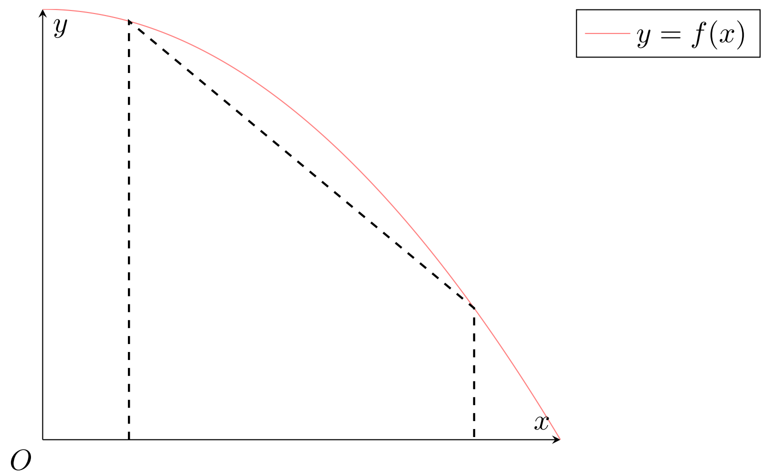


Figure 17.5

17.5.2 Simpson's Rule

Previously, we explored how Riemann sums approximate definite integrals using horizontal lines (i.e. degree 0 polynomials). We also saw how the trapezium rule improves this approximation by using sloped lines (i.e. degree 1 polynomials). Now, we introduce Simpson's rule, which takes this a step further by using quadratics (i.e. degree 2 polynomials) to achieve even greater accuracy in approximating definite integrals.

Consider the curve $y = f(x)$, which is non-negative over the interval $[a, b]$. Suppose the area represented by $\int_a^b f(x)\,\mathrm{d}x$ is divided by the ordinates y_0, y_1, y_2 into two strips each of width h as shown below. A particular parabola can be found passing through the three points on the curve with ordinates y_0, y_1, y_2. Simpson's rule uses the area under the parabola to approximate the area represented by $\int_a^b f(x)\,\mathrm{d}x$.

To deduce the area under the parabola, we consider the case where $y = f(x)$ is translated x_1 units to the left, i.e. the line $x = x_1$ is now the y-axis.

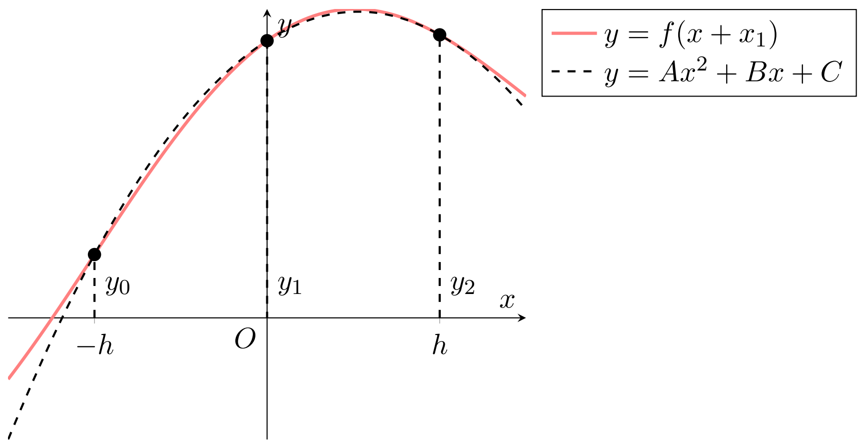


Figure 17.6

Under this translation,

$$\int_a^b f(x)\,\mathrm{d}x = \int_{-h}^{h} f(x + x_1)\,\mathrm{d}x.$$

This area will now be approximated by a parabola $y = g(x) = Ax^2 + Bx + C$, where A, B and C are constants. The area under the parabola is given by

$$\int_{-h}^{h} \left(Ax^2 + Bx + C\right)\,\mathrm{d}x = \left[\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right]_{-h}^{h} = \frac{h}{3}\left(2Ah^2 + 6C\right).$$

Now, observe that the parabola $y = g(x)$ intersects the curve at $(-h, y_1)$, $(0, y_2)$ and (h, y_3). Hence,

$$g(-h) = Ah^2 - Bh + C = y_0, \quad g(0) = C = y_1, \quad g(h) = Ah^2 + Bh + C = y_2.$$

Thus,

$$\frac{h}{3}\left(2Ah^2 + 6C\right) = \frac{h}{3}\left[\left(Ah^2 - Bh + C\right) + 4C + \left(Ah^2 + Bh + C\right)\right] = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

We hence arrive at Simpson's rule with 2 strips:

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}(y_0 + 4y_1 + y_2).$$

We can extend Simpson's rule to cover any even number of strips. In general,

Recipe 17.5.3 (Simpson's Rule). Simpson's rule with $2n$ strips (or $2n+1$ ordinates) gives the approximation

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &\approx \sum_{k=0}^{n} \frac{h}{3}(y_{2k} + 4y_{2k+1} + y_{2k+2}) \\ &= \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}\right].\end{aligned}$$

Sample Problem 17.5.4. Use Simpson's rule with 4 strips to find an approximation for

$$\int_0^2 \ln(x+2)\,\mathrm{d}x.$$

Find the percentage error of the approximation.

Solution. Let $f(x) = \ln(x+2)$. By the trapezium rule,

$$\begin{aligned}\int_0^2 \ln(x+2)\,\mathrm{d}x &\approx \frac{1}{3}\cdot\frac{2-0}{4}\Big[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)\Big] \\ &= 2.15881 \text{ (5 d.p.)}.\end{aligned}$$

As previously mentioned in Sample Problem 17.5.2 the actual value of the integral is 2.15888 (5 d.p.). Hence, the percentage error is

$$\left|\frac{2.15888 - 2.15881}{2.15888}\right| = 0.00324\%.$$

□

In the previous example, the trapezium rule gave an estimate of 2.15369 (5 d.p.), which has an error of 0.240%. In the case of Simpson's rule, the error is 0.00324%, vastly better than that of the trapezium rule's.

In general, Simpson's rule gives a better approximation than the trapezium rule as the quadratics used account for the concavity of the curve.

18 Functions of Two Variables

In Chapter §3, we learnt that functions can be described as a machine that takes in an input and produces an output according to a rule. Some examples of functions that we have encountered thus fare are $f(x) = x^2$, $g(x) = \cos x$, etc. These are functions of one variable, also called **univariate functions**.

However, in real life, there are functions that depend on more than one variable (i.e. the domain is not a subset of the real numbers). For instance, the cost (output) of a taxi ride may depend on variables (input) like time, distance travelled, traffic conditions, demand, etc. In this case, the function is called a **multivariate function**. The input with many variables can be expressed as a vector.

Similarly, the codomain of a function does not necessarily need to be a subset of the real numbers. Consider the following function $f(s,t)$:

$$f(s,t) = \begin{pmatrix} s+t \\ t \\ 2s-1 \end{pmatrix}.$$

Here, $f(s,t)$ takes in two inputs (s and t), and spits out three outputs ($s+t$, t and $2s-1$).

For the rest of this chapter, we will only study scalar-valued functions of two variables, of the form

$$z = f(x,y),$$

which we can visualize in 3D space. We will see how the ideas from univariate functions can be extended to two variable functions and how concepts of vectors can be useful in studying these functions.

18.1 Functions of Two Variables and Surfaces

18.1.1 Functions of Two Variables

Definition 18.1.1. A (scalar) **function of two variables**, f, is a rule that assigns each ordered pair of real numbers (x,y) in its domain to a unique real number.

Recall that the domain of a function $g(x)$ is a subset of the real number line, i.e. $D_g \subseteq \mathbb{R}$. Generalizing this to scalar functions of two variables, the domain of f is a subset of the xy-plane, denoted $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$. Mathematically,

$$D_f \subseteq \mathbb{R}^2.$$

If the domain of $f(x,y)$ is not well specified, then we will take its domain to be the set of all pairs $(x,y) \in \mathbb{R}^2$ for which the given expression is a well-defined real number.

Example 18.1.2 (Domain of $f(x,y)$)**.** Let $f(x,y) = \ln\left(y^2 - x\right)$. For $f(x,y)$ to be well-defined, the argument of the natural logarithm must be positive. That is, we require $y^2 - x > 0$. The domain of f is hence

$$D_f = \left\{(x,y) \in \mathbb{R}^2 \mid y^2 - x > 0\right\}.$$

18.1.2 Surfaces

Recall that we defined the graph of a function $g(x)$ to be the collection of all points (x, y) in the xy-plane such that the values x and y satisfy $y = g(x)$. We can extend this notion to functions of two variables:

Definition 18.1.3. The **graph** of $z = f(x, y)$, or **surface** with equation $z = f(x, y)$, is the collection of all points (x, y, z) in 3D Cartesian space such that the values x, y and z satisfy $z = f(x, y)$.

Visualizing and illustrating a 3D surface can be challenging, especially as surfaces become complicated. We can study the surface by fixing or changing the variables one at a time. This is the idea behind traces, or level curves.

Definition 18.1.4. **Horizontal traces** (or **level curves**) are the resulting curves when we intersect the surface $z = f(x, y)$ with horizontal planes.

This is like fixing the value of z, giving the 2D graph of the equation $f(x, y) = c$ for some constant c.

Definition 18.1.5. **Vertical traces** are the resulting curves when we intersect the surface $z = f(x, y)$ with vertical planes.

This is like fixing the value of x or y (or a combination of both, e.g. $y = x$).

Definition 18.1.6. A **contour plot** of $z = f(x, y)$ is a graph of numerous horizontal traces $f(x, y) = c$ for representative values of c (usually spaced-out values).

We may identify a surface by examining these traces to visualize graphs of two variables.

Example 18.1.7 (Graph of $z = f(x, y)$)**.** Let $f(x, y) = \ln(x^2 + y^2)$. Consider the horizontal traces of $z = f(x, y)$. Setting $z = c$, we get

$$\ln(x^2 + y^2) = c \implies x^2 + y^2 = e^c.$$

Hence, the horizontal trace of $z = f(x, y)$ at $z = c$ corresponds to a circle centred at the origin with radius e^c. Thus, the graph of $z = \ln(x^2 + y^2)$ looks like

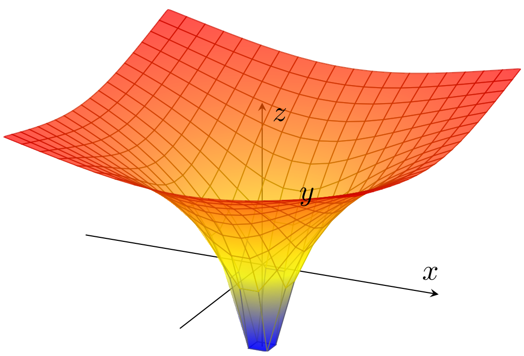


Figure 18.1

18.1.3 Cylinders and Quadric Surfaces

Exploring the traces of a surface allows us to visualize the shape of the surface. We can now look at some of the common surfaces, such as cylinders and quadric surfaces.

Definition 18.1.8. A surface is a **cylinder** if there is a plane P such that all planes parallel to P intersect the surface in the same curve (when viewed in 2D).

Examples of cylinders include the graphs of $x^2 + z^2 = 1$ and $z = y^2$, as shown below:

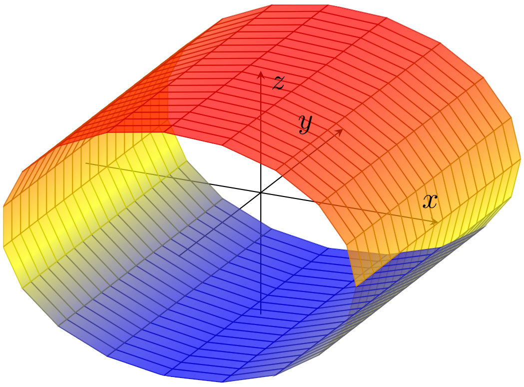


Figure 18.2: Graph of $x^2 + z^2 = 1$.

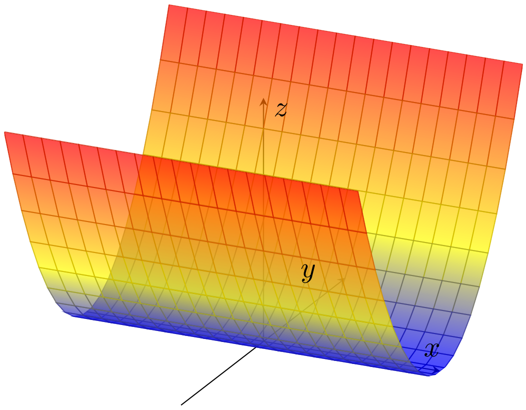


Figure 18.3: Graph of $z = y^2$.

Observe that $x^2 + z^2 = 1$ is a special case of a function of two variables $z = f(x, y)$ that can be reduced to $z = f(x)$ since z is independent of y. Similarly, $z = y^2$ can be reduced to $z = f(y)$ since z is independent of x. Indeed, if a function $z = f(x, y)$ can be reduced to a univariate function, then its surface must be cylindrical.

Another common surface is a quadric surface, which is a 3D generalization of 2D conic sections. Recall that a conic section in 2D has the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

We can generalize this into 3D to get a quadric surface.

Definition 18.1.9. A **quadric surface** has the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J,$$

where $A, B, \ldots, J \in \mathbb{R}$ and at least one of A, B and C is non-zero.

An example of a quadric surface is the ellipsoid, which is a generalization of an ellipse and has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

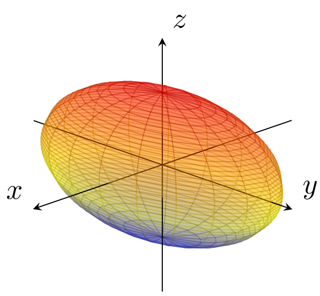


Figure 18.4: An ellipsoid.

When $a = b = c = r$, we get the equation

$$x^2 + y^2 + z^2 = r^2.$$

This represents a sphere centred at the origin with radius r. Observe the similarity between the equation of a circle ($x^2 + y^2 = r^2$) and the equation of a sphere.

18.2 Partial Derivatives

Recall that for a function f of one variable x, we defined the derivative function as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The usual notations are $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}f}{\mathrm{d}x}$ if $y = f(x)$.

The notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives some insight into how derivatives are derived. We can view

- "$\mathrm{d}x$" as a small change in x, and
- "$\mathrm{d}y$" as the change in y as a result of the small change in x.

Hence, the notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ actually represents the "rise over run", which is a measure of gradient at the point (x, y) on the graph.

We can extend this notion to functions of two variables $z = f(x, y)$. There are now two variables that will affect the change in the value of f. We can choose to vary x slightly (Δx) or vary y slightly Δy and see how f changes (Δf). This gives us some notion of a derivative. However, because we are only varying one independent variable at a time, we are only differentiating the function $f(x, y)$ "partially". We hence call these derivatives the partial derivatives of f.

Definition 18.2.1. The (first-order) **partial derivatives** of $f(x, y)$ are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$
$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

In Liebniz notation,

$$f_x(x, y) = \frac{\partial f}{\partial x}, \quad f_y(x, y) = \frac{\partial f}{\partial y}.$$

Recipe 18.2.2 (Partial Differentiation)**.** To partially differentiate a function $f(x, y)$ with respect to x, we differentiate $f(x, y)$ as we normally would, treating y as a constant. Similarly, if we are partially differentiating with respect to y, we treat x as a constant.

Sample Problem 18.2.3. Given $f(x, y) = \cos(xy + y^2)$, find $f_x(x, y)$.

Solution. To partially differentiate it with respect to x, we treat y as a constant. Using the chain rule,

$$f_x(x, y) = -\sin(xy + y^2) \frac{\partial}{\partial x} \left[xy + y^2\right].$$

Since y is a constant,

$$\frac{\partial}{\partial x}(xy) = y, \quad \frac{\partial}{\partial x} y^2 = 0.$$

Hence,

$$f_x(x, y) = -y \sin(xy + y^2).$$

□

18.2.1 Geometric Interpretation

Consider a surface S given by the equation $z = f(x, y)$. Let $P(a, b, c)$ be a point on S.

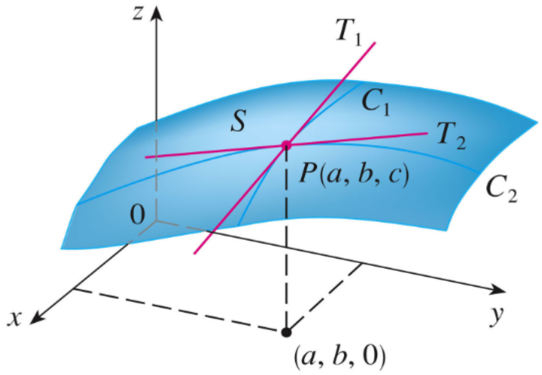


Figure 18.5: Partial derivatives as slopes of tangent lines.[1]

The curve C_1 is the graph of the function $g(x) = f(x, b)$, which is the intersection curve of the surface and the vertical plane $y = b$. The slope of its tangent T_1 at P is $g'(x) = f_x(a, b)$.

Similarly, the curve C_2 is the graph of the function $h(y) = f(a, y)$, which is the intersection curve of the surface and the vertical plane $x = a$. The slope of its tangent T_2 at P is $h'(y) = f_y(a, b)$.

We can hence visualize partial derivatives at the point P on S as slopes to the tangent lines T_1 and T_2 at that point.

18.2.2 Gradient

To represent the "full" derivative of a function, we simply collect its partial derivatives.

Definition 18.2.4. The **gradient** of a function $f(x, y)$, denoted as ∇f, is the collection of all its partial derivatives into a vector.

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}.$$

Example 18.2.5 (Gradient)**.** Let $f(x, y) = xy^2 + x^3$. Then its gradient is

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} y^2 + 3x^2 \\ 2xy \end{pmatrix}.$$

18.2.3 Second Partial Derivatives

Similar to second-order derivatives for univariate functions, we can also consider the partial derivatives of partial derivatives:

$$(f_x)_x, \quad (f_x)_y, \quad (f_y)_x, \quad (f_y)_y.$$

[1]Source: https://www2.victoriacollege.edu/~myosko/m2415sec143notes(7).pdf

If $z = f(x, y)$, we use the following notation for the second partial derivatives:

$$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2},$$
$$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x},$$
$$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y},$$
$$(f_y)_y = f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.$$

Thus, the notation f_{xy} means that we first partially differentiate with respect to x and then with respect to y. Notice that the order the variables appear in the denominator is reversed when using Liebniz notation, similar to the idea of composite functions:

$$(f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}.$$

Example 18.2.6 (Second Partial Derivatives)**.** Consider the function $f(x, y) = xy^2 + x^3 + \ln y$. Its partial derivatives are

$$f_x = y^2 + 3x^2, \quad f_y = 2xy + \frac{1}{y},$$

and its second partial derivatives are

$$f_{xx} = 6x, \quad f_{xy} = 2y, \quad f_{yx} = 2y, \quad f_{yy} = 2x - \frac{1}{y^2}.$$

Notice in the above example that $f_{xy} = f_{yx}$. This symmetry of second partial derivatives is known as Clairaut's theorem.

Theorem 18.2.7 (Clairaut's Theorem)**.** If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$

18.2.4 Multivariate Chain Rule

Recall that for a univariate function $y = f(x)$, where the variable x is a function of t, i.e. $x = g(t)$, the chain rule states

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

We can generalize this result to multivariate functions using partial derivatives:

Proposition 18.2.8 (Multivariate Chain Rule)**.** Consider the function $f(x, y)$, where x and y are functions of t. Then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

To see why this is morally true, we return to the definition of a partial derivative:

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$
$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

Rewriting these equations, we get

$$f(x+\Delta x, y) = f(x,y) + \Delta x f_x(x,y), \tag{1}$$
$$f(x, y+\Delta y) = f(x,y) + \Delta y f_y(x,y), \tag{2}$$

where Δx and Δy should be thought of as infinitesimally small changes in x and y.

We now consider the quantity $f(x+\Delta x, y+\Delta y)$. Applying (1) and (2) sequentially, we get

$$\begin{aligned} f(x+\Delta x, y+\Delta y) &= f(x, y+\Delta y) + \Delta x f_x(x, y+\Delta y) \\ &= f(x,y) + \Delta y f_y(x,y) + \Delta x f_x(x, y+\Delta y). \end{aligned} \tag{3}$$

Observe that if we partially differentiate (2) with respect to x, we get

$$f_x(x, y+\Delta y) = f_x(x,y) + \Delta y f_{yx}(x,y).$$

Substituting this into (3) yields

$$\begin{aligned} f(x+\Delta x, y+\Delta y) &= f(x,y) + \Delta y f_y(x,y) + \Delta x \left[f_x(x,y) + \Delta y f_{yx}(x,y)\right] \\ &= f(x,y) + \Delta y f_y(x,y) + \Delta x f_x(x,y) + \Delta x \Delta y f_{yx}(x,y). \end{aligned} \tag{4}$$

Since Δx and Δy are both infinitesimally small, the quantity $\Delta x \Delta y$ is negligible and can be disregarded. We thus have

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y) = \Delta x f_x(x,y) + \Delta y f_y(x,y).$$

Dividing throughout by Δt and writing f_x, f_y in Liebniz notation, we have

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t}.$$

In the limit as $\Delta t \to 0$, we have

$$\frac{\Delta f}{\Delta t} \to \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \frac{\Delta x}{\Delta t} \to \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \frac{\Delta y}{\Delta t} \to \frac{\mathrm{d}y}{\mathrm{d}t}.$$

Thus,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

□

Observe that if we had applied (2) before (1) on $f(x+\Delta x, y+\Delta y)$, we would have got

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta y f_y(x,y) + \Delta x f_x(x,y) + \Delta x \Delta y f_{xy}(x,y).$$

However, by Clairaut's theorem, we know $f_{xy} = f_{yx}$, so we would still have ended up with (4).

18.2.5 Directional Derivative

So far, we only know how to find the instantaneous rate of change of $f(x,y)$ in two special cases:

- The first case is when we vary x and hold y constant, in which the partial derivative $f_x(x,y)$ represents the instantaneous rate of change of $f(x,y)$.
- The second case is when we vary y and hold x constant, in which the partial derivative $f_y(x,y)$ represents the instantaneous rate of change of $f(x,y)$.

We wish to construct a more general "derivative" which represents the instantaneous rate of change of $f(x, y)$ where x and y are both allowed to vary.

To simplify matters, we assume that x and y are changing at a constant rate. That is, every time x increases by u_x, y will increase by u_y. We can represent this change with a unit vector $\mathbf{u}$ along the xy-plane:

$$\mathbf{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

Because we are measuring the instantaneous rate of change of $f(x, y)$ along a direction, we call this quantity the "directional derivative".

Definition 18.2.9. The **directional derivative** of $f(x, y)$ in the direction of the unit vector $\mathbf{u} = \langle u_x,\, u_y\rangle$ is denoted $D_\mathbf{u} f(x, y)$ and is defined as

$$D_\mathbf{u} f(x, y) = \lim_{h\to 0} \frac{f(x + hu_x, y + hu_y) - f(x, y)}{h}.$$

We now relate the directional derivative with the gradient of f.

Proposition 18.2.10.

$$D_\mathbf{u} f(x, y) = \nabla f \cdot \mathbf{u} = u_x f_x(x, y) + u_y f_y(x, y).$$

Proof. In §18.2.4, we derived the equation

$$f(x + \Delta x, y + \Delta y) - f(x, y) = \Delta x f_x(x, y) + \Delta y f_y(x, y),$$

where Δx and Δy are infinitesimally small. If we take $\langle \Delta x,\, \Delta y\rangle$ to be in the same direction as $\langle u_x,\, u_y\rangle$, i.e.

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \lim_{h\to 0} h \begin{pmatrix} u_x \\ u_y \end{pmatrix},$$

then we have

$$f(x + hu_x, y + hu_y) - f(x, y) = hu_x f_x(x, y) + hu_y f_y(x, y),$$

keeping in mind that we are taking the limit $h \to 0$ on both sides. Dividing both sides throughout by h,

$$\lim_{h\to 0} \frac{f(x + hu_x, y + hu_y) - f(x, y)}{h} = u_x f_x(x, y) + u_y f_y(x, y),$$

which was what we wanted. □

With this relation, we can prove several neat results.

Proposition 18.2.11. Suppose f is differentiable at (x_0, y_0), and $\nabla f(x_0, y_0) \neq \mathbf{0}$. Then $\nabla f(x_0, y_0)$ is perpendicular to the level curve of f through (x_0, y_0).

Proof. Let $f(x, y) = (x(t), y(t))$. Note that the tangent to the level curve at (x_0, y_0) has direction vector $\mathbf{u} = \langle \mathrm{d}x/\mathrm{d}t,\, \mathrm{d}y/\mathrm{d}t\rangle$.

Let the level curve at (x_0, y_0) have equation $f(x, y) = c$. Implicitly differentiating this with respect to t, we get

$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \end{pmatrix} = \nabla f \cdot \mathbf{u} = 0.$$

Since both ∇f and $\mathbf{u}$ are non-zero vectors, they must be perpendicular to each other. □

Proposition 18.2.12. The greatest rate of change of f occurs in the direction of ∇f, while the smallest rate of change occurs in the direction of $-\nabla f$

Proof. Since $\mathbf{u}$ is a unit vector,

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}|\cos\theta = |\nabla f|\cos\theta,$$

where θ is the angle between ∇f and $\mathbf{u}$. Clearly, $D_{\mathbf{u}}f$ is maximal when $\theta = 0$, in which case $\mathbf{u}$ is in the same direction as ∇f. Similarly, $D_{\mathbf{u}}f$ is minimal when $\theta = \pi$, in which case $\mathbf{u}$ is in the opposite direction as ∇f. □

We say that $\nabla f(a,b)$ is the **direction of steepest ascent** at (a,b), while $-\nabla f(a,b)$ is the **direction of steepest descent**.

18.2.6 Implicit Differentiation

Consider the unit circle, which has equation

$$x^2 + y^2 = r^2.$$

Previously, we learnt that to find $\mathrm{d}y/\mathrm{d}x$, we can simply differentiate term by term, treating y as a function of x and using the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}g(y) = \frac{\mathrm{d}}{\mathrm{d}y}g(y)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Using our example of the unit circle, we get

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}.$$

While morally true, this approach to implicit differentiate is not entirely rigorous. For a more formal justification, we turn to partial derivatives.

Going back to our example of the unit circle, if we move all terms to one side of the equation, we get

$$x^2 + y^2 - r^2 = 0.$$

Now, observe that the LHS is simply a function of x and y, i.e.

$$f(x,y) = x^2 + y^2 - r^2.$$

Hence, we can define y implicitly as a function of x that satisfies

$$f(x,y) = 0.$$

If we differentiate the above equation with respect to x, by the multivariate chain rule, we get

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Clearly, $\mathrm{d}x/\mathrm{d}x = 1$. Rearranging, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{f_x(x,y)}{f_y(x,y)}.$$

Since

$$f_x(x,y) = 2x, \quad \text{and} \quad f_y(x,y) = 2y,$$

we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{2y} = -\frac{x}{y}$$

as expected.

More generally,

Proposition 18.2.13 (Implicit Differentiation for Univariate Functions)**.** If the equation

$$f(x, y) = 0$$

implicitly defines y as a function of x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{f_x(x, y)}{f_y(x, y)},$$

given that $f_y(x, y) \neq 0$.

We can extend this result to functions of two variables.

Proposition 18.2.14 (Implicit Differentiation for Functions of Two Variables)**.** If the equation

$$f(x, y, z) = 0$$

implicitly defines z as a function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{f_x(x, y, z)}{f_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{f_y(x, y, z)}{f_z(x, y, z)},$$

given that $f_z(x, y, z) \neq 0$.

To see this in action, consider the following sample problem:

Sample Problem 18.2.15. Find the value of $\partial^2 z / \partial x^2$ at $(0, 0, c)$ of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution. Let

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

Applying the above result, we have

$$\frac{\partial z}{\partial x} = -\frac{f_x(x, y, z)}{f_z(x, y, z)} = -\frac{2x/a^2}{2z/c^2} = -\frac{c^2}{a^2}\frac{x}{z}.$$

Partially differentiating with respect to x once more,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x}\left(-\frac{c^2}{a^2}\frac{x}{z}\right) = -\frac{c^2}{a^2 z}.$$

Hence,

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(0,0,c)} = -\frac{c}{a^2}.$$

□

18.3 Approximations

In §15, we learnt that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots.$$

If we want to approximate $f(x)$ for x near 0, we can truncate the Maclaurin series of $f(x)$. For instance, the linear approximation to x is

$$f(x) \approx f(0) + f'(0),$$

which is the tangent line at $x = 0$. If we want better approximations, we can simply take more terms. For instance, if we take one more term, then we get the quadratic approximation

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2.$$

In some sense, we can get a good approximation to $f(x)$ around $x = 0$ if we can find a simpler function which

- has the same value as f at $x = 0$, and
- has the same derivatives as f at $x = 0$ (up to the order of derivatives we prefer).

The same idea is extended to functions of two variables (or any multivariate functions) at a general point. The idea of approximation $f(x, y)$ at a point (a, b) is to find a simpler function which

- has the same value as f at (a, b), and
- has the same nth-order partial derivatives as f at (a, b) (where n is the highest order we prefer).

In this subsection, we look at the case where $n = 1$ (linear approximation) and $n = 2$ (quadratic approximation).

18.3.1 Tangent Plane

To find a linear approximation of $f(x, y)$ at (a, b) is to find a simpler function which

- has the same value as f at (a, b), and
- has the same partial derivatives as f at (a, b).

Let this approximation be $T(x, y)$. As the name suggests, $T(x, y)$ is linear and is hence of the form

$$T(x, y) = C_1 + C_2(x - a) + C_3(y - b),$$

where C_1, C_2 and C_3 are constants to be determined.

From the first condition, we require $f(a, b) = T(a, b)$. Hence,

$$f(a, b) = T(a, b) = C_1.$$

From the second condition, we require $f_x(a, b) = T_x(a, b)$ and $f_y(a, b) = T_y(a, b)$. This gives

$$f_x(a, b) = T_x(a, b) = C_2$$

and

$$f_y(a, b) = T_y(a, b) = C_3.$$

We hence have:

Proposition 18.3.1 (Linear Approximation)**.** The linear approximation at (a, b) is given by

$$T(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Recall that the linear approximation to a univariate function at $x = a$ is the tangent line at that point. Generalizing this up a dimension, the linear approximation $T(x, y)$ is the **tangent plane** to $f(x, y)$ at (a, b).

Using 3D vector geometry, we can find the normal vector to $z = f(x, y)$ at (a, b):

$$\mathbf{n} = \begin{pmatrix} f_x(a, b) \\ f_y(a, b) \\ -1 \end{pmatrix}.$$

18.3.2 Quadratic Approximation

To find a quadratic approximation of $f(x, y)$ at (a, b) is to find a simpler function which

- has the same value as f at (a, b), and
- has the same first and second partial derivatives as f at (a, b).

Remark. In univariate functions, the word "quadratic" refers to functions with terms of order 2, such as x^2. Similarly with multivariables, "quadratic" refers to terms with order 2, but it could be x^2, y^2 or xy; all variables contribute to the total order of the term. For instance, x^2y^3 is a term of order $2 + 3 = 5$.

To get the quadratic approximation $Q(x, y)$, we simply add terms of order 2 to the linear approximation $T(x, y)$:

$$Q(x, y) = T(x, y) + C_1(x - a)^2 + C_2(x - a)(y - b) + C_3(y - b)^2,$$

where C_1, C_2 and C_3 are constants. We can determine them by equating the second partial derivatives of $Q(x, y)$ with that of $f(x, y)$'s:

$$\begin{aligned} f_{xx}(a, b) &= Q_{xx}(a, b) = 2C_1, \\ f_{xy}(a, b) &= Q_{xx}(a, b) = \ \ C_2, \\ f_{yy}(a, b) &= Q_{xx}(a, b) = 2C_3. \end{aligned}$$

We hence have:

Proposition 18.3.2 (Quadratic Approximation)**.** The quadratic approximation at (a, b) is given by

$$\begin{aligned} Q(x, y) = f(a, b) &+ f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &+ \frac{1}{2} f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2} f_{yy}(a, b)(y - b)^2. \end{aligned}$$

Note that by Clairaut's theorem, we can interchange f_{xy} and f_{yx} in the formula above, so long as they are continuous.

18.4 Maxima, Minima and Saddle Points

One important application of calculus is the optimization of functions which have many dependent variables. For example, one may maximize the amount of profit based on parameters such as the cost of raw materials, workers' salaries, time needed for production, etc.

To find stationary points of a univariate function, we equate its gradient to 0. Similarly, for functions of two variables $f(x, y)$, if we want to find stationary points, we look for points where its gradient, ∇f, is the zero vector, i.e.

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In functions of two variables, the stationary points we often come across are maxima, minima and saddle points (so named because it looks like a horse saddle).

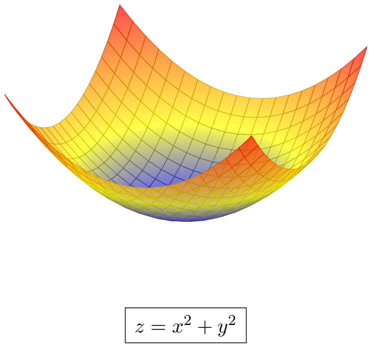


Figure 18.6: Minimum point at $(0, 0)$.

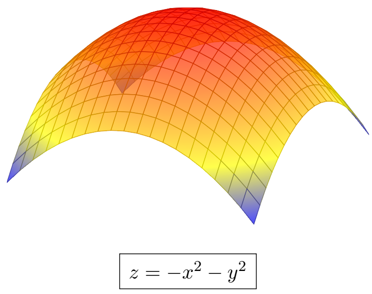


Figure 18.7: Minimum point at $(0, 0)$.

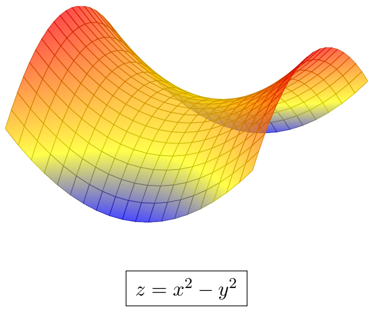


Figure 18.8: Saddle point at $(0, 0)$.

18.4.1 Global and Local Extrema

In optimization, we may distinguish between a **local extremum** (a collective term used to refer to the maximum and minimum) from a **global extremum**. Basically, a global maximum/minimum is the highest/lowest value which the function can achieve.

Local extrema are like the stationary points which we just discussed. For example, consider the following graph of $f(x, y) = x\mathrm{e}^{-x^2-y^2}$:

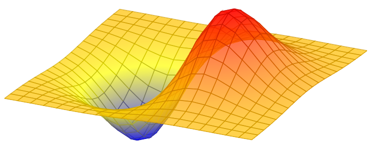


Figure 18.9

The intuitive idea behind local extrema is that when we move away from the maxima/minima in any direction, the value of the function will decrease/increase. However, this may not apply to global extrema. Consider the function $f(x, y) = x^2 + y^2$ with domain $-2 \leq x \leq 2$, $-2 \leq y \leq 2$.

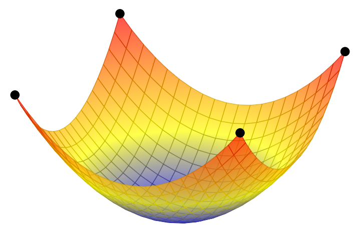


Figure 18.10

The global maxima occur at the corners of the domain. Note that these global maxima are also not stationary points.

Recipe 18.4.1 (Finding Global Extrema)**.** To find the global extrema of a function, we must

- check all local extrema (set $\nabla f = \mathbf{0}$), and
- check for extrema along the boundary of the function's domain.

18.4.2 Second Partial Derivative Test

We can determine the nature of the stationary points by the second partial derivative test:

Proposition 18.4.2 (Second Partial Derivative Test)**.** Let (a, b) be a stationary point of $f(x, y)$. Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 .$$

- If $D > 0$, and
  - $f_{xx}(a, b) > 0$ (or $f_{yy}(a, b) > 0$), then (a, b) is a minimum point.
  - $f_{xx}(a, b) < 0$ (or $f_{yy}(a, b) < 0$), then (a, b) is a maximum point.
- If $D < 0$, then (a, b) is a saddle point.
- If $D = 0$, the test is inconclusive.

The proof is similar to the proof of the second derivative test for univariate functions (see Proposition 14.3.5).

Proof. Consider the quadratic approximation $Q(x, y)$ of $f(x, y)$ at a stationary point (a, b). We have $f_x(a, b) = f_y(a, b) = 0$, hence

$$Q(x, y) = f(a, b) + \frac{1}{2}\left[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\right].$$

Let

$$P(x, y) = f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2.$$

We can view $P(x, y)$ as a quadratic in $(x - a)^2$. Consider the discriminant Δ of $P(x, y)$:

$$\begin{aligned}\Delta &= [2f_{xy}(a, b)(y - b)]^2 - 4f_{xx}(a, b)f_{yy}(a, b)(y - b)^2 \\ &= -4(y - b)^2\left(f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2\right).\end{aligned}$$

Let $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. We make the following observations:

- If $D > 0$, then $\Delta < 0$.

  - If $f_{xx}(a, b) > 0$, then $P(x, y) > 0$ (since $f_{xx}(a, b)$ is the leading coefficient of $P(x, y)$). Thus, $Q(x, y) \geq f(a, b)$, whence (a, b) is a minimum point.
  - If $f_{xx}(a, b) < 0$, then $P(x, y) < 0$. Thus, $Q(x, y) \leq f(a, b)$, whence (a, b) is a maximum point.

- If $D < 0$, then $\Delta > 0$. This means that $P(x, y)$ has zeroes elsewhere other than (a, b), and it is sometimes positive and negative. Hence, (a, b) is a saddle point.
- If $D = 0$, then $\Delta = 0$. Hence, $P(x, y)$ has zeroes elsewhere other than (a, b), and it is either always > 0 or < 0 outside the zeroes. Thus, the stationary point could be a maximum, a minimum or even a saddle point; the test is inconclusive.

□

19 Differential Equations

19.1 Definitions

Definition 19.1.1. A **differential equation** (DE) is an equation which involves one or more derivatives of a function y with respect to a variable x (i.e. y', y'', etc.). The **order** of a DE is determined by the highest derivative in the equation. The **degree** of a DE is the power of the highest derivative in the equation.

Example 19.1.2. The differential equation

$$x\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^3+x^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+y=0$$

has order 2 and degree 3.

Observe that the equations $y=x^2-2$, $y=x^2$ and $y=x^2+10$ all satisfy the property $y'=2x$ and are hence solutions of that DE. There are obviously many other possible solutions are we see that any equations of the form $y=x^2+C$, where C is an arbitrary constant, will be a solution to the DE $y'=2x$.

Definition 19.1.3. A **general solution** to a DE contains arbitrary constants, while a **particular solution** does not.

Hence, $y=x^2+C$ is the general solution to the DE $y'=2x$, while $y=x^2-2$, $y=x^2$ and $y=x^2+10$ are the particular solutions.

In general, the general solution of an nth order DE has n arbitrary constants.

19.2 Solving Differential Equations

In this section, we introduce methods to solve three special types of differential equations, namely

- separable DE,
- first-order linear DE, and
- second-order linear DE with constant coefficients.

We also demonstrate how to solve DEs using a given substitution, which is useful if the DE to be solved is not in one of the above three forms.

19.2.1 Separable Differential Equation

Definition 19.2.1. A **separable differential equation** is a DE that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f(x)g(y).$$

Recipe 19.2.2 (Solving via Separation of Variables).

1. Separate the variables.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \implies \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

2. Integrate both sides with respect to x.

$$\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,dx = \int f(x)\,\mathrm{d}x \implies \int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x.$$

Example 19.2.3 (Solving via Separation of Variables). Consider the separable DE

$$2x\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 1.$$

Separating variables,

$$\frac{2}{y^2+1}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}.$$

Integrating both sides with respect to x, we get

$$\int \frac{2}{y^2+1}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x.$$

Using the chain rule, we can rewrite the LHS as

$$\int \frac{2}{y^2+1}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x.$$

Thus,

$$2\arctan y = \ln|x| + C.$$

This is the general solution to the given DE.

19.2.2 First-Order Linear Differential Equation

Definition 19.2.4. A **first-order linear differential equation** is a DE that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x).$$

To solve a linear first-order DE, we first observe that the LHS looks like the product rule has been applied. This motivates us to multiply through by a new function $f(x)$ such that the LHS can be written as the derivative of a product:

$$f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)p(x)y = f(x)q(x). \tag{1}$$

Recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)y\right] = f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + f'(x)y.$$

Comparing this with (1), we want $f(x)$ to satisfy

$$f(x)p(x) = f'(x) \implies \frac{f'(x)}{f(x)} = p(x).$$

Observe that the LHS is simply the derivative of $\ln f(x)$. Integrating both sides, we get

$$\ln f(x) = \int p(x)\,\mathrm{d}x \implies f(x) = \exp \int p(x)\,\mathrm{d}x.$$

Going back to (1), we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y\mathrm{e}^{\int p(x)\,\mathrm{d}x}\right] = q(x)\mathrm{e}^{\int p(x)\,\mathrm{d}x}.$$

Once again, we get a separable DE, which we can solve easily:

$$y\mathrm{e}^{\int p(x)\,\mathrm{d}x} = \int q(x)\mathrm{e}^{\int p(x)\,\mathrm{d}x}\,\mathrm{d}x.$$

This is the general solution to the DE.

Definition 19.2.5. The function $f(x) = \mathrm{e}^{\int p(x)\,\mathrm{d}x}$ is called the **integrating factor**, sometimes denoted I. F..

Note that we do not need to derive the integrating factor like above every time we solve a linear first-order DE. We can simply quote the result $\text{I. F.} = \mathrm{e}^{\int p(x)\,\mathrm{d}x}$. The following list is a summary of the steps we need to solve a linear first-order DE.

Recipe 19.2.6 (Solving via Integrating Factor)**.**

1. Multiply the DE through by the $\text{I. F.} = \mathrm{e}^{\int p(x)\,\mathrm{d}x}$.

$$\mathrm{e}^{\int p(x)\,\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{\int p(x)\,\mathrm{d}x}p(x)y = \mathrm{e}^{\int p(x)\,\mathrm{d}x}q(x).$$

2. Express the LHS as the derivative of a product.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y\mathrm{e}^{\int p(x)\,\mathrm{d}x}\right] = \mathrm{e}^{\int p(x)\,\mathrm{d}x}q(x).$$

3. Integrating both sides with respect to x.

$$y\mathrm{e}^{\int p(x)\,\mathrm{d}x} = \int \mathrm{e}^{\int p(x)\,\mathrm{d}x}q(x)\,\mathrm{d}x.$$

Note that when finding the integrating factor, there is no need to include the arbitrary constant or consider $|x|$ when integrating $1/x$ with respect to x, as it does not contribute to the solution process in any way; the constants will cancel each other out.

Example 19.2.7 (Solving via Integrating Factor)**.** Consider the DE equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 5x^2.$$

Writing this in standard form,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{3}{x}\right)y = 5x.$$

The integrating factor is hence

$$\text{I. F.} = \mathrm{e}^{\int 3/x\,\mathrm{d}x} = \mathrm{e}^{3\ln x} = x^3.$$

Multiplying the integrating factor through the DE,

$$x^3\frac{\mathrm{d}y}{\mathrm{d}x}+3x^2y=\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3y\right)=5x^4.$$

Integrating both sides with respect to x, we get the general solution

$$x^3y=\int 5x^4\,\mathrm{d}x=x^5+C.$$

19.2.3 Second-Order Linear Differential Equations with Constant Coefficients

In this section, we look at second-order linear differential equations and constant coefficients, which has the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=f(x).$$

If $f(x)\equiv 0$, we call the DE **homogeneous**. Else, it is **non-homogeneous**. In general, a second-order DE will have two solutions.

Before looking at the methods to solve second-order DEs, we introduce two important concepts, namely the superposition principle and linear independence.

Theorem 19.2.8 (Superposition Principle)**.** Let y_1 and y_2 be solutions to a linear, homogeneous differential equation. Then Ay_1+By_2 is also a solution to the DE.

Proof. We consider the case where the DE has order 2, though the proof easily generalizes to higher orders.

Suppose y_1 and y_2 are solutions to

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0.$$

Substituting $y=Ay_1+By_2$ into the DE, we get

$$\begin{aligned}a\left(Ay_1''+By_2''\right)&+b\left(Ay_1'+By_2'\right)+c\left(Ay_1+By_2\right)\\&=A\left(ay_1''+by_1'+cy_1\right)+B\left(ay_2''+by_2'+cy_2\right)\\&=0.\end{aligned}$$

Hence, Ay_1+By_2 satisfies the DE and is hence a solution. □

Definition 19.2.9. Two functions y_1 and y_2 are **linearly independent** if the only solution to

$$Ay_1+By_2=0$$

is the trivial solution $A=B=0$. If there exists non-zero solutions to A and B, then the two functions are **linearly dependent**.

We are now ready to solve second-order DEs.

Homogeneous Second-Order Linear Differential Equations with Constant Coefficients

Consider a homogeneous first-order linear differential equation with constant coefficients which has the form

$$a\frac{\mathrm{d}y}{\mathrm{d}x}+by=0.$$

Using the method of integrating factor, we can show that the general solution is of the form

$$y = Ce^{-\frac{b}{a}x}.$$

We can extend this to the second-order case, i.e.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

by looking for solutions of the form $y = e^{mx}$, where m is a constant to be determined. Substituting $y = e^{mx}$ into the differential equation, we get

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0.$$

Dividing by e^{mx}, we get the quadratic

$$am^2 + bm + c = 0.$$

This is known as the **characteristic equation** of the DE.

If we can solve for m in the characteristic equation, we can find the solution $y = e^{mx}$. Since the characteristic equation is quadratic, it has, in general, two roots, say m_1 and m_2. We thus have the following three scenarios to consider:

- The roots are real and distinct.
- The roots are real and equal.
- The roots are complex conjugates.

Real and Distinct Roots If m_1 and m_2 are real and distinct, $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$ will both be solutions to the DE. Hence, by the superposition principle, the general solution is

$$y = Ae^{m_1x} + Be^{m_2x},$$

where A and B are constants.

Real and Equal Roots If the two roots are equal, i.e. $m_1 = m_2 = m$, then $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$ are no longer linearly independent. Hence, we effectively only get one solution $y_1 = e^{mx}$. To obtain the general solution, we have to find another solution that is not a constant multiple of e^{mx}. By intelligently guessing a solution, we see that $y_2 = xe^{mx}$ satisfies the DE. Hence, by the superposition principle, the general solution is

$$y = Ae^{mx} + Bxe^{mx} = (A + Bx)e^{mx}.$$

Complex Roots If the two roots are complex, then they are conjugates, and we can write them as

$$m_1 = p + iq, \quad m_2 = p - iq.$$

Hence,

$$y_1 = e^{(p+iq)x} = e^{px}(\cos qx + i\sin qx)$$

and

$$y_2 = e^{(p-iq)x} = e^{px}(\cos qx - i\sin qx).$$

By the superposition principle, we get the general solution

$$\begin{aligned} y &= Ce^{px}(\cos qx + i\sin qx) + De^{px}(\cos qx - i\sin qx) \\ &= e^{px}(A\cos qx + B\sin qx), \end{aligned}$$

where $A = C + D$ and $B = i(C - D)$ are arbitrary constants.

In summary,

Recipe 19.2.10 (Homogeneous Second-Order Linear DE with Constant Coefficients). To solve the second-order DE

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0,$$

1. Form the characteristic equation $am^2+bm+c=0$.
2. Find the roots m_1 and m_2 of this characteristic equation.
3.
   - If m_1 and m_2 are real and distinct, then
   $$y=A\mathrm{e}^{m_1x}+B\mathrm{e}^{m_2x}.$$
   - If m_1 and m_2 are real and equal, i.e. $m_1=m_2=m$, then
   $$y=(A+Bx)\mathrm{e}^{mx}.$$
   - If m_1 and m_2 are complex, i.e. $m_1=p+\mathrm{i}q$ and $m_2=p-\mathrm{i}q$, then
   $$y=\mathrm{e}^{px}\left(A\cos qx+B\sin qx\right).$$

Non-Homogeneous Second-Order Linear Differential Equations with Constant Coefficients

We now consider the non-homogeneous second-order linear DE with constant coefficients, which takes the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=f(x).$$

In order to solve this DE, we apply the following result:

Theorem 19.2.11. If y_c is the general solution of

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0$$

and y_p is a particular solution of

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=f(x),$$

then

$$y=y_c+y_p$$

is the general solution to

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=f(x).$$

Proof. We want to solve

$$ay''+by'+cy=f(x). \tag{1}$$

Let y_c be the solution to $ay''+by'+cy=0$. Then

$$ay_c''+by_c'+cy_c=0.$$

Let y_p be a particular solution to (1). Then

$$ay_p''+by_p'+cy_p=f(x).$$

Substituting $y = y_c + y_p$ into (1), we get

$$\begin{aligned} a\left(y_c'' + y_p''\right) + b\left(y_c' + y_p'\right) + c\left(y_c + y_p\right) \\ = \left(ay_c'' + by_c' + cy_c\right) + \left(ay_p'' + by_p' + cy_p\right) \\ = 0 + f(x) = f(x). \end{aligned}$$

□

Note that y_c is called the **complementary function** while y_p is called the **particular integral** or **particular solution**.

We know how to solve the homogeneous DE, so getting y_c is easy. The hard part is getting a particular solution y_p. However, if we make some intelligent guesses, we can determine the general form of y_p. This is called the **method of undetermined coefficients**. We demonstrate this method with the following example:

Example 19.2.12 (Method of Undetermined Coefficients). Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 3 + 8x^2.$$

y_c can easily be obtained:

$$y_c = A\mathrm{e}^x + B\mathrm{e}^{-4x}.$$

Now, observe that $f(x) = 3 + 8x^2$ is a polynomial of degree 2. Thus, we guess that y_p is also a polynomial of degree 2, i.e. $y_p = Cx^2 + Dx + E$, where C, D and E are coefficients to be determined (hence the name "method of undetermined coefficients"). Substituting this into the DE yields

$$(2C) + 3(2Cx + D) - 4\left(Cx^2 + Dx + E\right) = 3 + 8x^2.$$

Comparing coefficients, we get the system

$$\begin{cases} -4C & = 8 \\ \quad 6C - 4D & = 0 \\ \quad 2C + 3D - 4E & = 3 \end{cases},$$

whence $C = -2$, $D = -3$ and $E = -4$. Thus, the particular solution is

$$y_p = -2x^2 - 3x - 4$$

and the general solution is

$$y = y_c + y_p = A\mathrm{e}^x + B\mathrm{e}^{-4x} - 2x^2 - 3x - 4.$$

In our syllabus, we are only required to solve non-homogeneous DEs where $f(x)$ is a polynomial of degree n (as above), of the form $p\mathrm{e}^{kx}$, or of the form $p\cos kx + q\sin kx$. The "guess" for y_p in each of the three cases is tabulated below:

$f(x)$	**"Guess" for** y_p
Polynomial of degree n	Polynomial of degree n
$p\mathrm{e}^{kx}$	$C\mathrm{e}^{kx}$
$p\cos kx + q\sin kx$	$C\cos kx + D\sin kx$

In the event where our "guess" for y_p appears in the complementary function y_c, we need to make some adjustments to our "guess" (similar to the case where $m_1 = m_2$ when solving a homogeneous DE). Typically, we multiply the guess by powers x until the guess no longer appears in the complementary function.

Example 19.2.13 (Adjusting y_p).

- If $ay'' + by' + cy = e^{2x}$ has complementary function $y_c = Ae^{-5x} + Be^{2x}$, we try $y_p = Cxe^{2x}$.
- If $ay'' + by' + cy = e^{2x}$ has complementary function $y_c = (A + Bx)e^{2x}$, we try $y_p = Cx^2e^{2x}$.

19.2.4 Solving via Substitution

Sometimes, we are given a DE that is not of the forms described in this section. We must then use the given substitution function to simplify the original DE into one of the standard forms. Similar to integration by substitution, all instances of the dependent variable (including its derivatives) must be substituted.

Recipe 19.2.14 (Solving via Substitution).

1. Differentiate the given substitution function.
2. Substitute into the original DE and simplify to obtain another DE that we know how to solve.
3. Obtain the general solution of the new DE with new dependent variables.
4. Express the solution in terms of the original variables.

Sample Problem 19.2.15. By using the substitution $y = ux^2$, find the general solution of the differential equation

$$x^2 \frac{dy}{dx} - 2xy = y^2, \quad x > 0.$$

Solution. From $y = ux^2$, we see that

$$\frac{dy}{dx} = 2ux + x^2 \frac{du}{dx}.$$

Substituting this into the original DE,

$$x^2 \left(2ux + x^2 \frac{du}{dx} \right) - 2x \left(ux^2 \right) = \left(ux^2 \right)^2.$$

Simplifying, we get the separable DE

$$\frac{du}{dx} = u^2,$$

which we can easily solve:

$$\int \frac{1}{u^2} \, du = \int 1 \, dx \implies -\frac{1}{u} = x + C.$$

Re-substituting y back in, we have the general solution

$$-\frac{x^2}{y} = x + C.$$

□

19.3 Family of Solution Curves

Graphically, the general solution of a differential equation is represented by a family of solution curves which contains infinitely many curves as the arbitrary constant c can take any real number.

A particular solution of the differential equation is represented graphically by one member of that family of solution curves (i.e. one value of the arbitrary constant).

When sketching a family of curves, we choose values of the arbitrary constant that will result in qualitatively different curves. We also need to sketch sufficient members (usually at least 3) of the family to show all the general features of the family.

Example 19.3.1. The following diagram shows three members of the family of solution curves for the general solution $y = A\mathrm{e}^{x^2}$.

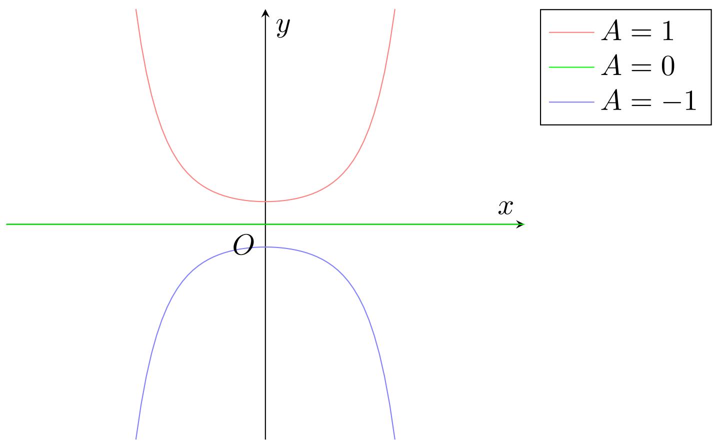


Figure 19.1

19.4 Approximating Solutions

Most of the time, a first-order differential equation of the general form $\mathrm{d}y/\mathrm{d}x = f(x, y)$ cannot be solved exactly and explicitly by analytical methods like those discussed in the earlier sections. In such cases, we can use numerical methods to approximate solutions to differential equations.

Different methods can be used to approximate solutions to a differential equation. A sequence of values $y_1, y_2, \ldots$ is generated to approximate the exact solutions at the points $x_1, x_2, \ldots$. It must be emphasized that the numerical methods do not generate a formula for the solution to the differential equation. Rather, they generate a sequence of approximations to the actual solution at the specified points.

In this section, we look at Euler's Method, as well as the improved Euler's Method.

19.4.1 Euler's Method

The key principle in Euler's method is the use of a linear approximation for the tangent line to the actual solution curve $y(t)$ to approximate a solution.

Derivation

Given an initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad y(t_0) = y_0,$$

we start at (t_0, y_0) on the solution curve as shown in the figure below. By the point-slope formula, the equation of the tangent line through (t_0, y_0) is given as

$$y - y_0 = \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=t_0} (t - t_0) = f(t_0, y_0)(t - t_0). \tag{1}$$

If we choose a step size of Δt on the t-axis, then $t_1 = t_0 + \Delta t$. Using (1) at $t = t_1$, we can obtain an approximate value y_1 from

$$y_1 = y_0 + (t_1 - t_0) f(t_0, y_0). \tag{2}$$

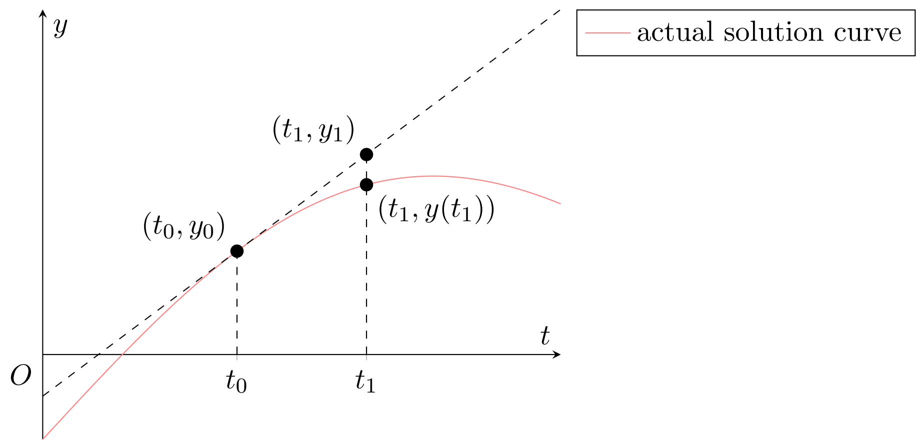


Figure 19.2

The point (t_1, y_1) on the tangent line is an approximation to the point $(t_1, y(t_1))$ on the actual solution curve. That is, $y_1 \approx y(t_1)$. From the above figure, it is observed that the accuracy of the approximation depends heavily on the size of Δt. Hence, we must choose an increment Δt which is "reasonably small".

We can extend (2) further. In general, at $t = t_{n+1}$, it follows that

$$y_{n+1} = y_n + (t_{n+1} - t_n) f(t_n, y_n).$$

Recipe 19.4.1 (Euler's Method)**.** Euler's method, with step size Δt, gives the approximation

$$y(t_n) \approx y_{n+1} = y_n + (t_{n+1} - t_n) f(t_n, y_n).$$

Example 19.4.2 (Euler's Method)**.** Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2y - 1, \quad y(0) = \frac{3}{2},$$

which can be verified to have solution $y = \mathrm{e}^{2t} + 1/2$. Suppose we wish to approximate the value of $y(0.3)$ (which we know to be $\mathrm{e}^{2(0.3)} + 1/2 = 2.322$). Using Euler's method with step size $\Delta t = 0.1$, we get

$$\begin{aligned}
y_1 &= y_0 + \Delta t\,(2y_0 - 1) = 1.5 + 0.1\,[2(1.5) - 1] = 1.7 \\
y_2 &= y_1 + \Delta t\,(2y_1 - 1) = 1.7 + 0.1\,[2(1.7) - 1] = 1.94 \\
y_3 &= y_2 + \Delta t\,(2y_2 - 1) = 1.94 + 0.1\,[2(1.94) - 1] = 2.228
\end{aligned}$$

Hence, $y(0.3) \approx y_3 = 2.228$, which is a decent approximation (4.04% error).

Error in Approximations

Similar to the trapezium rule, the nature of the estimates given by Euler's method depends on the concavity of the actual solution curve.

- If the actual solution curve is concave upwards (i.e. lies above its tangents), the approximations are under-estimates.
- If the actual solution curve is concave downwards (i.e. lies below its tangents), the approximations are over-estimates.

Also note that the smaller the step size Δt, the better the approximations. However, in doing so, more calculations must be made. This is a situation that is typically of numerical methods: there is a trade-off between accuracy and speed.

19.4.2 Improved Euler's Method

In the previous section, we saw how Euler's method over- or under-estimates the actual solution curve due to the curve's concavity. The improved Euler's method address this.

Derivation

Suppose the actual solution curve is concave upward. Let T_0 and T_1 be the tangent lines at $t = t_0$ and $t = t_1$ respectively. Let the gradients of T_0 and T_1 be m_0 and m_1 respectively. We wish to find the optimal gradient m such that the line with gradient m passing through $(t_0, y(t_0))$ also passes through $(t_1, y(t_1))$.

Since the actual solution curve is concave upward, both T_0 and T_1 lie below the actual solution curve for all $t \in [t_0, t_1]$. This is depicted in the diagram below.

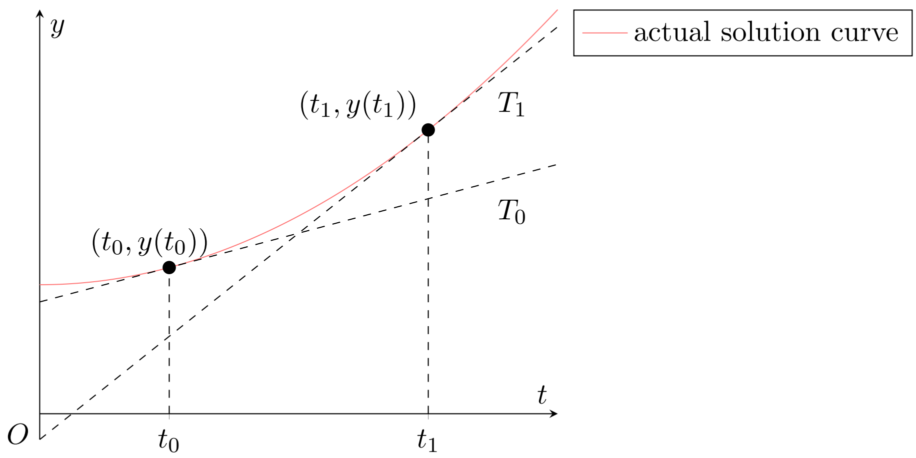


Figure 19.3

Now, observe what happens when we translate T_1 such that it passes through $(t_0, y(t_0))$:

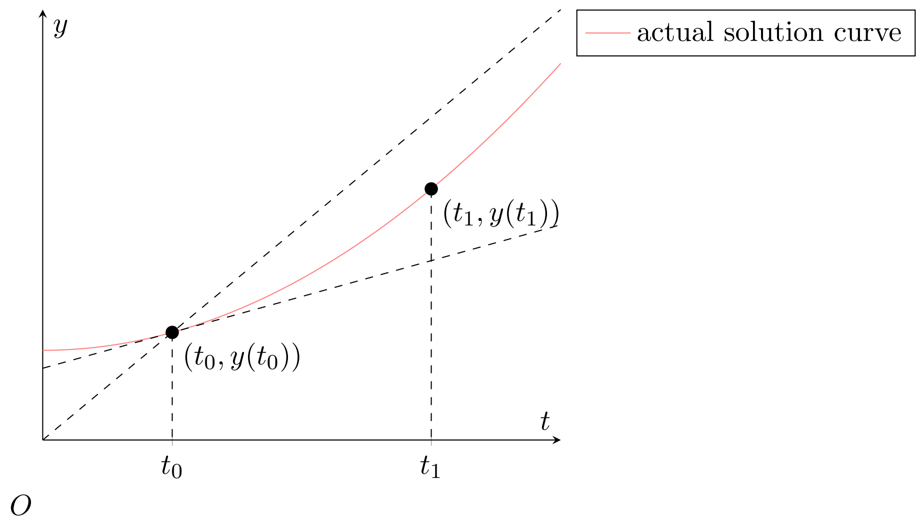


Figure 19.4

The translated T_1 is now overestimating the actual solution curve at $t = t_1$! Hence, the optimal gradient m is somewhere between m_0 and m_1. This motivates us to approximate m by taking the average of m_0 and m_1:

$$m \approx \frac{m_0 + m_1}{2}.$$

We now find m_0 and m_1. Note that

$$m_0 = f(t_0, y(t_0)) \quad \text{and} \quad m_1 = f(t_1, y(t_1)).$$

This poses a problem, as the value of $y(t_1)$ is not known to us. However, we can estimate it using the Euler method:

$$y(t_1) \approx \widetilde{y}_1 = y_0 + \Delta t f(t_0, y_0).$$

Note that we denote this approximation as $\widetilde{y}_0$. We thus have

$$m \approx \frac{m_0 + m_1}{2} = \frac{f(t_0, y_0) + f(t_1, \widetilde{y}_1)}{2}.$$

We are now ready to approximate $y(t_1)$. By the point-slope formula, the line with gradient m passing through (t_0, y_0) has equation

$$y - y_0 = m(t - t_0) \approx \frac{f(t_0, y_0) + f(t_1, \widetilde{y}_1)}{2}(t - t_0).$$

When $t = t_1$, we get

$$y(t_1) \approx y_1 = y_0 + \Delta t \left[\frac{f(t_0, y_0) + f(t_1, \widetilde{y}_1)}{2}\right]. \tag{1}$$

A similar derivation can be obtained when the actual solution curve is concave downwards.

Extending (1), we get the usual statement of the improved Euler's method:

Recipe 19.4.3 (Improved Euler's Method). The improved Euler's method, with step size Δt, gives the approximation

$$y_{n+1} = y_n + \Delta t \left[\frac{f(t_n, y_n) + f(t_{n+1}, \widetilde{y}_{n+1})}{2}\right],$$

where

$$\widetilde{y}_{n+1} = y_n + \Delta t f(t_n, y_n).$$

Definition 19.4.4. $\widetilde{y}_{n+1}$ is called the **predictor**, while y_{n+1} is called the **corrector**.

Example 19.4.5 (Improved Euler's Method). Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2y - 1, \quad y(0) = \frac{3}{2},$$

which we previously saw in Example 19.4.2. Suppose we wish to approximate the value of $y(0.3)$. Using the improved Euler's method with step size $\Delta t = 0.1$,

$$\widetilde{y}_1 = y_0 + \Delta t f(t_0, y_0) = 1.7$$
$$y_1 = y_0 + \Delta t \left[\frac{f(t_0, y_0) + f(t_1, \widetilde{y}_1)}{2}\right] = 1.72$$

$$\widetilde{y}_2 = y_1 + \Delta t f(t_1, y_1) = 1.964$$
$$y_2 = y_1 + \Delta t \left[\frac{f(t_1, y_1) + f(t_2, \widetilde{y}_2)}{2}\right] = 1.9884$$

$$\widetilde{y}_3 = y_2 + \Delta t f(t_2, y_2) = 2.28608$$
$$y_3 = y_2 + \Delta t \left[\frac{f(t_2, y_2) + f(t_3, \widetilde{y}_3)}{2}\right] = 2.35848$$

Hence, $y(0.3) \approx y_3 = 2.35848$, which gives an error of 0.270%, much better than the 4.04% achieved by Euler's method.

19.4.3 Relationship with Approximations to Definite Integrals

Recall that solving differential equations analytically required us to integrate. It is thus no surprise that approximating solutions to differential equations is related to approximating the values of definite integrals. As we will see, the Euler method is akin to approximating definite integrals using a Riemann sum, while the improved Euler method is akin to using the trapezium rule.

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$. By the fundamental theorem of calculus, the area under the graph of $f(t, y)$ from $t = t_0$ to $t = t_1$ is given by

$$\int_{t_0}^{t_1} f(t, y)\,\mathrm{d}t = \int_{t_0}^{t_1} \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = y(t_1) - y(t_0). \tag{1}$$

Note that we know $y(t_0)$. Hence, the better the approximation of the integral, the better the approximation of $y(t_1)$, which is what we want.

We can approximate this integral using a Riemann sum with one rectangle. Note that this rectangle has width Δt and height $f(t_0, y_0)$. Hence,

$$\int_{t_0}^{t_1} f(t, y)\,\mathrm{d}t = y(t_1) - y(t_0) \approx \Delta t f(t_0, y_0).$$

Rewriting, we get the statement of the Euler method:

$$y(t_1) \approx y(t_0) + \Delta t f(t_0, y_0).$$

We now approximate the integral in (1) using the trapezium rule with 2 ordinates. Note that the area of this trapezium is given by $\frac{1}{2}\Delta t\,[f(t_0, y_0) + f(t_1, y_1)]$. Hence,

$$\int_{t_0}^{t_1} f(t, y)\,\mathrm{d}t = y(t_1) - y(t_0) \approx \Delta t \left[\frac{f(t_0, y_0) + f(t_1, y_1)}{2}\right].$$

Rewriting, we (almost) get the statement of the improved Euler method:

$$y(t_1) \approx y(t_0) + \Delta t \left[\frac{f(t_0, y_0) + f(t_1, y_1)}{2} \right].$$

Recall that generally, the trapezium rule is a much better approximation than a Riemann sum. Correspondingly, it follows that the improved Euler method is a much better approximation than the Euler method.

19.5 Modelling Populations with First-Order Differential Equations

Populations, however defined, generally change their magnitude as a function of time. The main goal here is to provide some mathematical models as to how these populations change, construct the corresponding solutions, analyse the properties of these solutions, and indicate some applications.

For the case of living biological populations, we assume that all environment and/or cultural factors operate on a timescale which is much longer than the intrinsic timescale of the population of interest. If this holds, then the mathematical model takes the following form of a simple population:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = f(P), \quad P(0) = p_0 \geq 0,$$

where $P(t)$ is the value of the population P at time t. The function $f(P)$ is what distinguishes one model from another.

We would expect the model to have the same structure

$$\frac{\mathrm{d}P}{\mathrm{d}t} = g(P) - d(P),$$

where $g(P)$ and $d(P)$ are the growth and decline factors respectively. Also, we assume $g(0) = d(0) = 0$, whence $f(0) = 0$. This is related to the **axiom of parenthood**, which states the "every organism must have parents; there is no spontaneous generation of organisms".

In this section, we will look at two common population growth models, namely the exponential growth model and the logistic growth model.

19.5.1 Exponential Growth Model

A biological population with plenty of food, space to grow, and no threat from predators, tend to grow at a rate that is proportional to the population. That is, in each unit of time, a certain percentage of the individuals produce new individuals (similar for death too). If reproduction (and death) takes place more or less continuously, then the growth rate is represented by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP,$$

where k is the **proportionality constant**.

We know that all solutions of this differential equation have the form

$$P(t) = p_0 \mathrm{e}^{kt}.$$

As such, this model is known as the **exponential growth model**. Depending on the value of k, the model results in either an exponential growth, decay, or constant value function as seen in the diagram below.

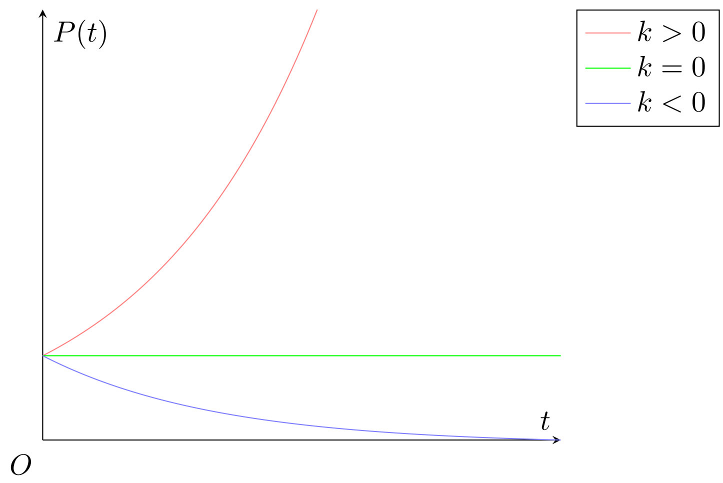


Figure 19.5

While the cases where $k \leq 0$ are possible to happen in real life, the case where $k > 0$ is not realistically possible as most populations are constrained by limitations of resources.

19.5.2 Logistic Growth Model

The following figure shows two possible courses for growth of a population. The red curve follows the exponential model, while the blue curve is constrained so that the population is always less than some number N. When the population is small relative to N, the two curves are identical. However, for the blue curve, when P gets closer to N, the growth rate drops to 0.

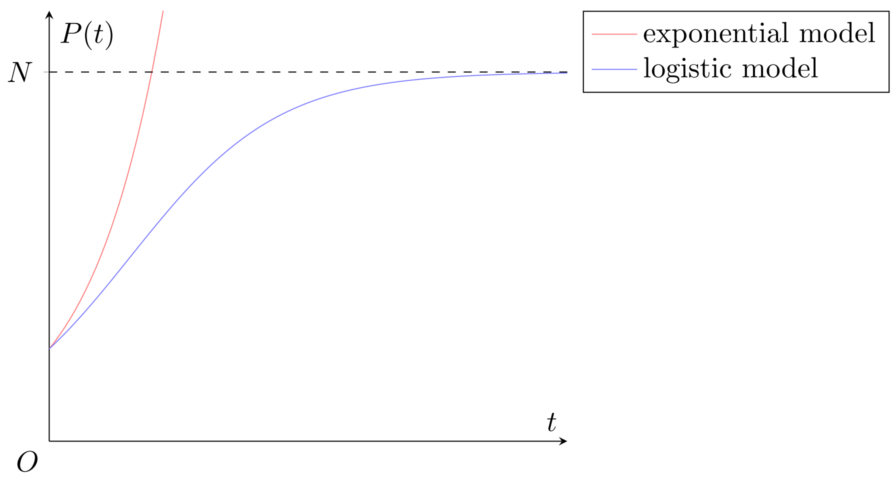


Figure 19.6

We may account for the growth rate declining to 0 by including in the model a factor $1 - P/N$, which is close to 1 (i.e. no effect) when P is much smaller than N, and close to 0 when P is close to N. The resulting model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right),$$

is called the **logistic growth model**. k is called the **intrinsic growth rate**, while N is called the **carrying capacity**.

Given the initial condition $P(0) = p_0$, the solution of the logistic equation is

$$P(t) = \frac{p_0 N}{[}p_0 + (N - p_0)\mathrm{e}^{-kt}].$$

Long-Term Behaviour

We now analyse the long-term behaviour of the model, which is determined by the value of P_0.

Notice that the derivative of the logistic growth model, $\mathrm{d}P/\mathrm{d}t = kP(1 - P/N)$, is 0 at $P = 0$ and $P = N$. Also notice that these are also solutions to the differential equation. These two values are the **equilibrium points** since they are constant solutions to the differential equation.

Consider the case where $k > 0$.

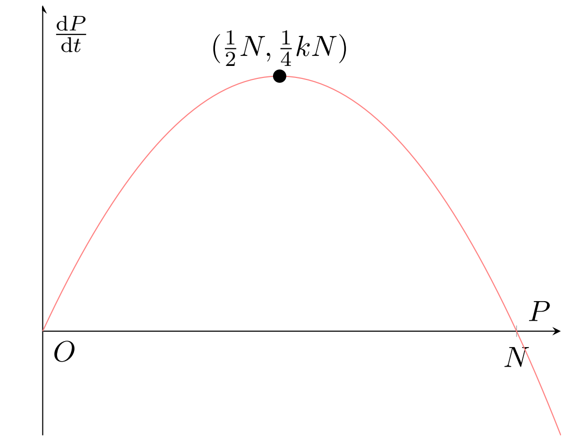


Figure 19.7

From the above diagram, we observe that

- if $0 < P_0 < N$, then P will increase towards N since $\mathrm{d}P/\mathrm{d}t > 0$.
- if $P_0 > N$, then P will decrease towards N since $\mathrm{d}P/\mathrm{d}t < 0$.

Since any population value in the neighbourhood of 0 will move away from 0, the equilibrium point at $P = 0$ is known as an **unstable equilibrium point**. On the contrary, since any population value in the neighbourhood of N will move towards N, the equilibrium point at $P = N$ is known as a **stable equilibrium point**.

Now consider the case where $k < 0$.

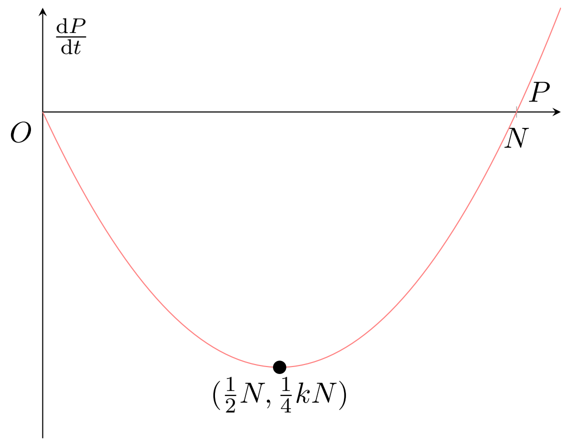


Figure 19.8

From the above diagram, we observe that

- if $0 < p_0 < N$, then P will decrease towards N since $\mathrm{d}P/\mathrm{d}t < 0$.
- if $p_0 > N$, then P will increase indefinitely since $\mathrm{d}P/\mathrm{d}t > 0$.

In this case, the equilibrium point at $P = 0$ is stable, while the equilibrium point at $P = N$ is unstable.

Thus, we see that what happens to the population in the long-run depends very much on the value of the initial population, P_0.

19.5.3 Harvesting

There are many single population systems for which harvesting takes place. **Harvesting** is a removal of a certain number of the population during each time period that the harvesting takes place. Below are some variants of the basic logistic model.

Constant Harvesting

The most direct way of harvesting is to use a strategy where a constant number, $H \geq 0$, of individuals are removed during each time period. For this situation, the logistic equation gets modified to the form

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right) - H,$$

where H is known as the **harvesting rate**.

Observe that the equilibrium solutions to this modified logistic equation are:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right) - H = 0 \implies P = \frac{N}{2} \pm \sqrt{\frac{N^2}{4} - \frac{NH}{k}}.$$

With the equilibrium solutions, we can do the same analysis above to determine the long-term behaviour of the model.

Variable Harvesting

The model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right) - HP$$

results by harvesting at a non-constant rate proportional to the present population P. The effect is to decrease the natural growth rate k by a constant amount H in the standard logistic model.

Restocking

The equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right) - H\sin(\omega t)$$

models a logistic equation that is periodically harvested and restocked with maximal rate H. For sufficiently large p_0, the equation models a stable population that oscillates about the carrying capacity N with period $T = 2\pi/\omega$.

Part VI

Statistics

20 Permutations and Combinations

20.1 Counting Principles

Fact 20.1.1 (The Addition Principle). Let E_1 and E_2 be two mutually exclusive events. If E_1 and E_2 can occur in n_1 and n_2 different ways respectively, then E_1 or E_2 can occur in $(n_1 + n_2)$ ways.

Fact 20.1.2 (The Multiplication Principle). Consider a task S that can be broken down into two independent ordered stages S_1 and S_2. If S_1 and S_2 can occur in n_1 and n_2 ways respectively, then S_1 and S_2 can occur in succession in $n_1 n_2$ ways

Note that both the Addition and Multiplication Principles can be extended to any finite number of events.

20.2 Permutations

Definition 20.2.1. A **permutation** is an arrangement of a number of objects in which the **order is important**.

Example 20.2.2. ABC, BAC and CBA are possible permutations of the letters 'A', 'B' and 'C'.

Definition 20.2.3 (Factorial). The **factorial** of a non-negative integer n is given by the recurrence relation

$$n! = n(n-1)!, \quad 0! = 1.$$

Equivalently,

$$n! = n(n-1)(n-2)\dots(3)(2)(1), \quad 0! = 1.$$

Proposition 20.2.4 (Permutations of Objects Taken from Sets of Distinct Objects). The number of permutations of n distinct objects, taken r at a time without replacement, is given by

$${}^nP_r = \underbrace{n(n-1)(n-2)\dots(n-r+1)}_{r \text{ consecutive integers}} = \frac{n!}{(n-r)!},$$

where $0 \leq r \leq n$.

Proof. Suppose we have n distinct objects that we want to fill up r ordered slots with. This operation can be done in r stages

- **Stage 1.** The number of ways to fill in the first slot is n.
- **Stage 2.** After filling in the first slot, the number of ways to fill in the second slot is $n - 1$.
- **Stage 3.** After filling in the first and second slots, the number of ways to fill in the third slot is $n - 2$.

This continues until we reach the last stage:

- **Stage r.** After filling all previous $r-1$ slots, the number of ways to fill in the last slot is $n-(r-1)=n-r+1$.

Thus, by the Multiplication Principle, the number of ways to fill up the r slots are

$$n(n-1)(n-2)\dots(n-r+1)=\frac{n!}{(n-r)!}.$$

□

Corollary 20.2.5 (Permutations of Distinct Objects in a Row)**.** The number of ways to arrange n distinct objects in a row, taken all at a time without replacement, is given by $n!$.

Proof. Take $r=n$. □

Proposition 20.2.6 (Permutations of Non-Distinct Objects in a Row)**.** The number of permutations of n objects in a row, taken all at a time without replacement, of which n_1 are of the 1st type, n_2 are of the 2nd type, ..., n_k are of the kth type, where $n=n_1+n_2+\dots+n_k$, is given by

$$\frac{n!}{n_1!\,n_2!\dots n_k!}.$$

Proof. Let A_i be the set of arrangements where objects in the first i groups are now distinguishable, while objects in the remaining groups remain indistinguishable. For instance, A_1 is the set of arrangements of n objects in a row, of which n_2 are of the 2nd type, n_3 are of the 3rd type, ..., n_k are of the kth type, while the objects previously of the 1st type are now distinct. We prove the above result by expressing $|A_0|$ in terms of $|A_k|$.

Suppose we make objects of the 1st type distinct. For each arrangement in A_0, the n_1 objects of the 1st type can be permuted among themselves in $n_1!$ ways. Hence,

$$|A_1|=n_1!\,|A_0|.$$

Next, suppose we make objects of the 2nd type distinct. For each arrangement in A_1, the n_2 objects of the 2nd type can be permuted among themselves in $n_2!$ ways. Hence,

$$|A_2|=n_2!\,|A_1|.$$

Continuing on, we see that

$$|A_k|=n_k!\,|A_{k-1}|=n_k!\,n_{k-1}!\,|A_{k-2}|=\dots=n_k!\,n_{k-1}!\dots n_1!\,|A_0|.$$

However, by definition, A_k is the set of arrangements of n distinct objects, which we know to be $n!$. Thus,

$$|A_0|=\frac{|A_k|}{n_1!\,n_2!\dots n_k!}=\frac{n!}{n_1!\,n_2!\dots n_k!}.$$

□

Remark. $\frac{n!}{n_1!n_2!\dots n_k!}$ is known as a **multinomial coefficient**, which is a generalization of the binomial coefficient and is related to the expansion of $(x_1+x_2+\dots+x_k)^n$.

Sample Problem 20.2.7. Find the number of different permutations of the letters in the word "BEEN".

Solution. Note that there is 1 'B', 2'E's and 1 'N' in "BEEN". Using the above result, the number of different permutations is given by

$$\frac{4!}{1!2!1!} = 12.$$

□

Proposition 20.2.8 (Circular Permutations). The number of permutations of n distinct objects in a circle is given by $(n-1)!$.

Proof. Fix one object as the reference point. The remaining $n-1$ objects have $(n-1)!$ possible ways to be arranged in the remaining $n-1$ positions around the circle. □

Proposition 20.2.9 (Permutations of Objects Taken from Sets of Distinct Objects with Replacement). The number of permutations of n distinct objects, taken r at a time with replacement, is given by n^r, where $0 \leq r \leq n$.

20.3 Combinations

Definition 20.3.1. A **combination** is a selection of objects from a given set where the order of selection does not matter.

Proposition 20.3.2 (Combinations of Objects Taken from Sets of Distinct Objects). The number of combinations of n distinct objects, taken r at a time without replacement, is given by

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!},$$

where $0 \leq r \leq n$.

Proof. Observe the number of ways to choose r objects from n distinct objects is equivalent to the number of permutations of n objects, where r objects are of the first type (chosen) while $n-r$ objects are of the second type (not chosen). Using the formula derived above, we have

$$^nC_r = \frac{n!}{r!(n-r)!}.$$

□

Corollary 20.3.3. For integers r and n, where $0 \leq r \leq n$,

$$^nP_r = {}^nC_r \cdot r!.$$

Proof. Rearrange the above result. □

Corollary 20.3.4. For integers r and n, where $0 \leq r \leq n$,

$$^nC_r = {}^nC_{n-r}.$$

Proof. Observe that

$$\frac{n!}{r!(n-r)!}$$

is invariant under $r \mapsto n-r$. □

20.4 Methods for Solving Combinatorics Problems

Some problems involving permutations and combinations may involve restrictions. When dealing with such problems, one should consider the restrictions first. There are four basic strategies that can be employed to tackle these restrictions.

Recipe 20.4.1 (Fixing Positions). When certain objects must be at certain positions, place those objects first.

Sample Problem 20.4.2. How many ways are there to arrange the letters of the word "SOCIETY" if the arrangements start and end with a vowel?

Solution. We first address the restriction by placing the vowels at the start and end of the arrangement. Since there are 3 vowels in "SOCIETY", there are $3 \cdot 2 = 6$ ways to do so. Next, observe there are 5! ways to arrange the remaining 5 letters. Thus, by the Multiplication Principle, there are

$$6 \cdot 5! = 720$$

arrangements that satisfy the given restriction. □

Recipe 20.4.3 (Grouping Method). When certain objects must be placed together, group them together as one unit.

Sample Problem 20.4.4. Find the number of ways the letters of the word "COMBINE" can be arranged if all the consonants are to be together.

Solution. Consider the consonants 'C', 'M', 'B' and 'N' as one unit:

$$\boxed{\text{C} \quad \text{M} \quad \text{B} \quad \text{N}} \quad \boxed{\text{O}} \quad \boxed{\text{I}} \quad \boxed{\text{E}}.$$

- **Stage 1.** There are 4! ways to arrange the 4 units.
- **Stage 2.** There are 4! ways to arrange 'C', 'M', 'B' and 'N' within the group.

Hence, by the Multiplication Principle, the total number of arrangements is

$$4! \cdot 4! = 576.$$

□

Recipe 20.4.5 (Slotting Method). When certain objects are to be separated, we first arrange the other objects to form barriers before slotting in those to be separated.

Sample Problem 20.4.6. Find the number of ways the letters of the word "COMBINE" can be arranged if all the consonants are to be separated.

Solution. We begin by arranging the vowels, of which there are 3! ways to do so.

$$\uparrow \quad \boxed{\text{O}} \quad \uparrow \quad \boxed{\text{I}} \quad \uparrow \quad \boxed{\text{E}} \quad \uparrow.$$

Next, we slot the 4 consonants into the 4 gaps in between the vowels (i.e. where the arrows are). There are 4! ways to do so. Thus, by the Multiplication Principle, the total number of arrangements is

$$3! \cdot 4! = 144.$$

□

Recipe 20.4.7 (Complementary Method)**.** If the direct method is too tedious, it is more efficient to count by taking all possibilities minus the complementary sets. This method can also be used for "at least/at most" problems.

Sample Problem 20.4.8. Find the number of ways the letters of the word "COMBINE" can be arranged if all the consonants are to be separated.

Solution. Note that, without restrictions, there are a total of 7! ways to arrange the letters in "COMBINE". From the previous example, we saw that the number of arrangements where all consonants are together is 576. Thus, by the complementary method, the number of arrangement where all consonants are separated is

$$\text{total} - \text{complementary} = 7! - 576 = 144,$$

which matches the answer given in the above example. □

21 Probability

21.1 Basic Terminology

Definition 21.1.1. A statistical or random **experiment** (or trial) refers to a process that generates a set of observable outcomes, and can be repeated under the same set of conditions.

Definition 21.1.2. The **sample space** (or possibility space) S of an experiment is the set of all possible outcomes of the experiment.

Definition 21.1.3. An **event** E is a subset of S. The **complement** of E, denoted by E', is the event that E does not occur, i.e. $E' = S \setminus E$.

Definition 21.1.4. Given a subset $G \subseteq S$, the function $n(G)$ returns the **number of possible outcomes** in G.

21.2 Probability

Definition 21.2.1 (Classical Probability)**.** If the sample space S consists of a finite number of equally likely outcomes, then the probability of an event E occurring (a measure of the likelihood that E occurs) is denoted $\mathrm{P}(E)$ and is defined as

$$\mathrm{P}(E) = \frac{n(E)}{n(S)}.$$

Proposition 21.2.2 (Range of Probabilities)**.** For any event E,

$$\mathrm{P}(E) \in [0, 1].$$

Proof. Let the sample space be S. Since $E \subseteq S$, we have

$$0 \leq n(E) \leq n(S) \implies 0 \leq \frac{n(E)}{n(S)} \leq \frac{n(S)}{n(S)} \implies 0 \leq \mathrm{P}(E) \leq 1.$$

□

Corollary 21.2.3. Let A and B be any two events. If $A \subseteq B$, then $\mathrm{P}(A) \leq \mathrm{P}(B)$.

Proof. Identical as above. □

Definition 21.2.4. When $\mathrm{P}(E) = 0$, we say that E is an **impossible** event. When $\mathrm{P}(E) = 1$, we say that P is a **sure** event.

Proposition 21.2.5 (Probability of Complement). For any event E,

$$\mathrm{P}(E) + \mathrm{P}\left(E'\right) = 1.$$

Proof. Let the sample space be S. By definition, $E' = S \setminus E$. Hence,

$$n(E') = n(S) - n(E) \implies \frac{n(E)}{n(S)} + \frac{n(E')}{n(S)} = \frac{n(S)}{n(S)} \implies \mathrm{P}(E) + \mathrm{P}\left(E'\right) = 1.$$

□

Definition 21.2.6. Let S be the sample space of a random experiment and A, B be any two events.

- The **intersection** of A and B, denoted by $A \cap B$, is the event that both A and B occur.
- The **union** of A and B, denoted by $A \cup B$, is the event that at least one occurs.

Proposition 21.2.7 (Inclusion-Exclusion Principle). Let A and B be any two events in a sample space S. Then

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B).$$

Proof. When we take the sum of the number of outcomes in events A and B, i.e. $n(A) + n(B)$, we will count the 'overlap', i.e. $n(A \cap B)$, twice. Hence,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Dividing throughout by $n(S)$ yields the desired result. □

Proposition 21.2.8 (Intersection of Complements). Let A and B be any two events. Then

$$\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}\left(A \cap B'\right).$$

Proof. By definition, $B' = S \setminus B$. Taking the intersection with A on both sides,

$$\mathrm{P}\left(A \cap B'\right) = \mathrm{P}(A \cap S) - \mathrm{P}(A \cap B) \implies \mathrm{P}(A \cap B) + \mathrm{P}\left(A \cap B'\right) = \mathrm{P}(A).$$

□

Proposition 21.2.9 ("Neither Nor"). Let A and B be any two events. Then

$$\mathrm{P}\left(A' \cap B'\right) = 1 - \mathrm{P}(A \cup B).$$

Proof. In layman terms, the above statement translates to

$$\mathrm{P}(\text{neither } A \text{ nor } B) = 1 - \mathrm{P}(A \text{ or } B),$$

which is clearly true. □

21.3 Mutually Exclusive Events

Definition 21.3.1. Two events A and B are said to be **mutually exclusive** if they cannot occur at the same time. Mathematically,

$$\mathrm{P}(A \cap B) = 0.$$

An equivalent criterion for mutual exclusivity is

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B),$$

which can easily be derived from $\mathrm{P}(A \cap B) = 0$ via the inclusion-exclusion principle.

21.4 Conditional Probability and Independent Events

Proposition 21.4.1 (Conditional Probability)**.** The probability of an event A occurring, given that another event B has already occurred, is given by

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Proof. Since B has already occurred, the sample space is reduced to B. Hence,

$$\mathrm{P}(A \mid B) = \frac{n(A \cap B)}{n(B)}.$$

Dividing the numerator and denominator by $n(S)$ completes the proof. □

Corollary 21.4.2. The event (A, given B) is the complement of the event (not A, given B), i.e.

$$\mathrm{P}(A \mid B) + \mathrm{P}\left(A' \mid B\right) = 1.$$

Proof.

$$\mathrm{P}(A \mid B) + \mathrm{P}\left(A' \mid B\right) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} + \frac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(B)}{\mathrm{P}(B)} = 1.$$

□

Definition 21.4.3 (Independent Events)**.** Let A and B be any two events. If either of the two occur without being affected by the other, then A and B are said to be **independent**. Mathematically,

$$\mathrm{P}(A \mid B) = \mathrm{P}(A), \qquad \mathrm{P}(B \mid A) = \mathrm{P}(B).$$

Proposition 21.4.4 (Multiplication Law)**.** A and B are independent events if and only if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B).$$

Proof. Since $\mathrm{P}(A) = \mathrm{P}(A \cap B)/\mathrm{P}(B)$ and $\mathrm{P}(A \mid B) = \mathrm{P}(A)$,

$$\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \mathrm{P}(A) \iff \mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B).$$

□

Proposition 21.4.5. If events A and B are independent, then so are the following pairs of events:

- A and B',
- A' and B,
- A' and B'.

Proof. We only prove that A' and B are independent. The proofs for the other pairs are almost identical.

Since A and B are independent events, we have $P(A \cap B) = P(A)P(B)$. Now consider $P(A' \cap B)$.

$$P\big(A' \cap B\big) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = P(B)\,[1 - P(A)] = P(B)P\big(A'\big).$$

Hence, A' and B are independent. □

21.5 Common Heuristics used in Solving Probability Problems

Recipe 21.5.1 (Table of Outcomes). Table of outcomes are useful as they serve as a systematic way of listing all the possible outcomes.

Sample Problem 21.5.2. Two fair dices are thrown. Find the probability that the sum of the two scores is odd and at least one of the two scores is greater than 4.

Solution. Consider the following table of outcomes.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

From the table of outcomes, the required probability is clearly $\frac{10}{36}$. □

Recipe 21.5.3 (Venn Diagrams). Venn diagrams are useful when we need to visualize how the events are interacting with each other.

Sample Problem 21.5.4. Let A and B be independent events. If $P(A' \cap B') = 0.4$, find the range of $P(A \cap B)$.

Solution. Consider the following Venn diagram.

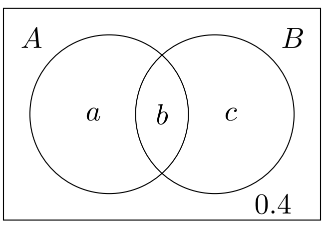


Figure 21.1

We see that

$$a + b + c = 0.6. \tag{$*$}$$

Further, since A and B are independent, we know

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) \implies b = (a+b)(c+b) = (a+b)(0.6-a).$$

Expanding, we get a quadratic in a:

$$a^2 + (b - 0.6)a + 0.4b = 0.$$

Since we want a to be real, the discriminant Δ is non-negative. Hence,

$$(b-0.6)^2 - 4(1)(0.4b) \geq 0 \implies b \leq 0.135 \quad \text{or} \quad b \geq 2.66.$$

Since $0 \leq b \leq 1$, we reject the latter. Thus, the range of $\mathrm{P}(A \cap B) = b$ is $[0, 0.135]$. □

Recipe 21.5.5 (Probability Trees)**.** A probability tree is a useful tool for sequential events, or events that appear in stages. The number indicated on each branch represents the conditional probability of the event at the end node given that all the events at the previous nodes have occurred.

Sample Problem 21.5.6. Peter has a bag containing 6 black marbles and 3 white marbles. He takes out two marbles at random from the bag. Find the probability that he has taken out a black marble and a white marble.

Solution. Consider the following probability tree.

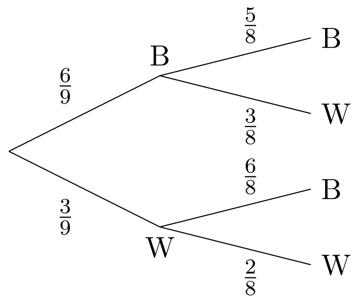


Figure 21.2

The required probability is thus

$$\left(\frac{6}{9}\right)\left(\frac{3}{8}\right) + \left(\frac{3}{9}\right)\left(\frac{6}{8}\right) = \frac{1}{2}.$$

□

Recipe 21.5.7 (Permutations and Combinations)**.** Using combinatorial methods is useful when the most direct way to calculate $\mathrm{P}(E)$ is to find $n(E)$ and $n(S)$.

Sample Problem 21.5.8. A choir has 7 sopranos, 6 altos, 3 tenors and 4 basses. At a particular rehearsal, three members of the choir are chosen at random. Find the probability that exactly one bass is chosen.

Solution. Note that there are a total of 20 people in the choir. Hence, the number of ways to choose three members of the choir, without restriction, is given by ${}^{20}C_3$. Meanwhile, the number of ways to choose exactly one bass is given by ${}^{4}C_1 \cdot {}^{16}C_2$: first choose one bass out of the four, then choose 2 members out of the remaining 16. Thus, the required probability is

$$\frac{{}^{4}C_1 \cdot {}^{16}C_2}{{}^{20}C_3} = \frac{8}{19}.$$

□

Part VII

Mathematical Proofs and Reasoning

22 Propositional Logic

Mathematics is a deductive science, where from a set of basic axioms, we prove more complex results. To do so, we often restate a sentence into **statements**, which are mathematical expressions. One important axiom that all statements obey is the law of the excluded middle.

Axiom 22.0.1 (Law of the Excluded Middle). The **law of the excluded middle** states that either a statement or its negation is true. Equivalently, a statement cannot be both true and false, nor can it be neither true nor false.

22.1 Statements

We call a sentence such as "x is even" that depends on the value of x a "statement about x". We can denote this statement more compactly as $P(x)$. For instance, $P(5)$ is the statement "5 is even", while $P(72)$ is the statement "72 is even", and so forth. We can also write $P(x)$ more compactly as P.

22.1.1 Structure of Statements

In this section, we examine how statements are linked together to form new statements. The first type of statement we will examine is the conditional statement.

Definition 22.1.1. A **conditional statement** has the form "if P then Q". Here, P is the **hypothesis** and Q is the **conclusion**, denoted by $P \implies Q$. This statement is defined to have the truth table

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

In words, the statement $P \implies Q$ also reads:

- P **implies** Q.
- P is a **sufficient condition** for Q.
- Q is a **necessary condition** for P.
- P **only if** Q.

To justify the truth table of $P \implies Q$, consider the following example:

Example 22.1.2 (Conditional Statement). Suppose I say

"If it is raining, then the floor is wet."

We can write this as $P \implies Q$, where P is the statement "it is raining" and Q is the statement "the floor is wet".

- Suppose both P and Q are true, i.e. it is raining, and the floor is wet. It is reasonable to say that I am telling the truth, whence $P \implies Q$ is true.
- Suppose P is true but Q is false, i.e. it is raining, and the floor is not wet. Clearly, I am not telling the truth; the floor would be wet if I was. Hence, $P \implies Q$ is false.
- Suppose P is false, i.e. it is not raining. Notice that the hypothesis of my claim is not fulfilled; I did not say anything about the floor when it is not raining. Hence, I am not lying, so $P \implies Q$ is true whenever P is false.

Examples of conditional statements in mathematics include

- If $|x - 1| < 4$, then $-3 < x < 5$.
- If a function f is differentiable, then f is continuous.

We now look at biconditional statements. As the name suggests, a biconditional statement comprises two conditional statements: $P \implies Q$ and $Q \implies P$. The conditional statement is much stronger than the conditional statement.

Definition 22.1.3. A **biconditional statement** has the form "P if and only if", denoted $P \iff Q$. This statement is defined to have the truth table

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

When $P \iff Q$ is true, we say that P and Q are **equivalent**, i.e. $P \equiv Q$.

An equivalent definition of $P \iff Q$ is the statement

$$(P \implies Q) \quad \text{and} \quad (Q \implies P).$$

This allows us to easily justify the truth table of $P \iff Q$:

P	Q	$P \implies Q$	$Q \implies P$	$P \iff Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Examples of conditional statements in mathematics include

- A triangle ABC is equilateral if and only if its three angles are congruent.
- a is a rational number if and only if $2a + 4$ is rational.

22.1.2 Quantifiers

We now introduce two important symbols, namely the universal quantifier ($\forall$) and the existential quantifier ($\exists$)

Definition 22.1.4. Let $P(x)$ be a statement about x, where x is a member of some set S (i.e. S is the **domain** of x). Then the notation

$$\forall x \in S,\, P(x)$$

means that $P(x)$ is true for every x in the set S. The notation

$$\exists x \in S,\, P(x)$$

means that there exists at least one element of x of S for which $P(x)$ is true.

Example 22.1.5. Let $P(x)$ be the statement "x is even". Clearly, the statement

$$\forall x \in \mathbb{Z},\, P(x)$$

is not true; not all integers are even. However, the statement

$$\exists x \in \mathbb{Z},\, P(x)$$

is true, because we can find an integer that is even (e.g. $x = 8$).

Note that a statement $P(x)$ does not necessarily have to mention x. For instance, we could define $P(x)$ as the statement "5 is even". Compare this with how a function $f(x)$ does not necessarily have to "mention" x, e.g. we could have $f(x) = 5$.

22.1.3 Types of Statements

Most of the statements we will encounter can be grouped into three classes, namely axioms, definitions and theorems.

Definition 22.1.6.

- An **axiom** is a mathematical statement that does not require proof.
- A **definition** is a true mathematical statement that gives the precise meaning of a word or phrase that represents some object, property or other concepts.
- A **theorem** is a true mathematical statement that can be proven mathematically.

22.2 Proofs

Mathematical proofs are convincing arguments expressed in mathematical language, i.e. a sequence of statements leading logically to the conclusion, where each statement is either an accepted truth, or an assumption, or a statement derived from previous statements. Occasionally there will be the clarifying remark, but this is just for the reader and has no logical bearing on the structure of the proof.

Definition 22.2.1. A **proof** is a deductive argument for a mathematical statement, showing that the stated assumptions logically guarantee the conclusion.

As an example, we will prove the following statement:

Statement 22.2.2. For all $n \in \mathbb{Z}^+$, both n and n^2 have the same parity.

We first define what it means for an integer to be even and odd:

Definition 22.2.3. An integer x is even if there exists some integer y such that $x = 2y$.

Definition 22.2.4. An integer x is odd if there exists some integer y such that $x = 2y + 1$.

We are now ready to prove Statement 22.2.2.

Proof of Statement 22.2.2. Since n can only be either odd or even, we just need to consider the following cases:

Case 1. Suppose n is even. By definition, there exists some $k \in \mathbb{Z}$ such that $n = 2k$. Then

$$n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right) = 2a,$$

where $a = 2k^2$. Since a is an integer, it follows from our definition that n^2 is even. Hence, n and n^2 have the same parity.

Case 2. Suppose n is odd. By definition, there exists some $h \in \mathbb{Z}$ such that $n = 2h+1$. Then

$$n^2 = (2h+1)^2 = 4h^2 + 4h + 1 = 2\left(2h^2 + 2h\right) + 1 = 2b + 1,$$

where $b = 2h^2 + 2h$. Since b is an integer, it follows from our definition that n^2 is odd. Hence, n and n^2 have the same parity. □

The above proof is an example of a direct proof.

Definition 22.2.5. A **direct proof** is an approach to prove a conditional statement $P \implies Q$. It is a series of valid arguments that starts with the hypothesis P, and ends with the conclusion Q.

In the case where we wish to prove a statement false, we can find a counter-example. In providing a counter-example, it must fulfil the hypothesis, but not the conclusion. That is, to show that $P \implies Q$ is false, we must show that P is true but Q is false.

Example 22.2.6 (Counter-Example)**.** Consider the statement $c \mid ab$, then $c \mid a$ or $c \mid b$, where $a, b, c \in \mathbb{Z}^+$. We can easily find a counter-example to this statement, e.g. $a = 3 \times 37$, $b = 7 \times 37$, $c = 3 \times 7$.

23 Number Theory

Definition 23.0.1. Let two integers a and b (with $b \neq 0$). If there exists some integer n such that $a = bn$, we say

- b divides a, and
- a is divisible by b.

We write this as $b \mid a$.

Proposition 23.0.2. For $a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$, then $a \mid (b \pm c)$.

Proof. From our definition, we there exists integers x and y such that $b = ax$ and $c = ay$. Hence,

$$b \pm c = ax \pm ay = a(x \pm y).$$

Since $x \pm y$ is an integer, $a \mid (b \pm c)$. □

Definition 23.0.3 (Congruence Modulo)**.** Let $a, b, n \in \mathbb{Z}$ with $n > 0$. We say that a is **congruent** to b **modulo** n, denoted as

$$a \equiv b \pmod{n},$$

iff n divides $a - b$. Equivalently, $a = b + nk$ for some $k \in ZZ$.

Example 23.0.4. $25 \equiv 7$ modulo 3 since $25 - 7 = 18$ is a multiple of 3.

Proposition 23.0.5 (Congruence is an Equivalence Relation)**.** Let $a, b, n \in \mathbb{Z}$.

- Congruence is reflexive, i.e. $a \equiv a$ modulo n.
- Congruence is symmetric, i.e. if $a \equiv b$ then $b \equiv a$ (modulo n).
- Congruence is transitive, i.e. if $a \equiv b$ and $b \equiv c$, then $a \equiv c$ (all modulo n).

Proof. Exercise. □

Proposition 23.0.6. For all integers a, b, c, d, k, n, with $n > 1$, suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then

- $a \pm c \equiv b \pm d \pmod{n}$.
- $a \cdot c \equiv b \cdot d \pmod{n}$.
- $a + k \equiv b + k \pmod{n}$.
- $ka \equiv kb \pmod{n}$.
- $a^m \equiv b^m \pmod{n}$ for all $m \in \mathbb{Z}^+$.

In other words, congruence modulo preserves addition, subtraction, multiplication, and exponentiation. Take not that congruence modulo does NOT always preserve division. That is, if $c \mid a$ and $d \mid b$, it is not always true that

$$\frac{a}{c} \equiv \frac{b}{d} \pmod{n}.$$

We now state an important result that formalizes our notion of remainders when dividing integers.

Theorem 23.0.7 (Euclid's Division Lemma)**.** Let $n \in \mathbb{Z}^+$. Then for any $m \in \mathbb{Z}$, there exists a unique integer r with $0 \leq r < n$ such that

$$m \equiv r \pmod{n}.$$

Equivalently, there exists an integer q such that

$$m = qn + r.$$

We will prove this statement for $m, n > 0$. We can take $m > n$ since if $0 < m < n$, we can simply take $q = 0$ and $r = m$.

Proof. We prove that such an r exists, and show that it must be unique.

Existence. Let q be the largest number such that $m \geq nq$ and let $r = m - nq \geq 0$. Seeking a contradiction, suppose $r \geq n$, i.e. $r = n + d$ for $d \geq 0$. Then

$$m = nq + r = nq + (n + d) = n(q + 1) + d \geq n(q + 1),$$

contradicting the maximality of q. Hence, $0 \leq r < n$, i.e. r exists.

Uniqueness. Suppose there exist r_1, r_2, with $0 \leq r_1, r_2 < n$ such that

$$m = q_1 n + r_1 = q_2 n + r_2.$$

Then $r_1 = (q_2 - q_1)n + r_2$. Since $0 \leq r_1, r_2 < n$, we must have $r_1 = r_2$. Hence, r must be unique. This concludes the proof. $\square$

EXERCISES

Part VIII

Group A

A1 Equations and Inequalities

Tutorial A1

Problem 1. Determine whether each of the following systems of equations has a unique solution, infinitely many solutions, or no solutions. Find the solutions, where appropriate.

(a) $$\begin{cases} a + 2b - 3c = -5 \\ -2a - 4b - 6c = 10 \\ 3a + 7b - 2c = -13 \end{cases}$$

(b) $$\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$$

(c) $$\begin{cases} x_1 + x_2 = 5 \\ 2x_1 + x_2 + x_3 = 13 \\ 4x_1 + 3x_2 + x_3 = 23 \end{cases}$$

(d) $$\begin{cases} 1/p + 1/q + 1/r = 5 \\ 2/p - 3/q - 4/r = -11 \\ 3/p + 2/q - 1/r = -6 \end{cases}$$

(e) $$\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\gamma = 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2\,, \text{ where } 0 \le \alpha \le 2\pi,\ 0 \le \beta \le 2\pi, \text{ and } 0 \le \gamma < \pi. \\ 6\sin\alpha - 3\cos\beta + \tan\gamma = 9 \end{cases}$$

Solution.

Part (a). Unique solution: $a = -9$, $b = 2$, $c = 0$.

Part (b). No solution.

Part (c). Infinitely many solutions: $x_1 = 8 - t$, $x_2 = t - 3$, $x_3 = t$.

Part (d). Solving, we obtain

$$\frac{1}{p} = 2, \quad \frac{1}{q} = -3, \quad \frac{1}{r} = 6.$$

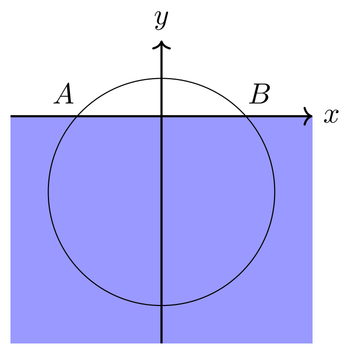
There is hence a unique solution: $p = 1/2$, $q = -1/3$, $r = 1/6$.

Part (e). Solving, we obtain

$$\sin\alpha = 1, \quad \cos\beta = -1, \quad \tan\gamma = 0.$$

There is hence a unique solution: $\alpha = \pi/2$, $\beta = \pi$, $\gamma = 0$.

Problem 2. The following figure shows the circular cross-section of a uniform log floating in a canal.



With respect to the axes shown, the circular outline of the log can be modelled by the equation

$$x^2 + y^2 + ax + by + c = 0.$$

A and B are points on the outline that lie on the water surface. Given that the highest point of the log is 1-cm above the water surface when AB is 40 cm apart horizontally, determine the values of a, b and c by forming a system of linear equations.

Solution. Since $AB = 40$, we have $A(-20, 0)$ and $B(20, 0)$. We also know $(0, 10)$ lies on the circle. Substituting these points into the given equation, we have the following system of equations:

$$\begin{cases} -20a \qquad + c = -400 \\ 20a \qquad + c = -400 \\ \qquad\quad 10b + c = -100 \end{cases}$$

Solving, we obtain $a = 0$, $b = 30$, $c = -400$.

* * * * *

Problem 3. Find the exact solution set of the following inequalities.

(a) $x^2 - 2 \geq 0$

(b) $4x^2 - 12x + 10 > 0$

(c) $x^2 + 4x + 13 < 0$

(d) $x^3 < 6x - x^2$

(e) $x^2(x-1)(x+3) \geq 0$

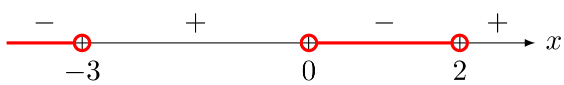
Solution.

Part (a). Note that $x^2 - 2 \geq 0 \implies x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$. The solution set is thus $\{x \in \mathbb{R} : x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}\}$.

Part (b). Completing the square, we see that $4x^2 - 12x + 10 > 0 \implies (x - \frac{3}{2})^2 + \frac{19}{4} > 0$. Since $(x - \frac{3}{2})^2 \geq 0$, all $x \in \mathbb{R}$ satisfy the inequality, whence the solution set is $\mathbb{R}$.

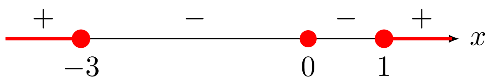
Part (c). Completing the square, we have $x^2 + 4x + 13 < 0 \implies (x+2)^2 + 9 < 0$. Since $(x+2)^2 \geq 0$, there is no solution to the inequality, whence the solution set is $\varnothing$.

Part (d). Note that $x^3 < 6x - x^2 \implies x(x+3)(x-2) < 0$.



The solution set is thus $\{x \in \mathbb{R}\colon x < -3 \text{ or } 0 < x < 2\}$.

Part (e).



The solution set is thus $\{x \in \mathbb{R}\colon x \leq -3 \text{ or } x = 0 \text{ or } x \geq 1\}$.

* * * * *

Problem 4. Find the exact solution set of the following inequalities.

(a) $|3x + 5| < 4$

(b) $|x - 2| < 2x$

Solution.

Part (a). If $3x + 5 < 4$, then $x < -\frac{1}{3}$. If $-(3x+5) < 4$, then $x > -3$. Combining both inequalities, we have $-3 < x < -\frac{1}{3}$. Thus, the solution set is $\left\{x \in \mathbb{R}\colon -3 < x < -\frac{1}{3}\right\}$.

Part (b). If $x - 2 < 2x$, then $x > -2$. If $-(x-2) < 2x$, then $x > \frac{2}{3}$. Combining both inequalities, we have $x > \frac{2}{3}$. Thus, the solution set is $\left\{x \in \mathbb{R}\colon x > \frac{2}{3}\right\}$.

* * * * *

Problem 5. It is given that $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c and d are constants. Given that the curve with equation $y = p(x)$ is symmetrical about the y-axis, and that it passes through the points with coordinates $(1, 2)$ and $(2, 11)$, find the values of a, b, c and d.

Solution. We know that $(1, 2)$ and $(2, 11)$ lie on the curve. Since $y = p(x)$ is symmetrical about the y-axis, we have that $(-1, 2)$ and $(-2, 11)$ also lie on the curve. Substituting these points into $y = p(x)$, we obtain the following system of equations:

$$\begin{cases} a + b + c + d = 1 \\ a - b + c - d = -1 \\ 8a + 4b + 2c + d = -5 \\ 8a - 4b + 2c - d = 5 \end{cases}$$

Solving, we obtain $a = 0$, $b = -2$, $c = 0$, $d = 3$.

* * * * *

Problem 6. Mr Mok invested \$50,000 in three funds A, B and C. Each fund has a different risk level and offers a different rate of return.

In 2016, the rates of return for funds A, B and C were 6%, 8%, and 10% respectively and Mr Mok attained a total return of \$3,700. He invested twice as much money in Fund A as in Fund C. How much did he invest in each of the funds in 2016?

Solution. Let a, b and c be the amount of money Mr Mok invested in Funds A, B and C respectively, in dollars. We thus have the following system of equations.

$$\begin{cases} a + b + c = 50000 \\ \frac{6}{100}a + \frac{8}{100}b + \frac{10}{100}c = 3700 \\ a = 2c \end{cases}$$

Solving, we have $a = 30000$, $b = 5000$ and $c = 15000$. Thus, Mr Mok invested \$30,000, \$5,000 and \$15,000 in Funds A, B and C respectively.

* * * * *

Problem 7. Solve the following inequalities with exact answers.

(a) $2x - 1 \geq \frac{6}{x}$

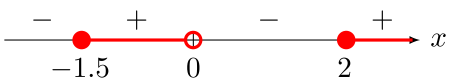
(b) $x - \frac{1}{x} < 1$

(c) $-1 < \frac{2x+3}{x-1} < 1$

Solution.

Part (a). Note that $x \neq 0$.

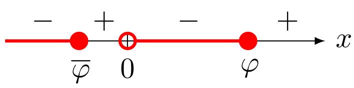
$$2x - 1 \geq \frac{6}{x} \implies x^2(2x-1) \geq 6x \implies x\left(2x^2 - x - 6\right) \geq 0 \implies x(2x+3)(x-2) \geq 0.$$



Thus, $-\frac{3}{2} \leq x < 0$ or $x \geq 2$.

Part (b). Note that $x \neq 0$.

$$x - \frac{1}{x} < 1 \implies x^3 - x < x^2 \implies x\left(x^2 - x - 1\right) < 0 \implies x(x-\varphi)(x-\overline{\varphi}) < 0.$$



Thus, $x \leq \bar{\varphi}$ or $0 < x \leq \varphi$.

Part (c).

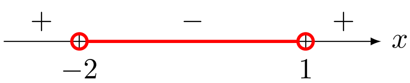
$$-1 < \frac{2x+3}{x-1} < 1 \implies -3 < \frac{5}{x-1} < -1 \implies -\frac{3}{5} < \frac{1}{x-1} < -\frac{1}{5} \implies -4 < x < -\frac{2}{3}.$$

* * * * *

Problem 8. Without using a calculator, solve the inequality $\frac{x^2+x+1}{x^2+x-2} < 0$.

Solution. Observe that $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$. The inequality thus reduces to $\frac{1}{x^2+x-2} < 0$.

$$\frac{1}{x^2+x-2} < 0 \implies x^2 + x - 2 < 0 \implies (x-1)(x+2) < 0.$$



Hence, $-2 < x < 1$.

Problem 9. Solve the following inequalities using a graphical method.

(a) $|3x+1| < (4x+3)^2$

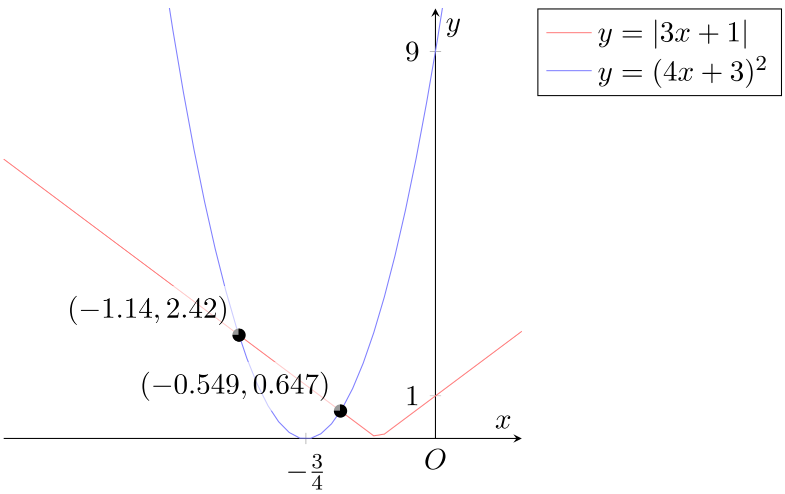
(b) $|3x+1| \geq |2x+7|$

(c) $|x-2| \geq x+|x|$

(d) $5x^2+4x-3 > \ln(x+1)$

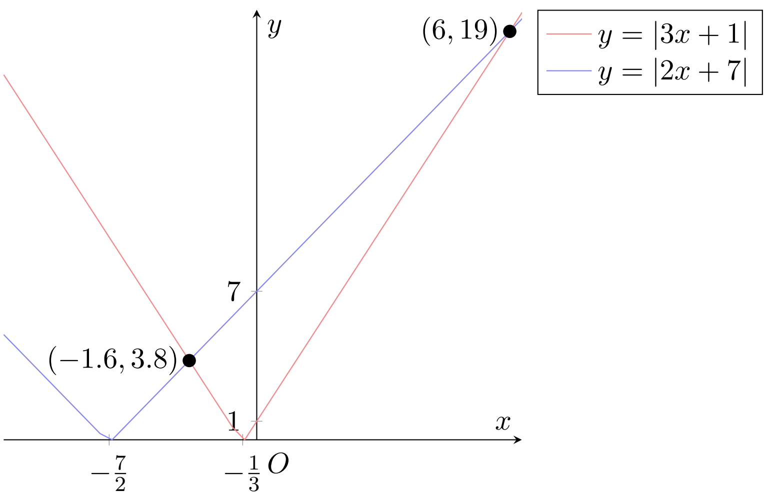
Solution.

Part (a).



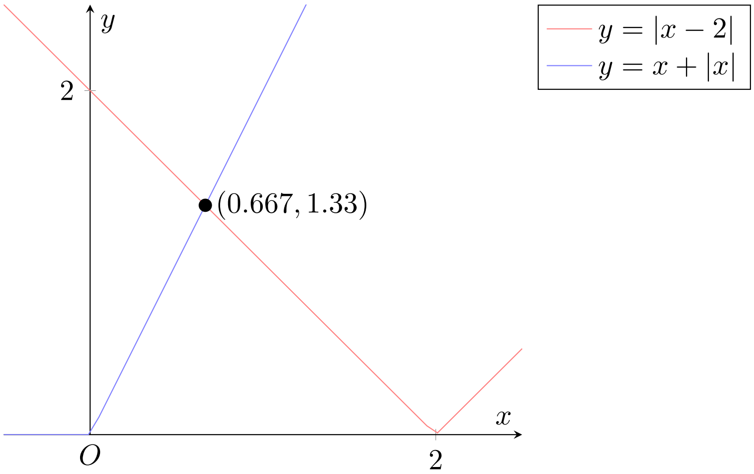
Thus, $x < -1.14$ or $x > -0.549$.

Part (b).



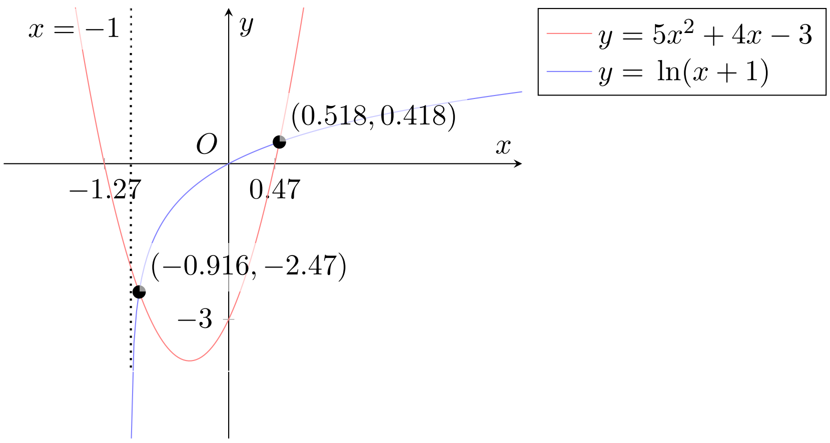
Thus, $x \leq -1.6$ or $x \geq 6$.

Part (c).



Thus, $x \leq 0.667$.

Part (d).

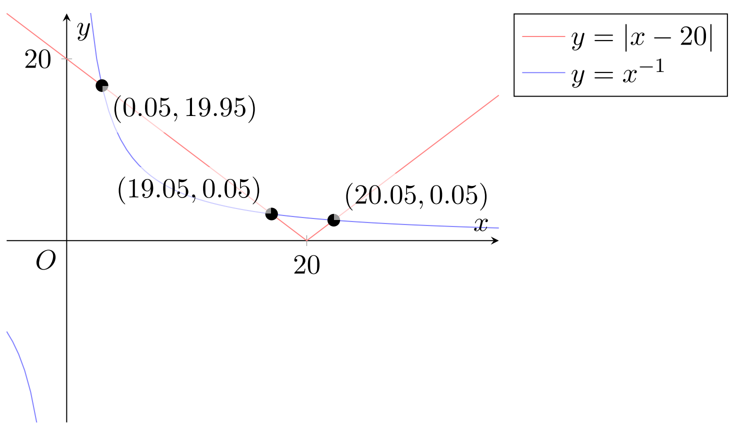


Thus, $-1 < x < -0.916$ or $x > 0.518$.

* * * * *

Problem 10. Sketch the graphs of $y = |x - 20|$ and $y = \frac{1}{x}$ on the same diagram. Hence or otherwise, solve the inequality $|x - 20| < \frac{1}{x}$, leaving your answers correct to 2 decimal places.

Solution.



Thus, $0 < x < 0.05$ or $19.95 < x < 20.05$.

* * * * *

Problem 11. Solve the inequality $\frac{x-9}{x^2-9} \leq 1$. Hence, solve the inequalities

(a) $\frac{|x|-9}{x^2-9} \leq 1$

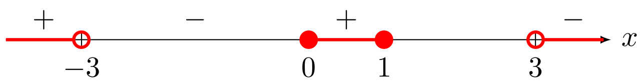
(b) $\frac{x+9}{x^2-9} \geq -1$

Solution. Note that $x^2 - 9 \neq 0 \implies x \neq \pm 3$.

$$\frac{x-9}{x^2-9} \leq 1 \implies (x-9)\left(x^2-9\right) \leq \left(x^2-9\right)^2.$$

Expanding and factoring, we get

$$x^4 - x^3 - 9x^2 + 9x = x(x+3)(x-1)(x-3) \geq 0.$$



Thus, $x < -3$ or $0 \leq x \leq 1$ or $x > 3$.
Part (a). Consider the substitution $x \mapsto |x|$. Then

$$|x| < -3 \text{ or } 0 \leq |x| \leq 1 \text{ or } |x| > 3.$$

This immediately gives us $x < -3$ or $-1 \leq x \leq 1$ or $x > 3$.
Part (b). Consider the substitution $x \mapsto -x$. Then

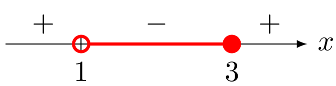
$$-x < -3 \text{ or } 0 \leq -x \leq 1 \text{ or } -x > 3.$$

This immediately gives us $x < -3$ or $-1 \leq x \leq 0$ or $x > 3$.

* * * * *

Problem 12. Solve the inequality $\frac{x-5}{1-x} \geq 1$. Hence, solve $0 < \frac{1-\ln x}{\ln x - 5} \leq 1$.
Solution. Note that $x \neq 1$.

$$\frac{x-5}{1-x} \geq 1 \implies (x-5)(1-x) \geq (1-x)^2 \implies 2x^2 - 8x + 6 \leq 0 \implies 2(x-1)(x-3) \leq 0.$$



Thus, $1 < x \leq 3$.

Consider the substitution $x \mapsto \ln x$. Taking reciprocals, we have our desired inequality $0 < \frac{1-\ln x}{\ln x - 5} \leq 1$. Hence,

$$1 < \ln x \leq 3 \implies e < x \leq e^3.$$

* * * * *

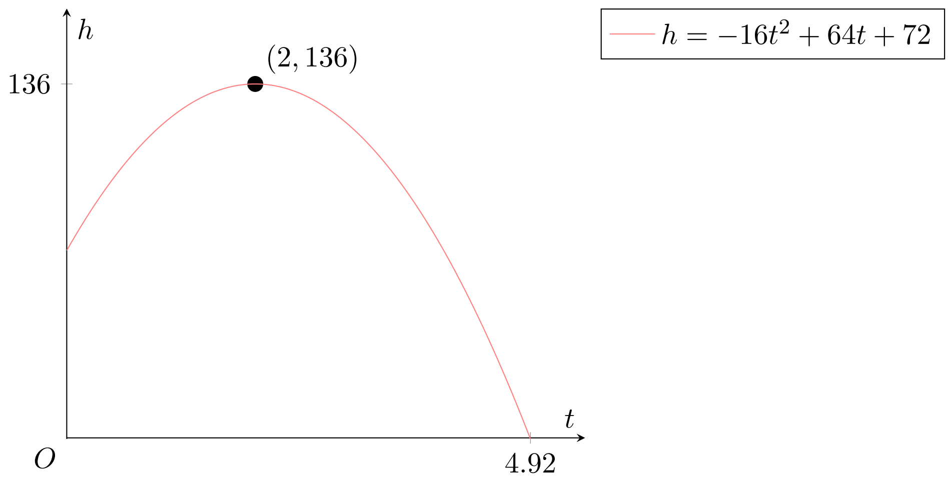
Problem 13. A small rocket is launched from a height of 72 m from the ground. The height of the rocket in metres, h, is represented by the equation $h = -16t^2 + 64t + 72$, where t is the time in seconds after the launch.

(a) Sketch the graph of h against t.

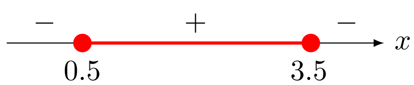
(b) Determine the number of seconds that the rocket will remain at or above 100 m from the ground.

Solution.

Part (a).



Part (b). Note that $-16t^2 + 64t + 72 \geq 100 \implies -4(2t-1)(2t-7) \geq 0$.

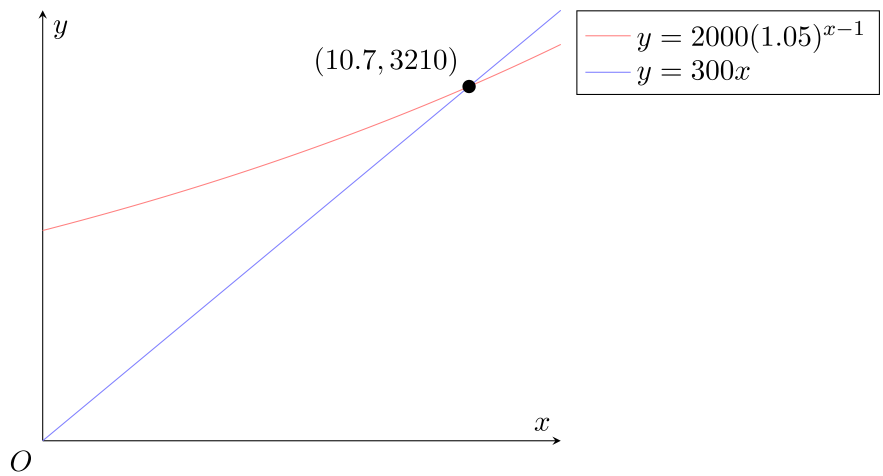


Thus, the rocket will remain at or above 100 m from the ground for 3 seconds.

* * * * *

Problem 14. Xinxin, a new graduate, starts work at a company with an initial monthly pay of \$2,000. For every subsequent quarter that she works, she will get a pay increase of 5%, leading to a new monthly pay of $2000(1.05)^{n-1}$ dollars in the nth quarter, where n is a positive integer. She also gives a regular donation of \$$300n$ in the nth quarter that she works. However, she will stop the donation when her monthly pay falls below the donation amount. At which quarter will this first happen?

Solution. Consider the curves $y = 2000(1.05)^{x-1}$ and $y = 300x$.



Hence, Xinxin will stop donating in the 11th quarter.

Self-Practice A1

Problem 1. On joining ABC International School, each of the 200 students is placed in exactly one of the four performing arts groups: Choir, Chinese Orchestra, Concert Band and Dance. The following table shows some information about each of the performing arts groups:

Performing Arts Group	Choir	Chinese Orchestra	Concert Band	Dance
Membership Fee (per student per month)	\$15	\$20	\$20	\$18
Instructor Fee (per student per month)	\$50	\$60	\$75	\$40
Costume Fee (one-time payment per student)	\$45	?	\$40	\$60
No. of Training Hours	5	6	8	7

In a typical month, the school collects a total of \$3,721 for membership fee from the students, and pays the instructors a total sum of \$11,830 (assuming that this sum of money is fully paid by the students). As for the training in a typical week, students from Chinese Orchestra and Concert Band spend in total 431 hours more than their peers in Choir and Dance. Find the enrolment in each of the performing arts groups.

Hence, find the costume fee paid by each student from Chinese Orchestra if a vendor charges a total of \$9,440 for all the costumes for the four performing arts groups.

Solution. Let a, b, c, d be the number of students in Choir, Chinese Orchestra, Concert Band and Dance respectively. From the given information, we have the following equations:

$$\begin{cases} a + b + c + d = 200 \\ 15a + 20b + 20c + 18d = 3721 \\ 50a + 60b + 75c + 40d = 11830 \\ -5a + 6b + 8c - 7d = 431 \end{cases}$$

Using G.C., we obtain the unique solution

$$a = 43, \quad b = 65, \quad c = 60, \quad d = 32.$$

Let the Chinese Orchestra's custom fee (per student) be x. From the given information, we have the following equation:

$$45a + xb + 40c + 60d = 9440.$$

Hence,

$$x = \frac{9440 - 45a - 40c - 60d}{b} = 49.$$

Thus, the costume fee paid by each student from Chinese Orchestra is \$49.

* * * * *

Problem 2. Solve the inequality $(x+2)^2\left(x^2+2x-8\right) \geq 0$.

Solution. Since $(x+2)^2 \geq 0$, we can remove it from the inequality, keeping in mind that $x = -2$ is a solution. We are hence left with

$$x^2 + 2x - 8 = (x+4)(x-2) \geq 0.$$

Since this quadratic is concave up, we clearly have $x \leq -4$ or $x \geq 2$. Altogether, we have

$$x \leq -4 \text{ or } x = -2 \text{ or } x \geq 2.$$

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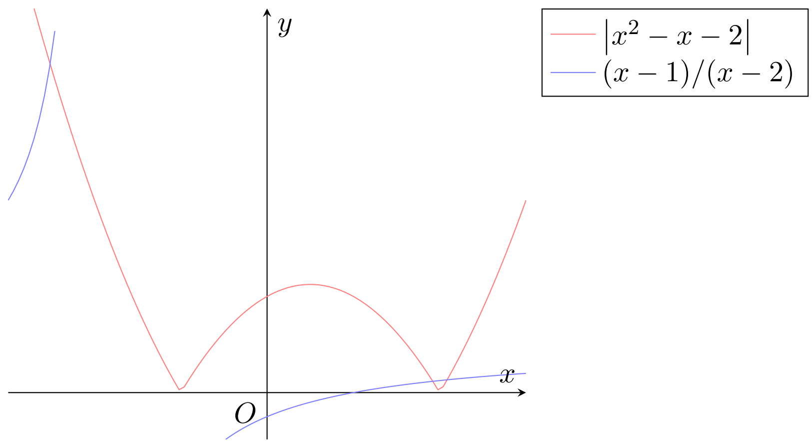
Problem 3. By using a graphical method, solve the inequality $\left|x^2 - x - 2\right| \geq \frac{x-1}{x+2}$.

Solution.



From the graph, the x-coordinates of the intersection points are -2.51, 1.92 and 2.09. Hence,

$$x \leq -2.51 \quad \text{and} \quad -2 \leq x \leq 1.92 \quad \text{and} \quad x \geq 2.09.$$

* * * * *

Problem 4. Show that $x^2 + 2x + 3$ is always positive for all real values of x. Hence, solve the inequality $\frac{x^2+2x+3}{3+2x-x^2} \leq 0$. Deduce the solution set of the inequality $\frac{x^2+2|x|+3}{3+2|x|-x^2} \leq 0$.

Solution. Note that the discriminant of $x^2 + 2x + 3 = 0$ is $\Delta = 2^2 - 4(1)(3) = -8 < 0$. Since the y-intercept is positive $(3 > 0)$, it follows that $x^2 + 2x + 3$ is always positive for real x.

Consider the inequality $\frac{x^2+2x+3}{3+2x-x^2} \leq 0$. Since $x^2 + 2x + 3$ is always positive, it suffices to solve $3 + 2x - x^2 \leq 0$. Observe that the roots of $3 + 2x - x^2 = 0$ are $x = 3$ and $x = -1$. Since $3 + 2x - x^2$ is concave down, we have

$$x \leq -1 \quad \text{or} \quad x \geq 3.$$

Replacing x with $|x|$, we get $|x| \leq -1$ (no solutions) and $|x| \geq 3$, whence $x \leq -3$ or $x \geq 3$. The solution set is thus

$$\{x \in \mathbb{R} : x \leq -3 \text{ or } x \geq 3\}.$$

Problem 5. Without use of a graphing calculator, solve the inequality $\frac{2x^2-7x+6}{x^2-x-2} \geq 1$. Deduce the range of values of x such that

(a) $\frac{2(\ln x)^2-7\ln x+6}{(\ln x)^2-\ln x-2} > 1$

(b) $\frac{2-7x+6x^2}{1-x-2x^2} \geq 1$

Solution. Moving all terms to one side, we get

$$\frac{2x^2-7x+6}{x^2-x-2} \geq 1 \implies \frac{x^2-6x+8}{x^2-x-2} \geq 0.$$

Note that x^2-6x+8 factors as $(x-2)(x-4)$ while x^2-x-2 factors as $(x-2)(x+1)$. Hence,

$$\frac{x-4}{x+1} \geq 0 \implies (x-4)(x+1) \geq 0.$$

Thus, we clearly have

$$x < -1 \quad \text{or} \quad x \geq 4.$$

Note that $x \neq -1$ since $x^2-x-2 \neq 0$.

Part (a). Replacing x with $\ln x$, we get

$$\ln x < -1 \quad \text{or} \quad \ln x \geq 4,$$

whence

$$0 \leq x < \mathrm{e}^{-1} \quad \text{or} \quad x \geq \mathrm{e}^4.$$

Part (b). Replacing x with $1/x$, we get

$$\frac{1}{x} < -1 \quad \text{or} \quad \frac{1}{x} \geq 4.$$

Hence,

$$-1 < x < 0 \quad \text{or} \quad 0 < x \leq \frac{1}{4}.$$

Note that $x = 0$ also satisfies the inequality $(2 \geq 1)$. Hence,

$$-1 < x \leq \frac{1}{4}.$$

* * * * *

Problem 6. It is given that $y = \frac{x^2+x+1}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. Without using a calculator, find the set of values that y can take.

Solution. Clearing denominators, we have

$$y(x-1) = x^2+x+1 \implies x^2+(1-y)x+(1+y) = 0.$$

Since we are interested in the set of values that y can take, we want this quadratic to have roots. Hence, the discriminant Δ should be non-negative:

$$\Delta = (1-y)^2 - 4(1+y) = y^2-6y-3 \geq 0.$$

Completing the square,

$$(y-3)^2 \geq 12 \implies |y-3| \geq \sqrt{12} = 2\sqrt{3}.$$

Hence,

$$y \leq 3 - 2\sqrt{3} \text{ or } y \geq 3 + 2\sqrt{3},$$

whence the solution set is

$$\left\{ y \in \mathbb{R} : y \leq 3 - 2\sqrt{3} \text{ or } y \geq 3 + 2\sqrt{3} \right\}.$$

* * * * *

Problem 7 (🌶). Solve for x, in terms of a, the inequality

$$\left| x^2 - 3ax + 2a^2 \right| < \left| x^2 + 3ax - a^2 \right|,$$

where $x \in \mathbb{R}$, $a \neq 0$.

Solution. Squaring both sides, we get

$$\left(x^2 - 3ax + 2a^2 \right)^2 < \left(x^2 + 3ax - a^2 \right)^2.$$

Collecting terms on one side,

$$\left(x^2 + 3ax - a^2 \right)^2 - \left(x^2 - 3ax + 2a^2 \right)^2 = 3a\,(2x - a)\left(2x^2 + a^2 \right) > 0.$$

Clearly, $2x^2 + a^2 > 0$ for all x. We are hence left with $a(2x - a) > 0$.

Case 1. If $a > 0$, then $2x - a > 0$, whence $x > a/2$.

Case 2. If $a < 0$, then $2x - a < 0$, whence $x < a/2$.

* * * * *

Problem 8 (🌶). Find constants a, b, c and d such that $1 + 2^3 + 3^3 + \cdots + n^3 = an^4 + bn^3 + cn^2 + dn$.

Solution 1. Substituting $n = 1, 2, 3, 4$ into the equation, we get the system

$$\begin{cases} a + b + c + d = 1 \\ 16a + 8b + 4c + 2d = 9 \\ 81a + 27b + 9c + 3d = 36 \\ 256a + 64b + 16c + 4d = 100 \end{cases}.$$

Solving, we have

$$a = \frac{1}{4}, \quad b = \frac{1}{2}, \quad c = \frac{1}{4}, \quad d = 0.$$

Solution 2. Recall that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Now observe that

$$(k+1)^3 - 1 = \sum_{k=1}^{n} \left[(k+1)^3 - k^3 \right] = \sum_{k=1}^{n} \left(3k^2 + 3k + 1 \right).$$

Rearranging, we obtain

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Similarly, we have

$$(k+1)^4 - 1 = \sum_{k=1}^{n} \left[(k+1)^4 - k^4\right] = \sum_{k=1}^{n} \left(4k^3 + 6k^2 + 4k + 1\right),$$

whence we obtain, upon rearranging,

$$\sum_{k=1}^{n} k^3 = \frac{n^4 + 2n^3 + n^2}{4}.$$

Comparing coefficients, we have

$$a = \frac{1}{4}, \quad b = \frac{1}{2}, \quad c = \frac{1}{4}, \quad d = 0.$$

* * * * *

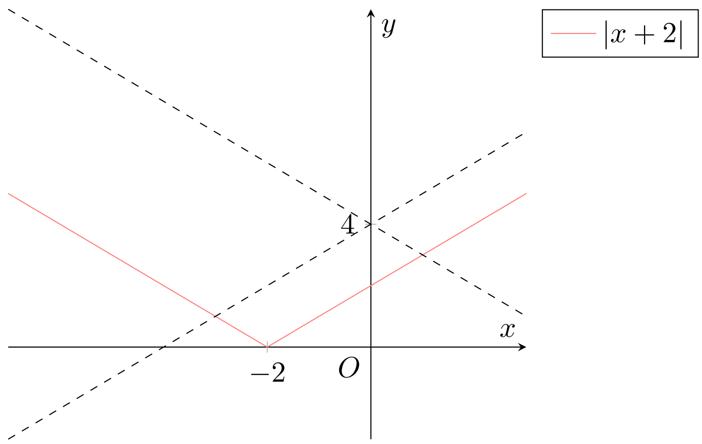
Problem 9 (🌶).

(a) By means of a sketch, or otherwise, state the range of values of a for which the equation $|x+2| = ax + 4$ has two distinct real roots.

(b) Solve the inequality $|x+2| < ax + 4$.

Solution.

Part (a).



Consider the figure above. Clearly, for 2 distinct roots (i.e. 2 distinct intersection points), we need $-1 < a < 1$.

Part (b). Note that the x-coordinate of the point of intersection between $y = ax + 4$ and $y = x + 2$ is:

$$x + 2 = ax + 4 \implies x = \frac{-2}{a-1}.$$

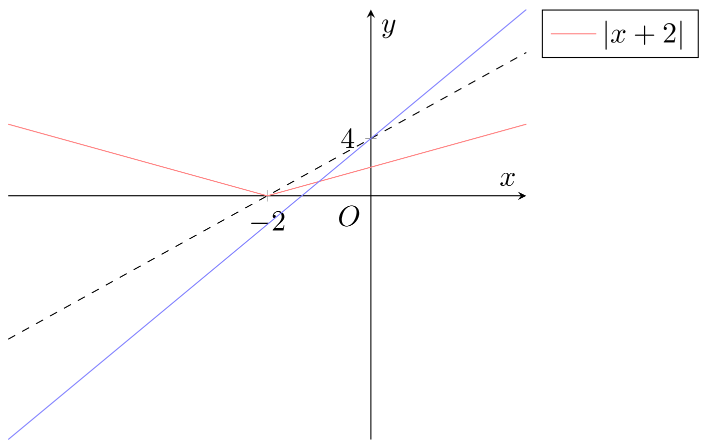
Similarly, the x-coordinate of the point of intersection between $y = ax+4$ and $y = -(x+2)$ is:

$$x + 2 = ax + 4 \implies x = \frac{-6}{a+1}.$$

Now consider the inequality $|x+2| < ax + 4$.

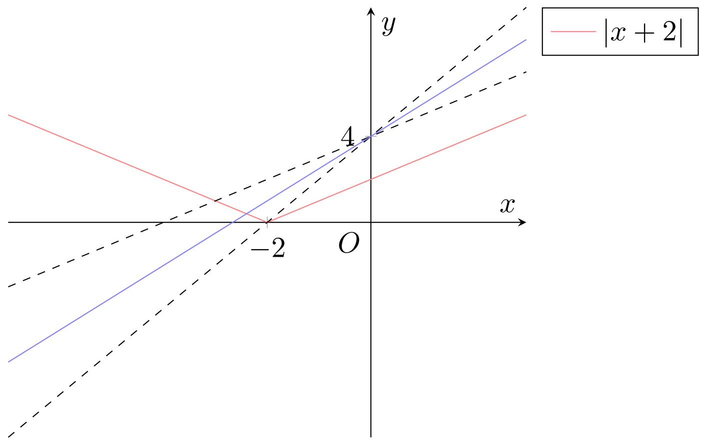
Case 1: $a > 2$. $y = ax + 4$ only intersects the line $y = x + 2$. Hence,

$$x > \frac{-2}{a-1}.$$



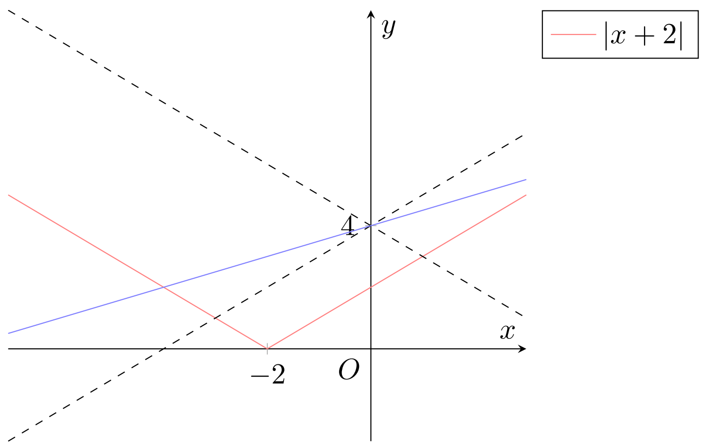
Case 2: $1 \le a \le 2$. $y = ax + 4$ only intersects the line $y = -(x+2)$. Hence,

$$x \ge \frac{-6}{a-1}.$$



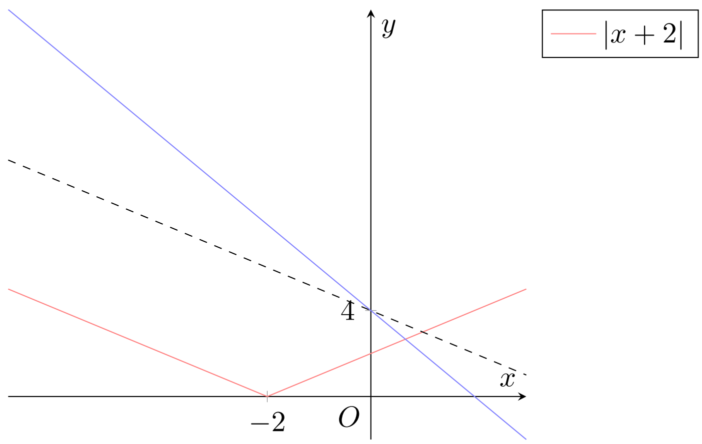
Case 3: $-1 < a < 1$. $y = ax + 4$ intersects both $y = x + 2$ and $y = -(x+2)$. Hence,

$$\frac{-6}{a-1} < x < \frac{-2}{a-1}.$$



Case 4: $a \le -1$. $y = ax + 4$ only intersects the line $y = x + 2$. Hence,

$$x \le \frac{-2}{a-1}.$$


$|x+2|$
y
x
4
-2
O

Assignment A1

Problem 1. A traveller just returned from Germany, France and Spain. The amount (in dollars) that he spent each day on housing, food and incidental expenses in each country are shown in the table below.

Country	Housing	Food	Incidental Expenses
Germany	28	30	14
France	23	25	8
Spain	19	22	12

The traveller's records of the trip indicate a total of \$191 spent for housing, \$430 for food and \$180 for incidental expenses. Calculate the number of days the traveller spent in each country.

He did his account again and the amount spent on food is \$337. Is this record correct? Why?

Solution. Let g, f and s represent the number of days the traveller spent in Germany, France and Spain respectively. From the table, we obtain the following system of equations:

$$\begin{cases} 23f + 28g + 19s = 391 \\ 25f + 30g + 22s = 430 \\ 8f + 14g + 12s = 180 \end{cases}$$

This gives the unique solution $g = 4$, $f = 8$ and $s = 5$. The traveller thus spent 4 days in Germany, 8 days in France and 5 days in Spain.

Consider the scenario where the amount spent on food is \$337.

$$\begin{cases} 23f + 28g + 19s = 391 \\ 25f + 30g + 22s = 337 \\ 8f + 14g + 12s = 180 \end{cases}$$

This gives the unique solution $g = 66$, $f = -27$ and $s = -44$. The record is hence incorrect as f and s must be positive.

* * * * *

Problem 2.

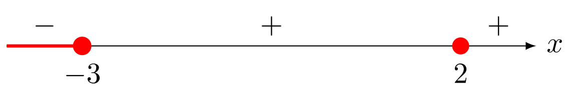
(a) Solve algebraically $x^2 - 9 \geq (x+3)\left(x^2 - 3x + 1\right)$.

(b) Solve algebraically $\frac{7-2x}{3-x^2} \leq 1$.

Solution.

Part (a).

$$\begin{aligned} & x^2 - 9 \geq (x+3)\left(x^2 - 3x + 1\right) \\ \implies & (x+3)(x-3) \geq (x+3)\left(x^2 - 3x + 1\right) \\ \implies & (x+3)\left(x^2 - 4x + 4\right) \leq 0 \\ \implies & (x+3)(x-2)^2 \leq 0 \end{aligned}$$



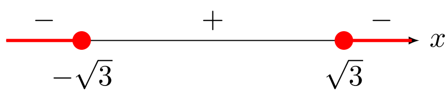
Thus, $x \leq -3$ or $x = 2$.

Part (b). Note that $3 - x^2 \neq 0 \implies x \neq \pm\sqrt{3}$.

$$\begin{aligned} \frac{7-2x}{3-x^2} &\leq 1 \\ \implies \frac{7-2x}{3-x^2} - \frac{3-x^2}{3-x^2} &\leq 0 \\ \implies \frac{x^2-2x+4}{3-x^2} &\leq 0 \end{aligned}$$

Observe that $x^2 - 2x + 4 = (x-1)^2 + 3 > 0$. Dividing through by $x^2 - 2x + 4$, we obtain

$$\begin{aligned} \frac{1}{3-x^2} &\leq 0 \\ \implies 3 - x^2 &\leq 0 \end{aligned}$$



Thus, $x < -\sqrt{3}$ or $x > \sqrt{3}$.

* * * * *

Problem 3.

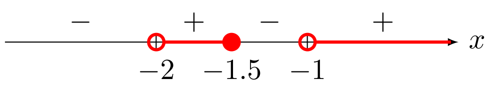
(a) Without using a calculator, solve the inequality $\frac{3x+4}{x^2+3x+2} \geq \frac{1}{x+2}$.

(b) Hence, deduce the set of values of x that satisfies $\frac{3|x|+4}{x^2+3|x|+2} \geq \frac{1}{|x|+2}$.

Solution.

Part (a). Note that $x^2 + 3x + 2 \neq 0$ and $x + 2 \neq 0$, whence $x \neq -1, -2$.

$$\begin{aligned} \frac{3x+4}{x^2+3x+2} &\geq \frac{1}{x+2} \\ \implies \frac{3x+4}{(x+2)(x+1)} &\geq \frac{1}{x+2} \\ \implies (3x+4)(x+2)(x+1) &\geq (x+2)(x+1)^2 \\ \implies (x+2)(x+1)(2x+3) &\geq 0 \end{aligned}$$



Thus, $-2 < x \leq -\frac{3}{2}$ or $x > -1$.

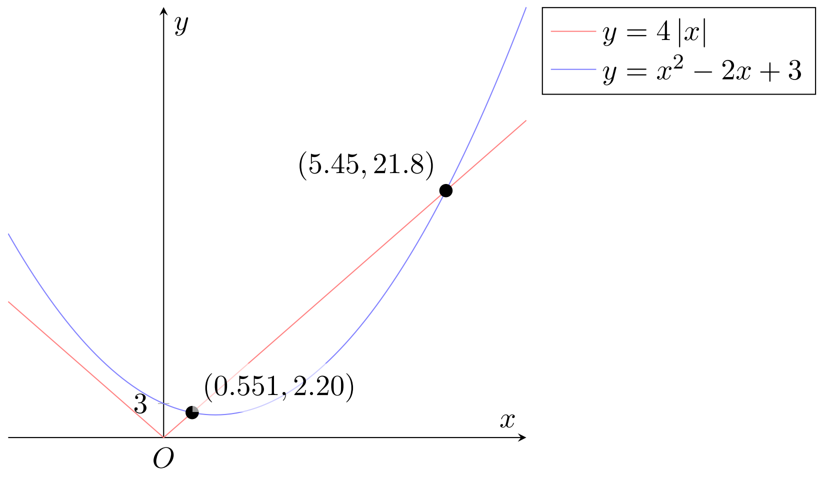
Part (b). Observe that $|x|^2 = x^2$. Hence, with the map $x \mapsto |x|$, we obtain

$$-2 < |x| \leq -\frac{3}{2} \text{ or } |x| > -1.$$

Since $|x| \geq 0$, we have that $|x| > -1$ is satisfied for all real x. Hence, the solution set is $\mathbb{R}$.

Problem 4. On the same diagram, sketch the graphs of $y = 4|x|$ and $y = x^2 - 2x + 3$. Hence or otherwise, solve the inequality $4|x| \geq x^2 - 2x + 3$.

Solution.



From the graph, we see that $0.551 \leq x \leq 5.45$.

A2 Numerical Methods of Finding Roots

Tutorial A2

Problem 1. Without using a graphing calculator, show that the equation $x^3+2x^2-2=0$ has exactly one positive root.

This root is denoted by α and is to be found using two different iterative methods, starting with the same initial approximation in each case.

(a) Show that α is a root of the equation $x = \sqrt{\frac{2}{x+2}}$, and use the iterative formula $x_{n+1} = \sqrt{\frac{2}{x_n+2}}$, with $x_1 = 1$, to find α correct to 2 significant figures.

(b) Use the Newton-Raphson method, with $x_1 = 1$, to find α correct to 3 significant figures.

Solution. Let $f(x) = x^3+2x^2-2$. Observe that for all $x > 0$, we have $f'(x) = 3x^2+4x > 0$. Hence, $f(x)$ is strictly increasing on $(0, \infty)$. Since $f(0)f(1) = (-2)(1) < 0$, it follows that $f(x)$ has exactly one positive root.

Part (a). We know $f(\alpha) = 0$. Hence,

$$\alpha^3 + 2\alpha^2 - 2 = 0 \implies \alpha^2(\alpha+2) = 2 \implies \alpha^2 = \frac{2}{\alpha+2} \implies \alpha = \sqrt{\frac{2}{\alpha+2}}.$$

Note that we reject the negative branch since $\alpha > 0$. We hence see that α is a root of the equation $x = \sqrt{\frac{2}{x+2}}$. Using the iterative formula $x_{n+1} = \sqrt{\frac{2}{x_n+2}}$ with $x_1 = 1$, we have

n	x_n
1	1
2	0.81650
3	0.84268
4	0.83879

Hence, $\alpha = 0.84$ (2 s.f.).

Part (b). Using the Newton-Raphson method ($x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$) with $x_1 = 1$, we have

n	x_n
1	1
2	0.857143
3	0.839545
4	0.839287
5	0.839287

Hence, $\alpha = 0.839$ (3 s.f.).

Problem 2.

(a) Show that the tangent at the point $(\mathrm{e}, 1)$ to the graph $y = \ln x$ passes through the origin, and deduce that the line $y = mx$ cuts the graph $y = \ln x$ in two points provided that $0 < m < 1/\mathrm{e}$.

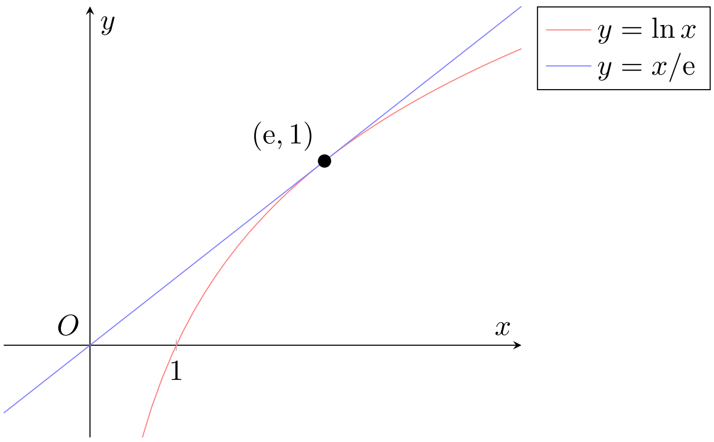
(b) For each root of the equation $\ln x = x/3$, find an integer n such that the interval $n < x < n + 1$ contains the root. Using linear interpolation, based on $x = n$ and $x = n + 1$, find a first approximation to the smaller root, giving your answer to 1 decimal place. Using your first approximation, obtain, by the Newton-Raphson method, a second approximation to the smaller root, giving your answer to 2 decimal places.

Solution.

Part (a). Note that the derivative of $y = \ln x$ at $x = \mathrm{e}$ is $1/\mathrm{e}$. Using the point slope formula, we see that the equation of the tangent at the point $(\mathrm{e}, 1)$ is given by

$$y - 1 = \frac{x - \mathrm{e}}{\mathrm{e}} \implies y = \frac{x}{\mathrm{e}}.$$

Since $x = 0$, $y = 0$ is clearly a solution, the tangent passes through the origin. From the graph below, it is clear that for $y = mx$ to intersect $y = \ln x$ twice, we must have $0 < m < 1/\mathrm{e}$.



Part (b). Consider $f(x) = x/3 - \ln x$. Let α and β be the smaller and larger root to $f(x) = 0$ respectively. Observe that $f(1)f(2) = (1)(-0.03) < 0$ and $f(4)f(5) = (-0.05)(0.06) < 0$. Thus, for the smaller root α, $n = 1$, while for the larger root β, $n = 4$.

Let x_1 be the first approximation to α. Using linear interpolation, we have

$$x_1 = \frac{f(2) - 2f(1)}{f(2) - f(1)} = 1.9 \text{ (1 d.p.)}$$

Note that $f'(x) = 1/3 - 1/x$. Using the Newton-Raphson method ($x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$), we have

n	x_n
1	1.9
2	1.85585
3	1.85718

Hence, $\alpha = 1.86$ (2 d.p.).

Problem 3. Find the exact coordinates of the turning points on the graph of $y = f(x)$ where $f(x) = x^3 - x^2 - x - 1$. Deduce that the equation $f(x) = 0$ has only one real root α, and prove that α lies between 1 and 2. Use the Newton-Raphson method applied to the equation $f(x) = 0$ to find a second approximation x_2 to α, taking x_1, the first approximation, to be 2. With reference to a graph of $y = f(x)$, explain why all further approximations to α by this process are always larger than α.

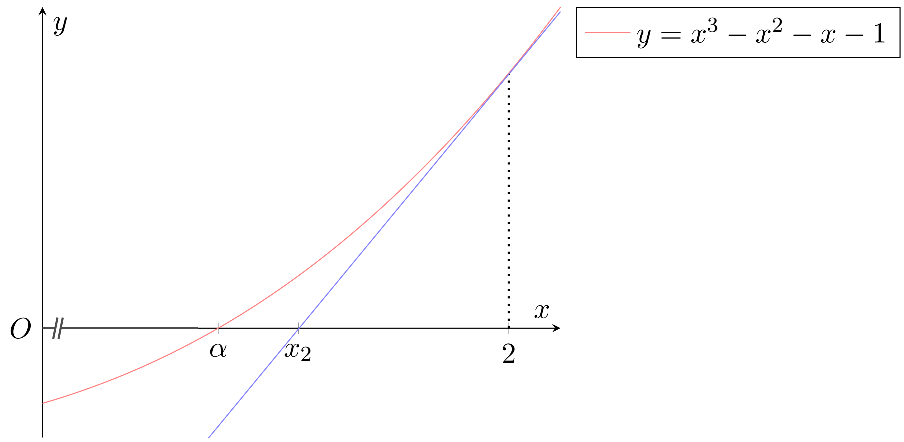
Solution. For turning points, $f'(x) = 0$.

$$f'(x) = 0 \implies 3x^2 - 2x - 1 = 0 \implies (3x+1)(x-1) = 0.$$

Hence, $x = -1/3$ or $x = 1$. When $x = -1/3$, we have $y = -0.815$, giving the coordinate $(-1/3, -0.815)$. When $x = 1$, we have $y = -2$, giving the coordinate $(1, -2)$.

Observe that $f(x)$ is strictly increasing for all $x > 1$. Further, since both turning points have a negative y-coordinate, it follows that $y < 0$ for all $x \leq 1$. Since $f(1)f(2) = (-2)(1) < 0$, the equation $f(x) = 0$ has only one real root.

Using the Newton-Raphson method with $x_1 = 2$, we have $x_2 = x_1 - f(x_1)/f'(x_1) = 13/7$.



Since x_2 lies on the right of α, the Newton-Raphson method gives an over-estimation given an initial approximation of 2. Thus, all further approximations to α will also be larger than α.

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Problem 4. A curve C has equation $y = x^5 + 50x$. Find the least value of $\mathrm{d}y/\mathrm{d}x$ and hence give a reason why the equation $x^5 + 50x = 10^5$ has exactly one real root. Use the Newton-Raphson method, with a suitable first approximation, to find, correct to 4 decimal places, the root of the equation $x^5 + 50x = 10^5$. You should demonstrate that your answer has the required accuracy.

Solution. Since $y = x^5 + 50x$, we have $\mathrm{d}y/\mathrm{d}x = 5x^4 + 50$. Since $x^4 \geq 0$ for all real x, the minimum value of $\mathrm{d}y/\mathrm{d}x$ is 50.

Let $f(x) = x^5 + 50x$. Since $\min \mathrm{d}f/\mathrm{d}x = 50 > 0$, it follows that $f(x)$ is strictly increasing. Hence, $f(x)$ will intersect only once with the line $y = 10^5$, whence the equation $x^5 + 50x = 10^5$ has exactly one real root.

Observe that $f(9)f(10) = (-40901)(50) < 0$. Thus, there must be a root in the interval $(9, 10)$. We now use the Newton-Raphson method ($x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$) with $x_1 = 9$ as the first approximation.

n	x_n
1	9
2	10.2178921
3	10.0017491
4	9.9901221
5	9.9899912
6	9.9899900

Thus, the root is 9.9900 (4 d.p.).

Observe that $f(9.98995)f(9.99005) = (-2.00)(3.00) < 0$. Hence, the root lies in the interval $(9.98995, 9.99005)$ whence the calculated root has the required accuracy.

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Problem 5.

(a) A function f is such that $f(4) = 1.158$ and $f(5) = -3.381$, correct to 3 decimal places in each case. Assuming that there is a value of x between 4 and 5 for which $f(x) = 0$, use linear interpolation to estimate this value.

For the case when $f(x) = \tan x$, and x is measured in radians, the value of $f(4)$ and $f(5)$ are as given above. Explain, with the aid of a sketch, why linear interpolation using these values does not give an approximation to a solution of the equation $\tan x = 0$.

(b) Show, by means of a graphical argument or otherwise, that the equation $\ln(x-1) = -2x$ has exactly one real root, and show that this root lies between 1 and 2.

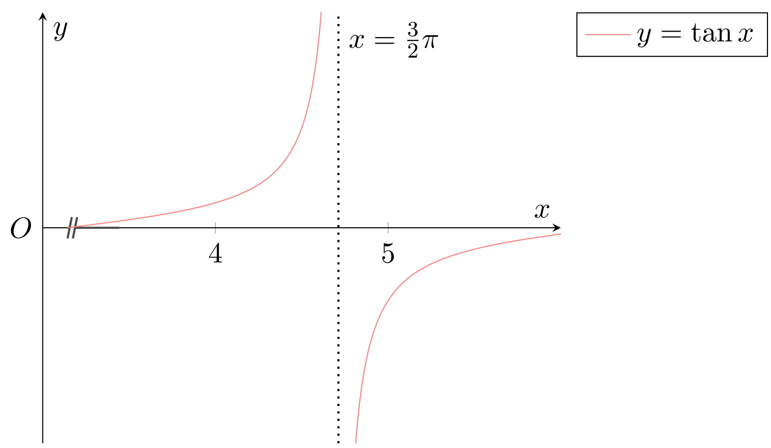
The equation may be written in the form $\ln(x-1) + 2x = 0$. Show that neither $x = 1$ nor $x = 2$ is a suitable initial value for the Newton-Raphson method in this case.

The equation may also be written in the form $x - 1 - \mathrm{e}^{-2x} = 0$. For this form, use two applications of the Newton-Raphson method, starting with $x = 1$, to obtain an approximation to the root, giving 3 decimal places in your answer.

Solution.

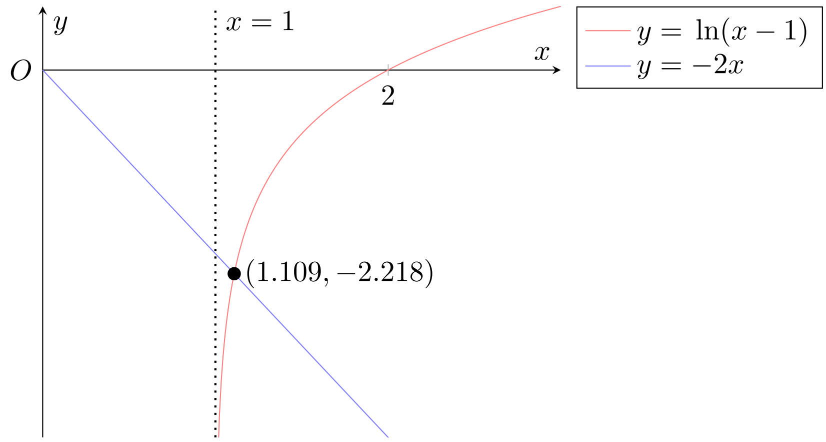
Part (a). Let the root of $f(x) = 0$ be α. Using linear interpolation on the interval $[4, 5]$, we have

$$\alpha = \frac{4f(5) - 5f(4)}{f(5) - f(4)} = 4.255 \text{ (3 d.p.)}.$$



Since $\tan x$ has a vertical asymptote at $x = 3\pi/2$, it is not continuous on $[4, 5]$. Thus, linear interpolation diverges when applied to the equation $\tan x = 0$.

Part (b).



Since there is only one intersection between the graphs $y = \ln(x-1)$ and $y = -2x$, there is only one real root to the equation $\ln(x-1) = -2x$. Furthermore, since $y = -2x$ is negative for all $x > 0$ and $y = \ln(x-1)$ is negative only when $1 < x < 2$, it follows that the root must lie between 1 and 2.

Let $f(x) = \ln(x-1) + 2x$. Then $f'(x) = \frac{1}{x-1} + 2$. Note that the Newton-Raphson method is given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Since $f'(1)$ is undefined, an initial approximation of $x_1 = 1$ cannot be used for the Newton-Raphson method, which requires a division by $f'(1)$.

Using the Newton-Raphson method with the initial approximation $x_2 = 2$, we see that $x_2 = 1$. Once again, because $f'(1)$ is undefined, $x_1 = 2$ is also not a suitable initial value.

Let $g(x) = x - 1 - e^{-2x}$. Then $g'(x) = 1 + 2e^{-2x}$. Using the Newton-Raphson method with the initial approximation $x_1 = 1$, we have

n	x_n
1	1
2	1.106507
3	1.108857

Hence, $x = 1.109$ (3 d.p.).

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Problem 6. The equation $x = 3\ln x$ has two roots α and β, where $1 < \alpha < 2$ and $4 < \beta < 5$. Using the iterative formula $x_{n+1} = F(x_n)$, where $F(x) = 3\ln x$, and starting with $x_0 = 4.5$, find the value of β correct to 3 significant figures. Find a suitable $F(x)$ for computing α.

Solution. Using the iterative formula $x_{n+1} = F(x_n)$, we have

n	x_n	n	x_n
0	4.5	5	4.53175
1	4.51223	6	4.53333
2	4.52038	7	4.53437
3	4.52579	8	4.53506
4	4.52937	9	4.53551

Hence, $\beta = 4.54$ (3 s.f.).

Note that $x = 3\ln x \implies x = \mathrm{e}^{x/3}$. Observe that $\mathrm{d}(\mathrm{e}^{x/3})/\mathrm{d}x = \frac{1}{3}\mathrm{e}^{x/3}$, which is between -1 and 1 for all $1 < x < 2$. Thus, the iterative formula $x_{n+1} = F(x_n)$ will converge, whence $F(x) = \mathrm{e}^{x/3}$ is suitable for computing α.

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Problem 7. Show that the cubic equation $x^3 + 3x - 15 = 0$ has only one real root. This root is near $x = 2$. The cubic equation can be written in any one of the forms below:

(a) $x = \frac{1}{3}(15 - x^3)$

(b) $x = \frac{15}{x^2+3}$

(c) $x = (15 - 3x)^{1/3}$

Determine which of these forms would be suitable for the use of the iterative formula $x_{r+1} = F(x_r)$, where $r = 1, 2, 3, \ldots$.

Hence, find the root correct to 3 decimal places.

Solution. Let $f(x) = x^3 + 3x - 15$. Then $f'(x) = 3x^2 + 3 > 0$ for all real x. Hence, f is strictly increasing. Since f is continuous, $f(x) = 0$ has only one real root.

Part (a). Let $g_1(x) = \frac{1}{3}(15 - x^3)$. Then $g_1'(x) = -x^2$. For values of x near 2, $|g_1'(x)| > 1$. Hence, the iterative formula $x_{n+1} = g_1(x_n)$ will diverge and $g_1(x)$ is unsuitable.

Part (b). Let $g_2(x) = \frac{15}{x^2+3}$. Then $g_2'(x) = \frac{-30x}{(x^2+3)^2}$. For values of x near 2, $|g_2'(x)| > 1$. Hence, the iterative formula $x_{n+1} = g_2(x_n)$ will diverge and $g_2(x)$ is unsuitable.

Part (c). Let $g_3(x) = (15 - 3x)^{1/3}$. Then $g_3'(x) = -(15 - 3x)^{-2/3}$. For values of x near 2, $|g_3'(x)| < 1$. Hence, the iterative formula $x_{n+1} = g_3(x_n)$ will converge and $g_3(x)$ is suitable.

Using the iterative formula $x_{r+1} = g_3(x_r)$, we get

r	x_r
1	2
2	2.080084
3	2.061408
4	2.065793
5	2.064765

Hence, $x = 2.065$ (3 d.p.).

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Problem 8. The equation of a curve is $y = f(x)$. The curve passes through the points $(a, f(a))$ and $(b, f(b))$, where $0 < a < b$, $f(a) > 0$ and $f(b) < 0$. The equation $f(x) = 0$ has precisely one root α such that $a < \alpha < b$. Derive an expression, in terms of a, b, $f(a)$ and $f(b)$, for the estimated value of α based on linear interpolation.

Let $f(x) = 3\mathrm{e}^{-x} - x$. Show that $f(x) = 0$ has a root α such that $1 < \alpha < 2$, and that for all x, $f'(x) < 0$ and $f''(x) > 0$. Obtain an estimate of α using linear interpolation to 2 decimal places, and explain by means of a sketch whether the value obtained is an over-estimate or an under-estimate.

Use one application of the Newton-Raphson method to obtain a better estimate of α, giving your answer to 2 decimal places.

Solution. Using the point-slope formula, the equation of the line that passes through both $(a, f(a))$ and $(b, f(b))$ is

$$y - f(a) = \frac{f(a) - f(b)}{a - b}(x - a).$$

Note that $(\alpha, 0)$ is approximately the solution to the above equation. Thus,

$$0 - f(a) \approx \frac{f(a) - f(b)}{a - b}(\alpha - a) \implies \alpha \approx \frac{bf(a) - af(b)}{f(a) - f(b)}.$$

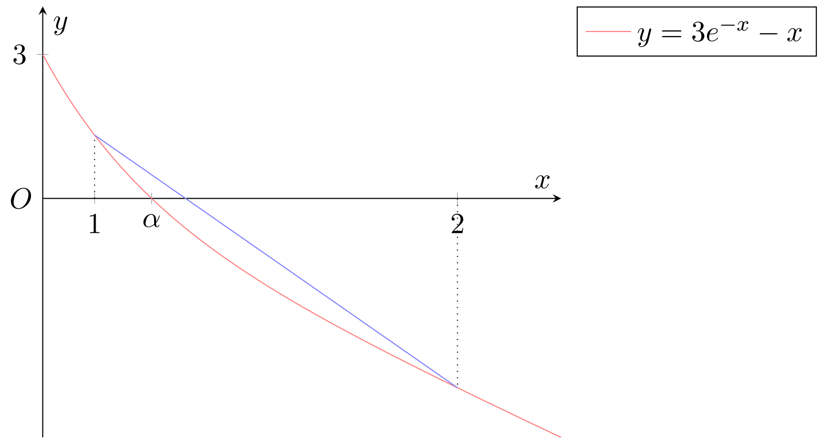
Since $f(x)$ is continuous, and $f(1)f(2) = (0.10)(-1.6) < 0$, there exists a root $\alpha \in (1, 2)$.

Note that $f'(x) = -3\mathrm{e}^{-x} - 1$ and $f''(x) = 3\mathrm{e}^{-x}$. Since $\mathrm{e}^{-x} > 0$ for all x, we have that $f'(x) < 0$ and $f''(x) > 0$ for all x.

Using linear interpolation on the interval $(1, 2)$, we have

$$\alpha = \frac{2f(1) - f(2)}{f(1) - f(2)} = 1.06 \text{ (2 d.p.)}.$$

Since $f'(x) < 0$ and $f''(x) > 0$, we know that $f(x)$ is strictly decreasing and is concave upwards. $f(x)$ hence has the following shape:



From the graph, we see that the value obtained is an over-estimate.

Using the Newton-Raphson method with the initial approximation $x_1 = 1.06$, we get

$$\alpha = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.05 \text{ (2 d.p.)}.$$

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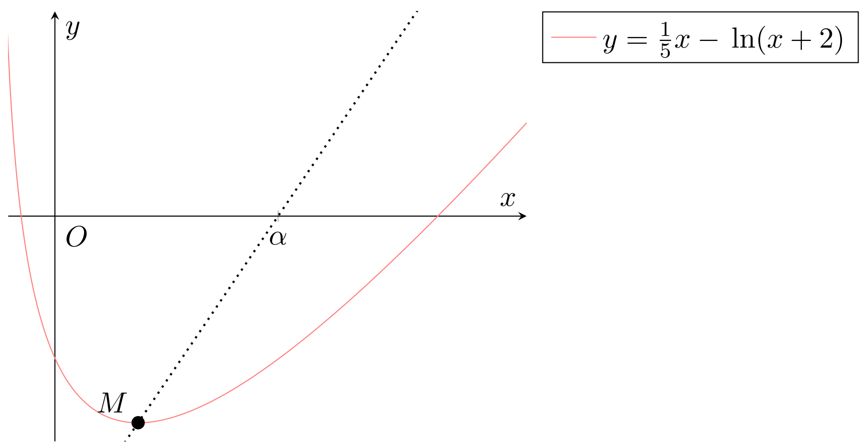
Problem 9.



The diagram shows a sketch of the graph $y = x/3 - \ln(x + 2)$. Find the x-coordinate of the minimum point M on the graph, and verify that y is positive when $x = 20$.

Show that the gradient of the curve is always less than $1/5$. Hence, by considering the line through M having gradient $1/5$, show that the positive root of the equation $x/3 - \ln(x+2) = 0$ is greater than 8.

Use linear interpolation, once only, on the interval $[8, 20]$, to find an approximate value a for this positive root, giving your answer to 1 decimal place.

Using a as an initial value, carry out one application of the Newton-Raphson method to obtain another approximation to the positive root, giving your answer to 2 decimal places.

Solution. For stationary points, $y' = 0$.

$$y' = 0 \implies \frac{1}{5} - \frac{1}{x+2} \implies x = 3.$$

By the second derivative test, we see that $y''(x) = \frac{1}{(x+2)^2} > 0$. Hence, the x-coordinate of M is 3. Substituting $x = 20$ into the equation of the curve gives $y = 4 - \ln 22 = 0.909 > 0$.

We know that $y' = 1/5 - 1/(x+2)$, hence $y' < 1/5$ for all $x > -2$. Since the domain of the curve is $x > -2$, y' is always less than $1/5$.

Let $(\alpha, 0)$ be the coordinates of the root of the line through M having gradient $\frac{1}{5}$. We know that the coordinates of M are $(3, 3/5 - \ln 5)$. Taking the gradient of the line segment joining M and $(\alpha, 0)$, we get

$$\frac{(3/5 - \ln 5) - 0}{3 - \alpha} = \frac{1}{5} \implies \alpha = 5\ln 5 = 8.05 > 8.$$

Since the gradient of the curve is always less than $1/5$, α represents the lowest possible value of the positive root of the curve. Hence, the positive root of the equation $x/5 - \ln(x+2) = 0$ is greater than 8.

Let $f(x) = x/5 - \ln(x+2)$. Using linear interpolation on the interval $[8, 20]$, we have

$$\alpha = \frac{8f(20) - 20f(8)}{f(20) - f(8)} = 13.2 \text{ (1 d.p.)}.$$

Using the Newton-Raphson method with the initial approximation $x_1 = 13.2$, we have

$$\alpha = x_1 - \frac{f(x_1)}{f'(x_1)} = 13.81 \text{ (2 d.p.)}.$$

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Problem 10.

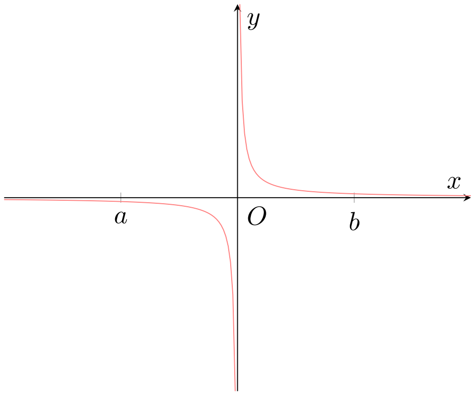
(a) The function f is such that $f(a)f(b) < 0$, where $a < b$. A student concludes that the equation $f(x) = 0$ has exactly one root in the interval (a, b). Draw sketches to illustrate two distinct ways in which the student could be wrong.

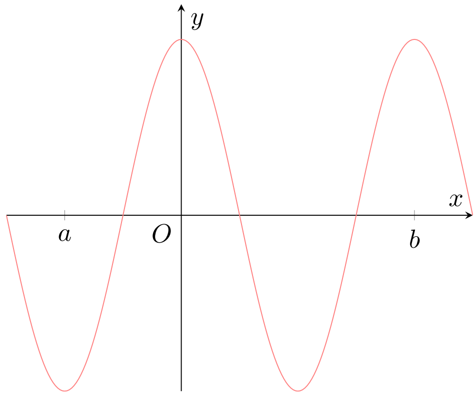
(b) The equation $\sec^2 x - e^2 = 0$ has a root α in the interval $[1.5, 2.5]$. A student uses linear interpolation once on this interval to find an approximation to α. Find the approximation to α given by this method and comment on the suitability of the method in this case.

(c) The equation $\sec^2 x - e^x = 0$ also has a root β in the interval $(0.1, 0.9)$. Use the Newton-Raphson method, with $f(x) = \sec^2 x - e^x$ and initial approximation 0.5, to find a sequence of approximations $\{x_1, x_2, x_3, \ldots\}$ to β. Describe what is happening to x_n for large n, and use a graph of the function to explain why the sequence is not converging to β.

Solution.

Part (a).





Part (b). Let $f(x) = \sec^2 x - e^x$. Using linear interpolation on the interval $[1.5, 2.5]$,

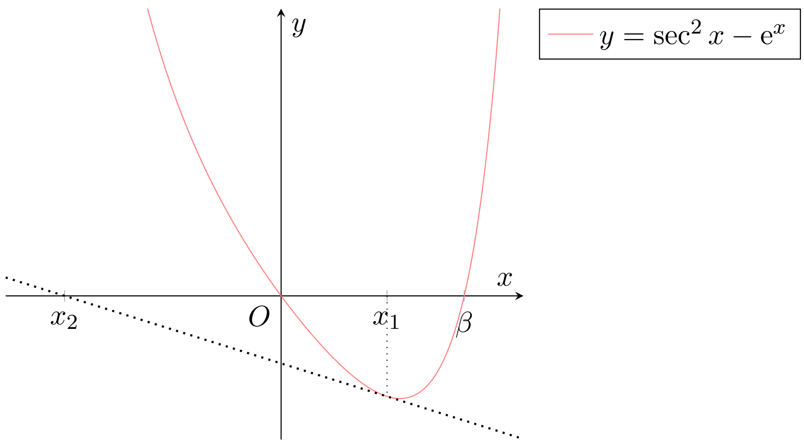
$$a = \frac{1.5f(2.5) - 2.5f(1.5)}{f(2.5) - f(1.5)} = 1.06 \text{ (2 d.p.)}.$$

$\sec^2 x$ is not continuous on the interval $[1.5, 2.5]$ due to the presence of an asymptote at $x = \pi/2$. Hence, linear interpolation is not suitable in this case.

Part (c). We know $f'(x) = 2\sec^2 x \tan x - e^x$. Using the Newton-Raphson method with the initial approximation $x_1 = 0.5$,

r	x_r
1	0.5
2	-1.02272
3	-0.75526
4	-0.40306
5	-0.09667
6	-0.00466
7	-0.00000

As $n \to \infty$, $x_n \to 0^-$.



From the above graph, we see that the initial approximation of $x_1 = 0.5$ is past the turning point. Hence, all subsequent approximations will converge to the root at 0 instead of the root at β. Thus, the sequence does not converge to β.

Problem 11. The function f is given by $f(x) = \sqrt{1-x^2} + \cos x - 1$ for $0 \le x \le 1$. It is known, from graphical work, that the equation $f(x) = 0$ has a single root $x = \alpha$.

(a) Express g(x) in terms of x, where $g(x) = x - \frac{f(x)}{f'(x)}$.

A student attempts to use the Newton-Raphson method, based on the form $x_{n+1} = g(x_n)$, to calculate the value of α correct to 3 decimal places.

(b) (i) The student first uses an initial approximation to α of $x_1 = 0$. Explain why this will be unsuccessful in finding a value for α.

(ii) The student next uses an initial approximation to α of $x_1 = 1$. Explain why this will also be unsuccessful in finding a value for α.

(iii) The student then uses an initial approximate to α of $x_1 = 0.5$. Investigate what happens in this case.

(iv) By choosing a suitable value for x_1, use the Newton-Raphson method, based on the form $x_{n+1} = g(x_n)$, to determine α correct to 3 decimal places.

Solution.

Part (a). We know $f'(x) = \frac{-x}{\sqrt{1-x^2}} - \sin x$. Hence,

$$g(x) = x - \frac{\sqrt{1-x^2} + \cos x - 1}{\frac{-x}{\sqrt{1-x^2}} - \sin x}.$$

Part (b).

Part (b)(i). Observe that $f'(0) = 0$. Hence, $g(0)$ is undefined. Thus, starting with an initial approximation of $x_1 = 0$ will be unsuccessful in finding a value for α.

Part (b)(ii). Observe that $\sqrt{1-x^2}$ is 0 when $x = 1$. Hence, $f'(0)$ is undefined. Thus, $g(0)$ is also undefined. Hence, starting with an initial approximation of $x_1 = 1$ will also be unsuccessful in finding a value for α.

Part (b)(iii). When $x_1 = 0.5$, we have $x_2 = g(x_1) = 1.20$. Since $g(x)$ is only defined for $0 \le x \le 1$, $x_3 = g(x_2)$ is undefined. Hence, an initial approximation of $x_1 = 0.5$ will also be unsuccessful in finding a value for α.

Part (b)(iv). Using the Newton-Raphson method with $x_1 = 0.9$, we have

r	x_r
1	0.9
2	0.92019
3	0.91928
4	0.91928

Thus, $\alpha = 0.919$ (3 d.p.).

Self-Practice A2

Problem 1.

(a) Sketch on the same diagram the graphs of $y = x - 1$ and $y = k\mathrm{e}^{-3x}$, where $-1 < k < 0$. State the number of real roots of the equation $k\mathrm{e}^{-3x} - (x-1) = 0$.

For the case $k = 1$, sketch appropriate graphs to show that the equation $\mathrm{e}^{-3x} - (x - 1) = 0$ has exactly one real root. Denoting this real root by α, find the integer n such that the interval $[n-1, n]$ contains α. Use linear interpolation, once only, on this interval to find an estimate for α, giving your answer correct to 2 decimal places.

(b) Let $f(x) = \mathrm{e}^{-3x} - (x-1)$. By considering the signs of $f'(x)$ and $f''(x)$ for all real values of x, explain with the aid of a simple diagram whether the value of α obtained in (a) is an over-estimate or an under-estimate.

(c) Taking the value of α obtained in (a) as the initial value, apply the Newton-Raphson method to find the value of α correct to 3 decimal places.

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Problem 2. The equation $f(x) = 0$ where $f(x) = \frac{1}{x} - 2 + \ln x$ has exactly two real roots α and β.

Verify that the larger root β lies between 6 and 7 and use one application of linear interpolation on the interval $[6, 7]$ to estimate this root, giving your answer correct to 2 decimal places.

Sketch the graph of $y = f(x)$, stating clearly the coordinates of the turning point. Using the graph of $y = f(x)$, deduce the integer N such that the interval $[N-1, N]$ contains the smaller root α.

An attempt to calculate the smaller root α is made. Explain why neither $x = 0$ nor $x = 1$ is a suitable initial value for the Newton-Raphson method in this case.

Taking $x = 0.3$ as the initial value, use the Newton-Raphson method to find a second approximation to the root α, giving your answer correct to three decimal places.

* * * * *

Problem 3. Sketch the graph of $y = (1+x)\mathrm{e}^{-x}$, indicating clearly the turning points and asymptotes (if any). State the transformation by which the graph of $y = x\mathrm{e}^{1-x}$ may be obtained from the graph of $y = (1+x)\mathrm{e}^{-x}$.

By means of a suitable sketch, deduce that $x\left(1 + \mathrm{e}^{1-x}\right) = 1$ has exactly one real root α. Show that α lies between 0.3 and 0.4.

Use linear interpolation once to obtain an approximation value, c, for α, giving your answer correct to 4 decimal places.

Using the Newton-Raphson method once with c as the first approximation, obtain a second approximation for α correct to 3 significant figures.

* * * * *

Problem 4. In this question, give all your final answers correct to 3 decimal places.

(a) Find, stating your reason, the value of the positive integer n such that

$$n - 1 \leq \sqrt[3]{100} \leq n.$$

Hence, use linear interpolation once only, to find an approximation, α, to the root of the equation $x^3 = 100$. Explain, with the aid of a suitable diagram, whether α is an overestimate or underestimate.

(b) Using the Newton-Raphson method with α as a first approximation, find $\sqrt[3]{100}$. Explain, using the same diagram as in (a), whether this method yields a series of overestimates or underestimates.

* * * * *

Problem 5. The roots of the quadratic equation $x^2 - 7x + 1 = 0$ are to be calculated by the use of the recurrence relation $x_{r+1} = \frac{1}{7-x_r}$. Sketch the graphs of $y = x$ and $y = \frac{1}{7-x}$ and hence show

(a) that the equation has 2 roots, which lie between 0 and 7.

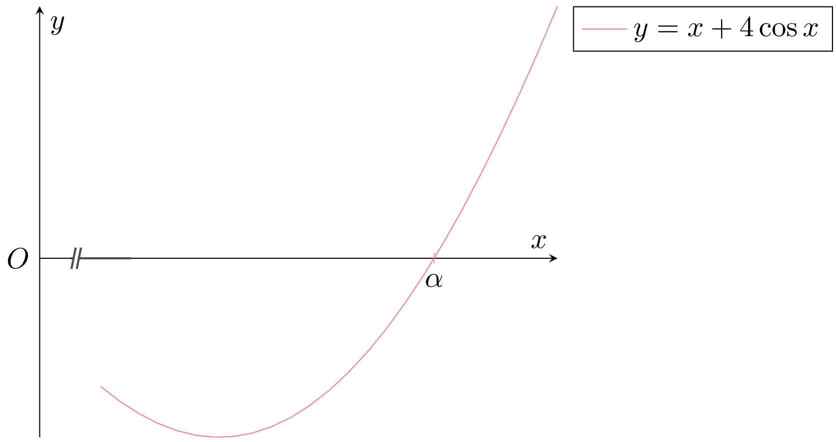
(b) if x_1 has a value lying between these roots, then the recurrence relation will always yield an approximation to the smaller root.

Taking $x_1 = 1$, find the smaller root correct to 3 decimal places. Obtain the value of the larger root to the same degree of accuracy.

Assignment A2

Problem 1. By considering the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, or otherwise, show that the equation $x + 4\cos x = 0$ has one negative root and two positive roots.

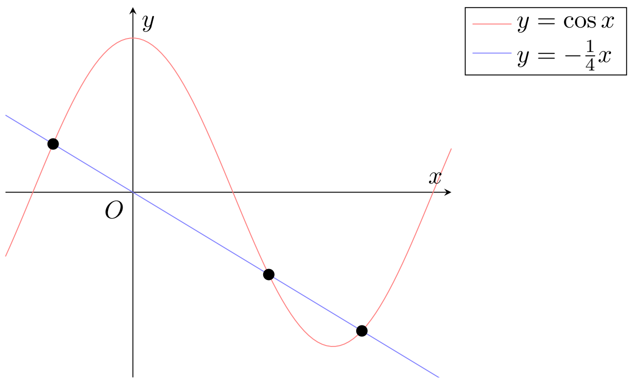
Use linear interpolation, once only, on the interval $[-1.5, 1]$ to find an approximation to the negative root of the equation $x + 4\cos x = 0$ correct to 2 decimal places.



The diagram shows part of the graph of $y = x + 4\cos x$ near the larger positive root, α, of the equation $x + 4\cos x = 0$. Explain why, when using the Newton-Raphson method to find α, an initial approximation which is smaller than α may not be satisfactory.

Use the Newton-Raphson method to find α correct to 2 significant figures. You should demonstrate that your answer has the required accuracy.

Solution.



Note that $x + 4\cos x = 0 \implies \cos x = -\frac{1}{4}x$. Plotting the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, we see that there is one negative root and two positive roots. Hence, the equation $x + 4\cos x = 0$ has one negative root and two positive roots.

Let $f(x) = x + 4\cos x$. Let β be the negative root of the equation $f(x) = 0$. Using linear interpolation on the interval $[-1.5. -1]$,

$$\beta = \frac{-1.5f(-1) - (-1)f(-1.5)}{f(-1) - f(1.5)} = -1.24 \text{ (2 d.p.)}.$$

There is a minimum at $x = m$ such that m is between the two positive roots. Hence, when using the Newton-Raphson method, an initial approximation which is smaller than m would result in subsequent approximations being further away from the desired root α. Hence, an initial approximation that is smaller than α may not be satisfactory.

We know from the above graph that $\alpha \in (\pi, 3\pi/2)$. We hence pick $3\pi/2$ as our initial approximation. Using the Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ with $x_1 = 3\pi/2$, we have

r	x_r
1	$\frac{3}{2}\pi$
2	3.7699
3	3.6106
4	3.5955
5	3.5953

Since $f(3.55)f(3.65) = (-0.1)(0.2) < 0$, we have $\alpha \in (3.55, 3.65)$. Hence, $\alpha = 3.6$ (2 s.f.).

* * * * *

Problem 2. Find the coordinates of the stationary points on the graph $y = x^3 + x^2$. Sketch the graph and hence write down the set of values of the constant k for which the equation $x^3 + x^2 = k$ has three distinct real roots.

The positive root of the equation $x^3 + x^2 = 0.1$ is denoted by α.

(a) Find a first approximation to α by linear interpolation on the interval $0 \leq x \leq 1$.

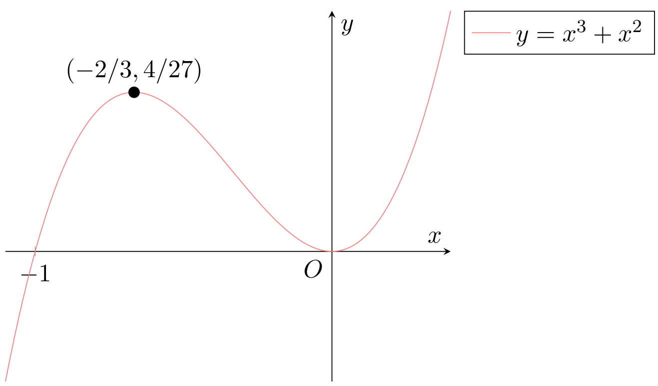
(b) With the aid of a suitable figure, indicate why, in this case, linear interpolation does not give a good approximation to α.

(c) Find an alternative first approximation to α by using the fact that if x is small then x^3 is negligible when compared to x^2.

Solution. For stationary points, $y' = 0$.

$$y' = 0 \implies 3x^2 + 2x = 0 \implies x(3x+2) = 0.$$

Hence, $x = 0$ or $x = -2.3$. When $x = 0$, $y = 0$. When $x = -2/3$, $y = 4/27$. Thus, the coordinates of the stationary points of $y = x^3 + x^2$ are $(0, 0)$ and $(-2/3, 4/27)$.

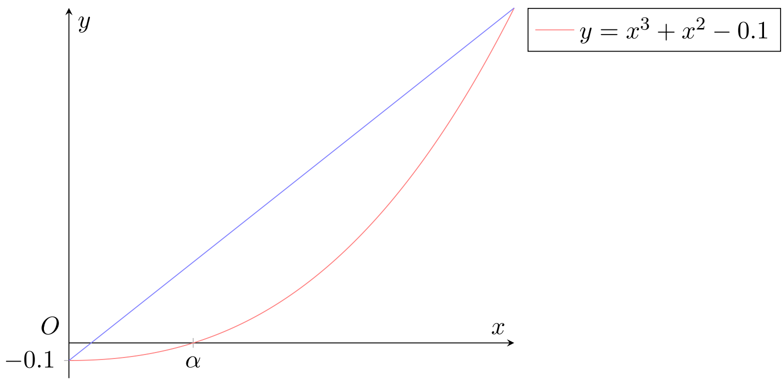


Therefore, $k \in (0, 4/27)$. The solution set of k is thus $\{k \in \mathbb{R} : 0 < k < 4/27\}$.

Part (a). Let $f(x) = x^2 + x^2 - 0.1$. Using linear interpolation on the interval $[0, 1]$,

$$\alpha = \frac{-f(0)}{f(1) - f(0)} = \frac{1}{20}.$$

Part (b).



On the interval $[0, 1]$, the gradient of $y = x^3 + x^2 - 0.1$ changes considerably. Hence, linear interpolation gives an approximation much less than the actual value.

Part (c). For small x, x^3 is negligible when compared to x^2. Consider $g(x) = x^2 - 0.1$. Then the positive root of $g(x) = 0$ is approximately α. Hence, an alternative approximation to α is $\sqrt{0.1} = 0.316$ (3 s.f.).

* * * * *

Problem 3. The equation $2\cos x - x = 0$ has a root α in the interval $[1, 1.2]$. Iterations of the form $x_{n+1} = F(x_n)$ are based on each of the following rearrangements of the equation:

(a) $x = 2\cos x$

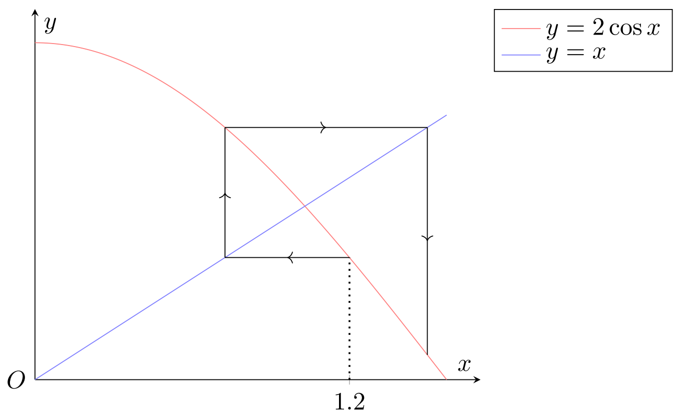
(b) $x = \cos x + \frac{1}{2}x$

(c) $x = \frac{2}{3}(\cos x + x)$

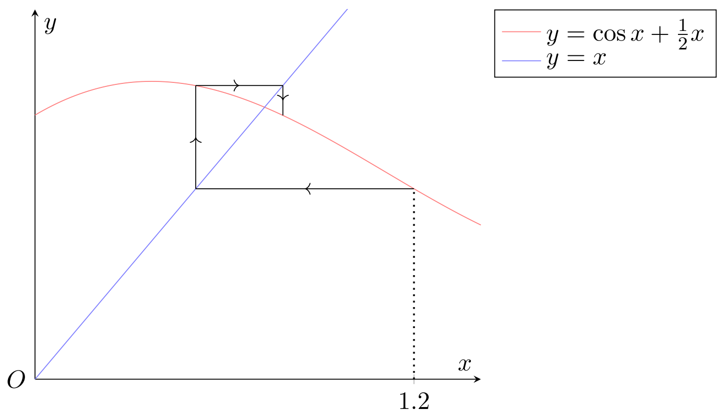
Determine which iteration will converge to α and illustrate your answer by a 'staircase' or 'cobweb' diagram. Use the most appropriate iteration with $x_1 = 1$, to find α to 4 significant figures. You should demonstrate that your answer has the required accuracy.

Solution.

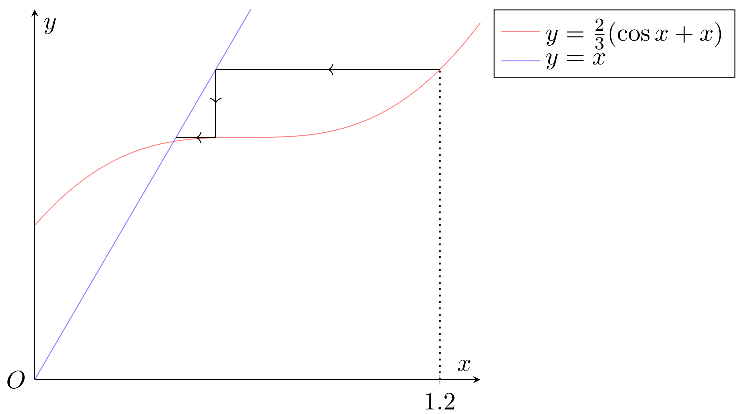
Part (a). Consider $f(x) = 2\cos x$. Then $f'(x) = -2\sin x$. Observe that $\sin x$ is increasing on $[1, 1.2]$. Since $\sin 1 > \frac{1}{2}$, $|f'(x)| > 1$ for all $x \in [1, 1.2]$. Thus, fixed-point iteration fails and will not converge to α.



Part (b). Consider $f(x) = \cos x + \frac{1}{2}x$. Then $f'(x) = -\sin x + \frac{1}{2} - \left(\sin x - \frac{1}{2}\right)$. Since $0 \leq \sin x \leq 1$ for $x \in \left[0, \frac{\pi}{2}\right]$, and $[1, 1.2] \subset \left[0, \frac{\pi}{2}\right]$, we know $-\frac{1}{2} \leq \sin x - \frac{1}{2} \leq \frac{1}{2}$ for $x \in [1, 1.2]$. Thus, $0 \leq \left|\sin x - \frac{1}{2}\right| \leq \frac{1}{2}$ for $x \in [1, 1.2]$. Hence, fixed-point iteration will work and converge to α.



Part (c). Consider $f(x) = \frac{2}{3}(\cos x + x)$. Then $f'(x) = \frac{2}{3}(-\sin x + 1)$. For fixed-point iteration to converge to α, we need $|f'(x)| < 1$ for x near α. It thus suffices to show that $|-\sin x + 1| < \frac{3}{2}$ for all $x \in [1, 1.2]$. Observe that $1 - \sin x$ is strictly decreasing and positive for $x \in \left[0, \frac{\pi}{2}\right]$. Since $1 - \sin 1 < \frac{3}{2}$, and $[1, 1.2] \subset \left[0, \frac{\pi}{2}\right]$, we have that $|-\sin x + 1| < \frac{3}{2}$ for all $x \in [1, 1.2]$. Thus, $|f'(x)| < 1$ for x near α. Hence, fixed-point iteration will work and converge to α.



For $x \in [1, 1.2]$, $\left|\frac{2}{3}(-\sin x + 1)\right| < \left|-\sin x + \frac{1}{2}\right| < 1$. Thus, $x_{n+1} = \frac{2}{3}(\cos x_n + x_n)$ is the most suitable iteration as it will converge to α the quickest. Using $F(x_{n+1}) = \frac{2}{3}(\cos x_n + x_n)$ with $x_1 = 1$,

r	x_r
1	1
2	1.02687
3	1.02958
4	1.02984
5	1.02986

Since $F(1.0295) > 1.0295$ and $F(1.0305) < 1.0305$, we have $\alpha \in (1.0295, 1.0305)$. Hence, $\alpha = 1.030$ (4 s.f.).

A3 Sequences and Series I

Tutorial A3

Problem 1. Determine the behaviour of the following sequences.

(a) $u_n = 3\left(\frac{1}{2}\right)^{n-1}$

(b) $v_n = 2 - n$

(c) $t_n = (-1)^n$

(d) $w_n = 4$

Solution.
Part (a). Decreasing, converges to 0.
Part (b). Decreasing, diverges.
Part (c). Alternating, diverges.
Part (d). Constant, converges to 4.

* * * * *

Problem 2. Find the sum of all even numbers from 20 to 100 inclusive.

Solution. The even numbers from 20 to 100 inclusive form an AP with common difference 2, first term 20 and last term 100. Since we are adding a total of $\frac{100-20}{2} + 1 = 41$ terms, we get a sum of $41\left(\frac{20+100}{2}\right) = 2460$.

* * * * *

Problem 3. A geometric series has first term 3, last term 384 and sum 765. Find the common ratio.

Solution. Let the nth term of the geometric series be ar^{n-1}, where $1 \leq n \leq k$. We hence have $3r^{k-1} = 384$, which gives $r^k = 128r$. Thus,

$$\frac{3(1-r^k)}{1-r} = 765 \implies \frac{3(1-128r)}{1-r} = 765 \implies r = 2.$$

* * * * *

Problem 4.

(a) Find the first four terms of the following sequence $u_{n+1} = \frac{u_n+1}{u_n+2}$, $u_1 = 0$, $n \geq 1$.

(b) Write down the recurrence relation between the terms of these sequences.

  (i) $-1, 2, -4, 8, -16, \ldots$

  (ii) $1, 3, 7, 15, 31, \ldots$

Solution.
Part (a). Using G.C., the first four terms of u_n are 0, $\frac{1}{2}$, $\frac{3}{5}$ and $\frac{8}{13}$.

Part (b).

Part (b)(i). $u_{n+1} = -2u_n$, $u_1 = -1$, $n \geq 1$.

Part (b)(ii). $u_{n+1} = 2u_n + 1$, $u_1 = 1$, $n \geq 1$.

* * * * *

Problem 5. The sum of the first n terms of a series, S_n, is given by $S_n = 2n(n+5)$. Find the nth term and show that the terms are in arithmetic progression.

Solution. We have

$$u_n = S_n - S_{n-1} = 2n(n+5) - 2(n-1)(n+4) = 4n + 8.$$

Observe that $u_n - u_{n-1} = [4n+8] - [4(n-1)+8] = 8$ is a constant. Hence, u_n is in AP.

* * * * *

Problem 6. The sum of the first n terms, S_n, is given by

$$S_n = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}.$$

(a) Find an expression for the nth term of the series.

(b) Hence or otherwise, show that it is a geometric series.

(c) State the values of the first term and the common ratio.

(d) Give a reason why the sum of the series converges as n approaches infinity and write down its value.

Solution.

Part (a). Note that

$$u_n = S_n - S_{n-1} = \left[\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}\right] - \left[\frac{1}{2} - \left(\frac{1}{2}\right)^{n}\right] = \left(\frac{1}{2}\right)^{n+1}.$$

Part (b). Since $\frac{u_{n+1}}{u_n} = \frac{(1/2)^{n+2}}{(1/2)^{n+1}} = \frac{1}{2}$ is constant, u_n is in GP.

Part (c). The first term is $\frac{1}{4}$ and the common ratio is $\frac{1}{2}$.

Part (d). As $n \to \infty$, we clearly have $\left(\frac{1}{2}\right)^{n+1} \to 0$. Hence, $S_\infty = \frac{1}{2}$.

* * * * *

Problem 7. The first term of an arithmetic series is $\ln x$ and the rth term is $\ln\left(xk^{r-1}\right)$, where k is a real constant. Show that the sum of the first n terms of the series is $S_n = \frac{n}{2}\ln\left(x^2k^{n-1}\right)$. If $k = 1$ and $x \neq 1$, find the sum of the series $\mathrm{e}^{S_1} + \mathrm{e}^{S_2} + \mathrm{e}^{S_3} + \ldots + \mathrm{e}^{S_n}$.

Solution. Let u_n be the nth term in the arithmetic series. Then

$$u_n = \ln\left(xk^{r-1}\right) = \ln x + (r-1)\ln k.$$

We thus see that the arithmetic series has first term $\ln x$ and common difference of $\ln k$. Thus,

$$S_n = n\left(\frac{\ln x + (\ln x + (r-1)\ln k)}{2}\right) = \frac{n}{2}\ln\left(x^2k^{r-1}\right).$$

When $k = 1$, we have $S_n = \ln(x^n)$, whence $e^{S_n} = x^n$. Thus,

$$e^{S_1} + e^{S_2} + e^{S_3} + \ldots + e^{S_n} = x + x^2 + x^3 + \ldots x^n = \frac{x(1 - x^{n+1})}{1 - x}.$$

* * * * *

Problem 8. A baker wants to bake a 1-metre tall birthday cake. It comprises 10 cylindrical cakes each of equal height 10 cm. The diameter of the cake at the lowest layer is 30 cm. The diameter of each subsequent layer is 4% less than the diameter of the cake below. Find the volume of this cake in cm^3, giving your answer to the nearest integer.

Solution. Let the diameter of the nth layer be d_n cm. We have $d_{n+1} = 0.96d_n$ and $d_1 = 30$, whence $d_n = 30 \cdot 0.96^{n-1}$. Let the nth layer have volume v_n cm^3. Then

$$v_n = 10\pi \left(\frac{d_n}{2}\right)^2 = 10\pi \left(\frac{900 \cdot 0.9216^{n-1}}{4}\right) = 2250\pi \cdot 0.9216^{n-1}.$$

The volume of the cake in cm^3 is thus given by

$$2250\pi \left(\frac{1 - 0.9216^{10}}{1 - 0.9216}\right) = 50309.$$

* * * * *

Problem 9. The sum to infinity of a geometric progression is 5 and the sum to infinity of another series is formed by taking the first, fourth, seventh, tenth, ... terms is 4. Find the exact common ratio of the series.

Solution. Let the nth term of the geometric progression be given by ar^{n-1}. Then, we have

$$\frac{a}{1 - r} = 5 \implies a = 5(1 - r). \tag{1}$$

Note that the first, fourth, seventh, tenth, ... terms forms a new geometric series with common ratio r^3: $a, ar^3, ar^6, ar^9, \ldots$. Thus,

$$\frac{a}{1 - r^3} = 4 \implies a = 4(1 - r^3). \tag{2}$$

Equating (1) and (2), we have

$$5(1 - r) = 4(1 - r^3) \implies 4r^3 + 5r + 1 = 0 \implies (r - 1)(4r^2 + 4r - 1) = 0.$$

Since $|r| < 1$, we only have $4r^2 + 4r - 1 = 0$, which has solutions $r = \frac{-1+\sqrt{2}}{2}$ or $r = \frac{-1-\sqrt{2}}{2}$. Once again, since $|r| < 1$, we reject $r = \frac{-1-\sqrt{2}}{2}$. Hence, $r = \frac{-1+\sqrt{2}}{2}$.

* * * * *

Problem 10. A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of the geometric series are equal to the first, fourth and sixth terms respectively of the arithmetic series.

(a) Show that $3r^2 - 5r + 2 = 0$

(b) Deduce that the geometric series is convergent and find, in terms of a, the sum of infinity.

(c) The sum of the first n terms of the arithmetic series is denoted by S. Given that $a > 0$, find the set of possible values of n for which S exceeds $4a$.

Solution.

Part (a). Let the nth term of the geometric series be $G_n = G_1 r^{n-1}$. Let the nth term of the arithmetic series be $A_n = a + (n-1)d$.

Since $G_1 = A_1$, we have $G_1 = a$. We can thus write $G_n = ar^{n-1}$. From $G_2 = A_4$, we have $ar = a + 3d$, which gives $a = \frac{3d}{r-1}$. From $G_3 = A_6$, we have $ar^2 = a + 5d$. Thus,

$$\frac{3d}{r-1} \cdot r^2 = \frac{3d}{r-1} + 5d \implies \frac{3r^2}{r-1} = \frac{3}{r-1} + 5 \implies 3r^2 - 5r + 2 = 0.$$

Part (b). Note that the roots to $3r^2 - 5r + 2 = 0$ are $r = 1$ and $r = 2/3$. Clearly, $r \neq 1$ since $a = 3d/(r-1)$ would be undefined. Hence, $r = 2/3$, whence the geometric series is convergent.

Let S_∞ be the sum to infinity of G_n. Then $S_\infty = a/(1-r) = 3a$.

Part (c). Note that $d = a(r-1)/3 = -\frac{a}{9}$. Hence,

$$S = n\left(\frac{a + [a + (n-1)d]}{2}\right) = n\left(\frac{2a + (n-1)\left(-\frac{a}{9}\right)}{2}\right) = \frac{an}{18}(19 - n).$$

Consider $S > 4a$.

$$S > 4a \implies \frac{n}{18}(19 - n) > 4 \implies -n^2 + 19n - 72 > 0.$$

Using G.C., we see that $5.23 < n < 13.8$. Since n is an integer, the set of values that n can take on is $\{n \in \mathbb{Z} : 6 \leq n \leq 13\}$.

* * * * *

Problem 11. Two musical instruments, A and B, consist of metal bars of decreasing lengths.

(a) The first bar of instrument A has length 20 cm and the lengths of the bars form a geometric progression. The 25th bar has length 5 cm. Show that the total length of all the bars must be less than 357 cm, no matter how many bars there are.

Instrument B consists of only 25 bars which are identical to the first 25 bars of instrument A.

(b) Find the total length, L cm, of all the bars of instrument B and the length of the 13th bar.

(c) Unfortunately, the manufacturer misunderstands the instructions and constructs instrument B wrongly, so that the lengths of the bars are in arithmetic progression with a common difference d cm. If the total length of the 25 bars is still L cm and the length of the 25th bar is still 5 cm, find the value of d and the length of the longest bar.

Solution.

Part (a). Let $u_n = u_1 r^{n-1}$ be the length of the nth bar. Since $u_1 = 20$, we have $u_n = 20r^{n-1}$. Since $u_{25} = 5$, we have $r = 4^{-\frac{1}{24}}$. Hence, $u_n = 20 \cdot 4^{-\frac{n-1}{24}}$. Now, consider the sum to infinity of u_n:

$$S_\infty = \frac{u_1}{1-r} = \frac{20}{1 - 4^{-1/24}} = 356.3 < 357.$$

Hence, no matter how many bars there are, the total length of the bars will never exceed 357 cm.

Part (b). We have

$$L = u_1 \left(\frac{1 - r^{25}}{1 - r} \right) = 20 \left(\frac{1 - 4^{-25/24}}{1 - 4^{-1/24}} \right) = 272.26 = 272 \text{ (3 s.f.)}.$$

Note that

$$u_{13} = 20 \cdot \left(4^{-1/24} \right)^{13-1} = 10.$$

The 13th bar is hence 10 cm long.

Part (c). Let $v_n = a + (n-1)d$ be the length of the wrongly-manufactured bars. Since the length of the 25th bar is still 5 cm, we know $v_{25} = a + 24d = 5$. Now, consider the total lengths of the bars, which is still L cm.

$$L = 25 \left(\frac{a+5}{2} \right) = 272.26.$$

Solving, we see that $a = 16.781$. Hence, $d = \frac{5-a}{24} = -0.491$, and the longest bar is $16.8 =$ cm long.

* * * * *

Problem 12. A bank has an account for investors. Interest is added to the account at the end of each year at a fixed rate of 5% of the amount in the account at the beginning of that year. A man a woman both invest money.

(a) The man decides to invest $\$x$ at the beginning of one year and then a further $\$x$ at the beginning of the second and each subsequent year. He also decides that he will not draw any money out of the account, but just leave it, and any interest, to build up.

  (i) How much will there be in the account at the end of 1 year, including the interest?

  (ii) Show that, at the end of n years, when the interest for the last year has been added, he will have a total of $\$21(1.05^n - 1)x$ in his account.

  (iii) After how many complete years will he have, for the first time, at least $\$12x$ in his account?

(b) The woman decides that, to assist her in her everyday expenses, she will withdraw the interest as soon as it has been added. She invests $\$y$ at the beginning of each year. Show that, at the end of n years, she will have received a total of $\$\frac{1}{40}n(n+1)y$ in interest.

Solution.

Part (a).

Part (a)(i). There will be $\$1.05x$ in the account at the end of 1 year.

Part (a)(ii). Let $\$u_n x$ be the amount of money in the account at the end of n years. Then, u_n satisfies the recurrence relation $u_{n+1} = 1.05(1 + u_n)$, with $u_1 = 1.05$. Observe that

$$u_1 = 1.05 \implies u_2 = 1.05 + 1.05^2 \implies u_3 = 1.05 + 1.05^2 + 1.05^3 \implies \cdots.$$

We thus have

$$u_n = 1.05 + 1.05^2 + \cdots + 1.05^n = 1.05 \left(\frac{1 - 1.05^n}{1 - 1.05} \right) = 21 \left(1.05^n - 1 \right).$$

Hence, there will be $\$21(1.05^n - 1)x$ in the account after n years.

Part (a)(iii). Consider the inequality $u_n \geq 12x$.

$$u_n \geq 12x \implies 21(1.05^n - 1) \geq 12 \implies n \geq 9.26.$$

Since n is an integer, the smallest value of n is 10. Hence, after 10 years, he will have at least \$$12x$ in his account for the first time.

Part (b). After n years, the woman will have \$$ny$ in her account. Hence, the interest she gains at the end of the nth year is $\frac{1}{20}ny$. Thus, the total interest she will gain after n years is

$$\frac{y}{20} + \frac{2y}{20} + \cdots + \frac{ny}{20} = \frac{y}{20}(1 + 2 + \cdots + n) = \frac{y}{20} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)y}{40}.$$

* * * * *

Problem 13. The sum, S_n, of the first n terms of a sequence $U_1, U_2, U_3, \ldots$ is given by

$$S_n = \frac{n}{2}(c - 7n),$$

where c is a constant.

(a) Find U_n in terms of c and n.

(b) Find a recurrence relation of the form $U_{n+1} = f(U_n)$.

Solution.

Part (a). Observe that

$$U_n = S_n - S_{n-1} = \frac{n}{2}(c - 7n) - \frac{n-1}{2}(c - 7(n-1)) = -7n + \frac{7+c}{2}.$$

Part (b). Observe that $U_{n+1} - U_n = -7$. Thus,

$$U_{n+1} = U_n - 7, \quad U_1 = \frac{7+c}{2}, \quad n \geq 1.$$

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Problem 14. The positive numbers x_n satisfy the relation

$$x_{n+1} = \sqrt{\frac{9}{2} + \frac{1}{x_n}}$$

for $n = 1, 2, 3, \ldots$.

(a) Given that $n \to \infty$, $x_n \to \theta$, find the exact value of θ.

(b) By considering $x_{n+1}^2 - \theta^2$, or otherwise, show that if $x_n > \theta$, then $0 < x_{n+1} < \theta$.

Solution.

Part (a). Observe that

$$\theta = \lim_{n\to\infty} \sqrt{\frac{9}{2} + \frac{1}{x_n}} = \sqrt{\frac{9}{2} + \frac{1}{\theta}} \implies 2\theta^3 - 9\theta - 2 = 0 \implies (\theta + 2)\left(2\theta^2 - 4\theta - 1\right) = 0.$$

We reject $\theta = -2$ since $\theta > 0$. We thus consider $2\theta^2 - 4\theta - 1 = 0$, which has roots $\theta = 1 + \sqrt{\frac{3}{2}}$ and $\theta = 1 - \sqrt{\frac{3}{2}}$. Once again, we reject $\theta = 1 - \sqrt{\frac{3}{2}}$ since $\theta > 0$. Thus, $\theta = 1 + \sqrt{\frac{3}{2}}$.

Part (b). Suppose $x_n > \theta$. Then

$$x_{n+1}^2 = \frac{9}{2} + \frac{1}{x_n} < \frac{9}{2} + \frac{1}{\theta} = \theta^2 \implies 0 < x_{n+1} < \theta.$$

Self-Practice A3

Problem 1. The sum of the first n terms of a sequence $\{u_n\}$ is given by the formula $S_n = 2n(n-3)$, where $n \in \mathbb{Z}^+$.

(a) Express u_n in terms of n, and show that the sequence $\{u_n\}$ follows an arithmetic progression.

(b) Three terms u_3, u_k and u_{38} of this sequence are consecutive terms in a geometric sequence. Find the value of k.

(c) Explain why the infinite series $\mathrm{e}^{-u_1} + \mathrm{e}^{-u_2} + \mathrm{e}^{-u_3} + \ldots$ exists, and determine the value of the infinite sum, leaving your answer in exact form.

Solution.

Part (a). Note that

$$u_n = S_n - S_{n-1} = 2n(n-3) - 2(n-1)(n-1-3) = 4n - 8.$$

Thus,

$$u_n - u_{n-1} = [4n-8] - [4(n-1)-8] = 4.$$

Since $u_n - u_{n-1}$ is a constant, the sequence $\{u_n\}$ follows an arithmetic progression with common difference 4.

Part (b). Note that $u_3 = 4$ and $u_{38} = 144$. Let the common ratio be r. Then

$$u_{38} = r^2 u_3 \implies r^2 = \frac{u_{38}}{u_3} = 36 \implies r = \pm 6.$$

Since $u_k > u_3 > 0$, the common ratio r must be positive. Hence, $r = 6$. Thus,

$$4k - 8 = u_k = ru_3 = 6(4) = 24,$$

whence $k = 8$.

Part (c). Observe that

$$\frac{\mathrm{e}^{-u_n}}{\mathrm{e}^{-u_{n-1}}} = \mathrm{e}^{u_{n-1}-u_n} = \mathrm{e}^{-4}.$$

Hence, $\{\mathrm{e}^{-u_n}\}$ is in geometric progression with common ratio e^{-4}. Since $\left|\mathrm{e}^{-4}\right| < 1$, the sum to infinity exists, and is given by

$$\sum_{n=1}^{\infty} \mathrm{e}^{-u_n} = \mathrm{e}^{-u_1}\left(\frac{1}{1-\mathrm{e}^{-4}}\right) = \frac{\mathrm{e}^4}{1-\mathrm{e}^{-4}}.$$

* * * * *

Problem 2. At the end of December 2010, the amount of water in a large tank was 43 000 litres. The tank was filled with 7000 litres of water at the start of every month. It was observed that 25% of the amount at the start of any month was lost by the end of that month.

(a) Show that at the end of February 2011, the amount of water in the tank was 33 375 litres.

(b) Find the amount of water in the tank, measured in litres, at the end of the nth month after the end of December 2010, expressing your answer in the form $A\left(\frac{3}{4}\right)^n + B$, where A and B are positive integers to be determined.

Solution.

Part (a). Let the amount of water in the tank, measured in litres, at the start of the nth month after the end of December 2010 be u_n. Clearly, $u_0 = 43000$ and

$$u_n = \frac{3}{4}\left(u_{n-1} + 7000\right).$$

Note that

$$u_1 = \frac{3}{4}\left(u_0 + 7000\right) = 37500, \quad u_2 = \frac{3}{4}\left(u_1 + 7000\right) = 33375.$$

Hence, at the end of February 2011, the amount of water in the tank was 33 375 litres.

Part (b). Let k be the constant such that

$$u_n - k = \frac{3}{4}\left(u_{n-1} - k\right).$$

It quickly follows that $k = 21000$. Then

$$u_n - 21000 = \frac{3}{4}\left(u_{n-1} - 21000\right) = \left(\frac{3}{4}\right)^n \left(u_0 - 21000\right).$$

Thus,

$$u_n = 22000\left(\frac{3}{4}\right)^n + 21000,$$

whence $A = 22000$ and $B = 21000$.

* * * * *

Problem 3.

(a) A runner wants to train for the marathon. He runs 8 km during the first day, and increases the distance he runs each subsequent day by 400 m. Find the minimum number of days, n, that he needs to take to complete at least 2000 km.

(b) A sequence of real numbers $\{u_1, u_2, u_3, \ldots\}$, where $u_1 \neq 0$, is defined such that the $(n+1)$th term of the sequence is equal to the sum of the first n terms, where $n \in \mathbb{Z}^+$. Prove that the sequence $\{u_2, u_3, u_4, \ldots\}$ follows a geometric progression. Hence, find $u_1 + u_2 + \cdots + u_{N+1}$ in terms of u_1 and N.

Solution.

Part (a). Let u_n be the distance ran on the nth day, measured in km. Clearly, $\{u_n\}$ is in arithmetic progression with common difference 0.4, and $u_1 = 8$. Thus,

$$u_n = 0.4(n-1) + 8 = 0.4n + 7.6.$$

Let S_n be the total distance ran in n days. We have

$$S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n} (0.4k + 7.6) = 0.4\left(\frac{n(n+1)}{2}\right) + 7.6n.$$

Consider

$$S_n = 0.4\left(\frac{n(n+1)}{2}\right) + 7.6n \geq 2000.$$

Using G.C., we have $n \geq 82.4$ or $n \leq -121.4$. Since n is a positive integer, the least n is 83. Thus, he needs at least 83 days to complete at least 2000 km.

Part (b). Note that $u_2 = u_1$. Observe that

$$S_n - S_{n-1} = u_n = S_{n-1} \implies S_n = 2S_{n-1}.$$

Hence,

$$\frac{u_{n+1}}{u_n} = \frac{S_n}{S_{n-1}} = 2,$$

whence $\{u_2, u_3, u_4, \ldots\}$ is geometric progression with common ratio 2. Thus,

$$u_1 + u_2 + \cdots + u_{N+1} = u_1 + u_2 \left(\frac{1-2^N}{1-2}\right) = u_1 + u_1 \left(2^N - 1\right) = u_1 2^N.$$

* * * * *

Problem 4.

(a) If the sum of the first n terms of a series is S_n, where $S_n = n - 3n^2$, write down an expression for S_{n-1}. Hence, prove that the series is in an arithmetic series.

(b) Each time a ball falls vertically onto a horizontal surface, it rebounds to two-thirds of the height from which it fell. The ball is initially dropped from a point 12 m above the surface.

Show that the distance the ball has travelled just before it touches the surface for the nth time is $60 - 72\left(\frac{2}{3}\right)^n$.

Hence, find the least number of times the ball has bounced to travel a total distance of more than 52 m.

Solution.

Part (a). Clearly,

$$S_{n-1} = (n-1) - 3(n-1)^2 = -3n^2 + 7n - 4.$$

Hence,

$$u_n = S_n - S_{n-1} = \left(n - 3n^2\right) - \left(-3n^2 + 7n - 4\right) = -6n + 4.$$

Observe that

$$u_n - u_{n-1} = [-6n + 4] - [-6(n-1) - 4] = -6.$$

Hence, $\{u_n\}$ is in arithmetic progression with common ratio -6.

Part (b). Let u_n be the height of the nth "drop" of the ball. We have $u_1 = 12$, and the recurrence relation $u_{n+1} = \frac{2}{3}u_n$. Quite clearly,

$$u_n = \left(\frac{2}{3}\right)^{n-1} u_1 = 18\left(\frac{2}{3}\right)^n.$$

Let D_n be the total distance travelled by the ball just before it touches the surface for the nth time. Observe that the after the initial 12 m, the ball travels up and down before touching the surface again. Hence,

$$D_n = u_1 + 2u_2 + 2u_3 + \cdots + 2u_n = u_1 + \sum_{k=2}^{n} 2u_n.$$

This evaluates as

$$D_n = u_1 + 2\sum_{k=2}^{n} 18\left(\frac{2}{3}\right)^n = 12 + 36 \cdot \left(\frac{2}{3}\right)^2 \left(\frac{1-(2/3)^{n-1}}{1-2/3}\right) = 60 - 72\left(\frac{2}{3}\right)^n.$$

Consider $D_n \geq 52$. Using G.C., we have $n \geq 5.4$. Thus, the ball must bounce at least 6 times.

$* * * * *$

Problem 5. The sequence $\{2^n, n = 0, 1, 2, \ldots\}$ is grouped into sets such that the rth bracket contains r terms: $\{1\}$, $\left\{2, 2^2\right\}$, $\left\{2^3, 2^4, 2^5\right\}$, $\left\{2^6, 2^7, 2^8, 2^9\right\}$, ... Find the total number of terms in the first n brackets. Hence, find the sum of numbers in the first n brackets. Deduce (in any order), in terms of n, the first and the last number in the nth bracket.

Solution. Clearly, the number of terms in the first n brackets is

$$1 + 2 + 3 + \cdots + n = \frac{n(n-1)}{2}.$$

Note that the kth number is given by 2^{k-1}. The sum of number in the first n brackets is hence given by

$$\sum_{k=0}^{n(n+1)/2-1} 2^k = \frac{1 - 2^{n(n+1)/2}}{1-2} = 2^{n(n+1)/2} - 1.$$

The last number in the nth bracket is clearly

$$2^{n(n+1)/2-1}.$$

Note that there are $n(n-1)/2$ terms in the first $(n-1)$ brackets. Thus, the first number in the nth bracket is

$$2^{n(n-1)/2}.$$

Assignment A3

Problem 1. A university student has a goal of saving at least \$1 000 000 (in Singapore dollars). He begins working at the start of the year 2019. In order to achieve his goal, he saves 40% of his annual salary at the end of each year. If his annual salary in the year 2019 is \$40800, and it increases by 5% (of his previous year's salary) every year, find

(a) his annual savings in 2027 (to the nearest dollar),

(b) his total savings at the end of n years.

What is the minimum number of complete years for which he has to work in order to achieve his goal?

Solution. Let $\$u_n$ be his annual salary in the nth year after 2019, with $n \in \mathbb{N}$. Then $u_{n+1} = 1.05 \cdot u_n$, with $u_0 = 40800$. Hence, $u_n = 40800 \cdot 1.05^n$. Let $\$v_n$ be the amount saved in the nth year after 2019. Then $v_n = 0.40 \cdot u_n = 16320 \cdot 1.05^n$.

Part (a). In 2027, $n = 8$. Hence, his annual savings in 2027, in dollars, is given by

$$v_8 = 16320 \cdot 1.05^8 = 24112 \text{ (to the nearest integer).}$$

Part (b). His total savings at the end of n years, in dollars, is given by

$$16320\left(1.05^0 + 1.05^1 + \cdots + 1.05^n\right) = 16320\left(\frac{1-1.05^n}{1-1.05}\right) = 326400\left(1.05^n - 1\right).$$

Consider $326400\left(1.05^n - 1\right) \geq 1000000$. Using G.C., we see that $n \geq 28.7$. Thus, he needs to work for a minimum of 29 complete years to reach his goal.

* * * * *

Problem 2.

(a) A rope of length 200π cm is cut into pieces to form as many circles as possible, whose radii follow an arithmetic progression with common difference 0.25 cm. Given that the smallest circle has an area of π cm^2, find the area of the largest circle in terms of π.

(b) The sum of the first n terms of a sequence is given by $S_n = \alpha^{-n} - 1$, where α is a non-zero constant, $\alpha \neq 1$.

  (i) Show that the sequence is a geometric progression and state its common ratio in terms of α.

  (ii) Find the set of values of α for which the sum to infinity of the sequence exists.

  (iii) Find the value of the sum to infinity.

Solution.

Part (a). Let the sequence r_n be the radius of the nth smallest circle, in centimetres. Hence, $r_n = \frac{1}{4} + r_{n-1}$. Since the smallest circle has area π cm^2, $r_1 = 1$. Thus, $r_n = 1 + \frac{1}{4}(n-1)$.

Consider the nth partial sum of the circumferences:

$$2\pi r_1 + 2\pi r_2 + \cdots + 2\pi r_n = 2\pi \cdot n\left(\frac{1 + \left[1 + \frac{1}{4}(n-1)\right]}{2}\right) = \frac{\pi(n^2+7n)}{4}.$$

Since the rope has length 200π cm, we have the inequality

$$\frac{\pi(n^2+7n)}{4} \leq 200\pi \implies n^2 - 7n - 800 \leq 0 \implies (n+32)(n-25) \leq 0.$$

Hence, $n \leq 25$. Since the rope is cut to form as many circles as possible, $n = 25$. Thus, the largest circle has area $\pi \cdot r_{25}^2 = 49\pi$ cm^2.

Part (b). Let the sequence being summed by $u_1, u_2, \ldots$. Observe that

$$u_n = S_n - S_{n-1} = \left(\alpha^{-n} - 1\right) - \left(\alpha^{-(n-1)} - 1\right) = \alpha^{-n}(1-\alpha).$$

Part (b)(i). Observe that

$$\frac{u_{n+1}}{u_n} = \frac{\alpha^{-(n+1)}(1-\alpha)}{\alpha^{-n}(1-\alpha)} = \alpha^{-1},$$

which is a constant. Thus, u_n is in GP with common ratio α^{-1}.

Part (b)(ii). Consider $S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty}(\alpha^{-n} - 1)$. For S_∞ to exist, we need $\lim_{n\to\infty} \alpha^{-n}$ to exist. Hence, $\left|\alpha^{-1}\right| < 1$, whence $|\alpha| > 1$. Thus, $\alpha < -1$ or $\alpha > 1$. The solution set of α is thus $\{x \in \mathbb{R} : x < -1 \text{ or } x > 1\}$.

Part (b)(iii). Since $\left|\alpha^{-1}\right| < 1$, we know $\lim_{n\to\infty} \alpha^{-n} = 0$. Hence, $S_\infty = -1$.

* * * * *

Problem 3. A sequence $u_1, u_2, u_3, \ldots$ is such that $u_{n+1} = 2u_n + An$, where A is a constant and $n \geq 1$.

(a) Given that $u_1 = 5$ and $u_2 = 15$, find A and u_3.

It is known that the nth term of this sequence is given by

$$u_n = a(2^n) + bn + c,$$

where a, b and c are constants.

(b) Find a, b and c.

Solution.

Part (a). Substituting $n = 1$ into the recurrence relation yields $u_2 = 2u_1 + A$. Thus, $A = u_2 - 2u_1 = 5$. Substituting $n = 2$ into the recurrence relation yields $u_3 = 2u_2 + 2A = 40$.

Part (b). Since $u_1 = 5$, $u_2 = 15$ and $u_3 = 40$, we have the following system

$$\begin{cases} 2a + b + c = 5 \\ 4a + 2b + c = 15 \\ 8a + 3b + c = 40 \end{cases}$$

which has the unique solution $a = \frac{15}{2}$, $b = -5$ and $c = -5$

Problem 4. The graphs of $y = 2^x/3$ and $y = x$ intersect at $x = \alpha$ and $x = \beta$ where $\alpha < \beta$. A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3} \cdot 2^{x_n}, \qquad n \geq 1.$$

(a) Prove algebraically that, if the sequence converges, then it converges to either α or β.

(b) By using the graphs of $y = \frac{1}{3} \cdot 2^x$ and $y = x$, prove that

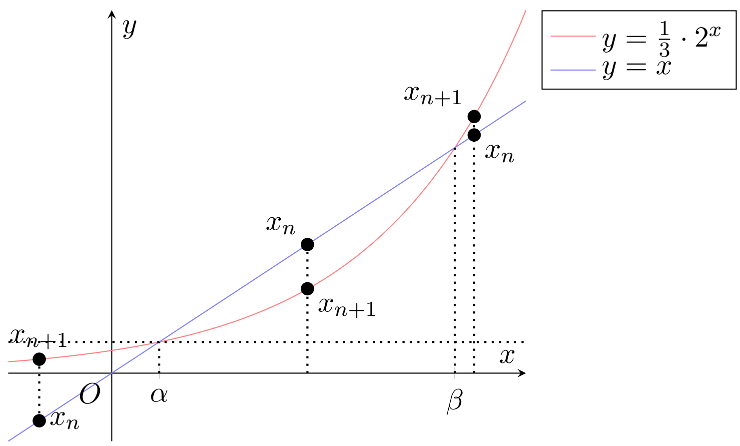
- if $\alpha < x_n < \beta$, then $\alpha < x_{n+1} < x_n$
- if $x_n < \alpha$, then $x_n < x_{n+1} < \alpha$
- if $x_n > \beta$, then $x_n < x_{n+1}$

Describe the behaviour of the sequence for the three cases.

Solution.

Part (a). Let $L = \lim_{n\to\infty} x_n$. Then $L = \frac{1}{3} \cdot 2^L$. Since $y = x$ and $y = \frac{1}{3} \cdot 2^x$ intersect only at $x = \alpha$ and $x = \beta$, then α and β are the only roots of $x = \frac{1}{3} \cdot 2^x$. Since L is also a root of $x = \frac{1}{3} \cdot 2^x$, L must be either α or β.

Part (b).



If $\alpha < x_n < \beta$, then x_n is decreasing and converges to α. If $x_n < \alpha$, then x_n is increasing and converges to α. If $x_n > \beta$, then x_n is increasing and diverges.

A4 Sequences and Series II

Tutorial A4

Problem 1. True or False? Explain your answers briefly.

(a) $\sum_{r=1}^{n}(2r+3) = \sum_{k=1}^{n}(2k+3)$

(b) $\sum_{r=1}^{n}\left(\frac{1}{r}+5\right) = \sum_{r=1}^{n}\frac{1}{r}+5$

(c) $\sum_{r=1}^{n}\frac{1}{r} = 1/\sum_{r=1}^{n} r$

(d) $\sum_{r=1}^{n} c = \sum_{r=0}^{n-1}(c+1)$

Solution.
Part (a). True: A change in index does not affect the sum.
Part (b). False: In general, $\sum_{r=1}^{n} 5$ is not equal to 5.
Part (c). False: In general, $\sum \frac{a}{b} \neq \sum a / \sum b$.
Part (d). False: Since c is a constant, $\sum_{r=1}^{n} c = nc \neq n(c+1) = \sum_{r=0}^{n-1}(c+1)$.

* * * * *

Problem 2. Write the following series in sigma notation twice, with $r = 1$ as the lower limit in the first and $r = 0$ as the lower limit in the second.

(a) $-2+1+4+\ldots+40$

(b) $a^2+a^4+a^6+\ldots+a^{50}$

(c) $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+n\text{th term}$

(d) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$ to n terms

(e) $\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{28\cdot 30}$

Solution.
Part (a).

$$-2+1+4+\ldots+40 = \sum_{r=1}^{15}(3r-5) = \sum_{r=0}^{14}(3r-2).$$

Part (b).

$$a^2+a^4+a^6+\ldots+a^{50} = \sum_{r=1}^{25} a^{2r} = \sum_{r=0}^{24} a^{2r+2}.$$

Part (c).

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+n\text{th term} = \sum_{r=1}^{n}\frac{1}{2r+1} = \sum_{r=0}^{n-1}\frac{1}{2r+3}.$$

Part (d).

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots \text{ to } n \text{ terms}=\sum_{r=1}^{n}\left(-\frac{1}{2}\right)^{r-1}=\sum_{r=0}^{n-1}\left(-\frac{1}{2}\right)^{r}.$$

Part (e).

$$\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{28\cdot 30}=\sum_{r=1}^{27}\frac{1}{(r+1)(r+3)}=\sum_{r=0}^{26}\frac{1}{(r+2)(r+4)}.$$

* * * * *

Problem 3. Without using the G.C., evaluate the following sums.

(a) $\sum_{r=1}^{50}(2r-7)$

(b) $\sum_{r=1}^{a}(1-a-r)$

(c) $\sum_{r=2}^{n}\left(\ln r+3^r\right)$

(d) $\sum_{r=1}^{\infty}\left(\frac{2^r-1}{3^r}\right)$

Solution.

Part (a).

$$\sum_{r=1}^{50}(2r-7)=2\sum_{r=1}^{50}r-7\sum_{r=1}^{50}1=2\left(\frac{50\cdot 51}{2}\right)-7(50)=2200.$$

Part (b).

$$\sum_{r=1}^{a}(1-a-r)=(1-a)\sum_{r=1}^{a}1-\sum_{r=1}^{a}r=(1-a)a-\frac{a(a+1)}{2}=\frac{a}{2}(1-3a).$$

Part (c).

$$\sum_{r=2}^{n}\left(\ln r+3^r\right)=\sum_{r=2}^{n}\ln r+\sum_{r=2}^{n}3^r=\ln n!+3^2\left(\frac{1-3^{n-2+1}}{1-3}\right)=\ln n!+\frac{9}{2}\left(3^{n-1}-1\right).$$

Part (d).

$$\sum_{r=1}^{\infty}\left(\frac{2^r-1}{3^r}\right)=\sum_{r=1}^{\infty}\left(\frac{2}{3}\right)^r-\sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^r=\frac{2/3}{1-2/3}-\frac{1/3}{1-1/3}=\frac{3}{2}.$$

* * * * *

Problem 4. The nth term of a series is $2^{n-2}+3n$. Find the sum of the first N terms.

Solution.

$$\begin{aligned}\sum_{n=1}^{N}\left(2^{n-2}+3r\right)&=\sum_{n=1}^{N}2^{n-2}+3\sum_{n=1}^{N}n\\&=2^{1-2}\left(\frac{\left(2^N-1\right)}{2-1}\right)+3\left(\frac{N(N+1)}{2}\right)\\&=\frac{1}{2}\left(2^N+3N^2+3N-1\right).\end{aligned}$$

Problem 5. The rth term, u_r, of a series is given by $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$. Express $\sum_{r=1}^{n} u_r$ in the form $A\left(1 - \frac{B}{27^n}\right)$, where A and B are constants. Deduce the sum to infinity of the series.

Solution. Observe that

$$u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1} = 12\left(\frac{1}{3}\right)^{3r} = 12\left(\frac{1}{27}\right)^{r}.$$

Hence,

$$\sum_{r=1}^{n} = 12 \cdot \frac{1}{27}\left(\frac{1 - 1/27^n}{1 - 1/27}\right) = \frac{6}{13}\left(1 - \frac{1}{27^n}\right),$$

whence $A = \frac{6}{13}$ and $B = 1$. In the limit as $n \to \infty$, $\frac{1}{27^n} \to 0$. Hence, the sum to infinity is $\frac{6}{13}$.

* * * * *

Problem 6. The rth term, u_r, of a series is given by $u_r = \ln\frac{r}{r+1}$. Find $\sum_{r=1}^{n} u_r$ in terms of n. Comment on whether the series converges.

Solution. Observe that $u_r = \ln\frac{r}{r+1} = \ln r - \ln(r+1)$. Hence,

$$\begin{aligned}\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n} (\ln r - \ln(r+1)) \\ &= [\ln 1 - \ln 2] + [\ln 2 - \ln 3] + \cdots + [\ln n - \ln(n+1)] \\ &= \ln 1 - \ln(n+1) = \ln\frac{1}{n+1}.\end{aligned}$$

As $n \to \infty$, $\ln\frac{1}{n+1} \to \ln 0$. Hence, the series diverges to negative infinity.

* * * * *

Problem 7. Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, without using the G.C., find the following sums.

(a) $\sum_{r=0}^{n}[r(r+4)+n]$

(b) $\sum_{r=n+1}^{2n}(2r-1)^2$

(c) $\sum_{r=-15}^{20} r(r-2)$

Solution.

Part (a).

$$\begin{aligned}\sum_{r=0}^{n}[r(r+4)+n] &= \sum_{r=0}^{n}\left(r^2 + 4r + n\right) \\ &= \frac{n}{6}(n+1)(2n+1) + 4\left[\frac{n(n+1)}{2}\right] + n(n+1) \\ &= \frac{n}{6}(n+1)(2n+19).\end{aligned}$$

Part (b).

$$\begin{aligned}\sum_{r=n+1}^{2n}(2r-1)^2 &= \sum_{r=1}^{n}(2(r+n)-1)^2 = \sum_{r=1}^{n}\left(4r^2+4(2n-1)r+(2n-1)^2\right)\\ &= 4\left[\frac{n}{6}(n+1)(2n+1)\right]+4(2n-1)\left[\frac{n(n+1)}{2}\right]+(2n-1)^2n\\ &= \frac{1}{3}n\left(28n^2-1\right)\end{aligned}$$

Part (c).

$$\begin{aligned}\sum_{r=-15}^{20}r(r-2) &= \sum_{r=1}^{36}(r-16)[(r-16)-2] = \sum_{r=1}^{36}\left(r^2-34r+288\right)\\ &= \frac{36}{6}\left[(36+1)(2\cdot 36+1)\right]-34\left[\frac{36\cdot 37}{2}\right]+288(36)\\ &= 3930\end{aligned}$$

* * * * *

Problem 8. Let $S=\sum_{r=0}^{\infty}\frac{(x-2)^r}{3^r}$ where $x\neq 2$. Find the range of values of x such that the series S converges. Given that $x=1$, find

(a) the value of S

(b) S_n, in terms of n, where $S_n=\sum_{r=0}^{n-1}\frac{(x-2)^r}{3^r}$

(c) the least value of n for which $|S_n-S|$ is less than 0.001% of S

Solution. Note that

$$S=\sum_{r=0}^{\infty}\frac{(x-2)^r}{3^r}=\sum_{r=0}^{\infty}\left(\frac{x-2}{3}\right)^r.$$

Hence, for S to converge, we must have $\left|\frac{x-2}{3}\right|<1$, which gives $-1<x<5$, $x\neq 2$.

Part (a). When $x=1$, we get

$$S=\sum_{r=0}^{\infty}\left(-\frac{1}{3}\right)^r=\frac{1}{1-(-\frac{1}{3})}=\frac{3}{4}.$$

Part (b). We have

$$S_n=\sum_{r=0}^{n-1}\left(-\frac{1}{3}\right)^r=\frac{1-(-\frac{1}{3})^n}{1-(-\frac{1}{3})}=\frac{3}{4}\left[1-\left(-\frac{1}{3}\right)^n\right].$$

Part (c). Observe that

$$|S_n-S|<0.001\%S \implies \left|\frac{S_n-S}{S}\right|<\frac{1}{100000} \implies \left|\frac{\frac{3}{4}(1-(-\frac{1}{3})^n)}{\frac{3}{4}}-1\right|<\frac{1}{100000}.$$

Using G.C., the least value of n that satisfies the above inequality is 11.

Problem 9. Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$,

(a) write down $\sum_{r=1}^{2k} r^2$ in terms of k

(b) find $2^2 + 4^2 + 6^2 + \ldots + (2k)^2$.

Hence, show that $\sum_{r=1}^{k}(2r-1)^2 = \frac{k}{3}(2k+1)(2k-1)$.

Solution.

Part (a).

$$\sum_{r=1}^{2k} r^2 = \frac{2k}{6}(2k+1)(2(2k)+1) = \frac{k}{3}(2k+1)(4k+1).$$

Part (b).

$$2^2 + 4^2 + 6^2 + \ldots + (2k)^2 = \sum_{r=1}^{k}(2r)^2 = \sum_{r=1}^{k} 4r^2 = \frac{2k}{3}(k+1)(2k+1).$$

From parts (a) and (b), we clearly have

$$\sum_{r=1}^{k}(2r-1)^2 = \sum_{r=1}^{2k} r^2 - \sum_{r=1}^{k}(2r)^2 = \frac{k}{3}(2k+1)(4k+1) - \frac{2k}{3}(k+1)(2k+1) = \frac{k}{3}(2k+1)(2k-1).$$

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Problem 10. Given that $u_n = e^{nx} - e^{(n+1)x}$, find $\sum_{n=1}^{N} u_n$ in terms of N and x. Hence, determine the set of values of x for which the infinite series $u_1 + u_2 + u_3 + \ldots$ is convergent and give the sum to infinity for cases where this exists.

Solution.

$$\sum_{n=1}^{N} u_n = \left(e^x - e^{2x}\right) + \left(e^{2x} - e^{3x}\right) + \cdots + \left(e^{Nx} + e^{(N+1)x}\right) = e^x - e^{(N+1)x}.$$

For the infinite series to converge, we require $|e^x| < 1$. Hence, $x \in \mathbb{R}_0^-$.

We now consider the sum to infinity.

Case 1. Suppose $x = 0$. Then $e^x = 1$, whence the sum to infinity is clearly 0.

Case 2. Suppose $x < 0$. Then $\lim_{N\to\infty} e^{(N+1)x} \to 0$. Thus, the sum to infinity is e^x.

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Problem 11. Given that r is a positive integer and $f(r) = \frac{1}{r^2}$, express $f(r) - f(r+1)$ as a single fraction. Hence, prove that $\sum_{r=1}^{4n}\left(\frac{2r+1}{r^2(r+1)^2}\right) = 1 - \frac{1}{(4n+1)^2}$. Give a reason why the series is convergent and state the sum to infinity. Find $\sum_{r=2}^{4n}\left(\frac{2r-1}{r^2(r-1)^2}\right)$.

Solution.

$$f(r) - f(r+1) = \frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}.$$

$$\begin{aligned}\sum_{r=1}^{4n}\left(\frac{2r+1}{r^2(r+1)^2}\right) &= \sum_{r=1}^{4n}[f(r) - f(r+1)] \\ &= [f(1) - f(2)] + [f(2) - f(3)] + \cdots + [f(4n) - f(4n-1)] \\ &= f(1) - f(4n+1) = 1 - \frac{1}{(4n+1)^2}\end{aligned}$$

As $n \to \infty$, $\frac{1}{(4n+1)^2} \to 0$. Hence, the series converges to 1.

$$\begin{aligned}\sum_{r=2}^{4n}\left(\frac{2r-1}{r^2(r-1)^2}\right) = \sum_{r=1}^{4n-1}\left(\frac{2r+1}{r^2(r+1)^2}\right) &= \sum_{r=1}^{4n-1}[f(r)-f(r+1)]\\ &= [f(1)-f(2)]+[f(2)-f(3)]+\cdots+[f(4n-1)-f(4n)]\\ &= 1-f(4n) = 1-\frac{1}{16n^2}\end{aligned}$$

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Problem 12.

(a) Express $\frac{1}{(2x+1)(2x+3)(2x+5)}$ in partial fractions.

(b) Hence, show that $\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$.

(c) Deduce the sum of $\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{3\cdot5\cdot7\cdot9} + \ldots + \frac{1}{41\cdot43\cdot45}$.

Solution.

Part (a). Using the cover-up rule, we obtain

$$\frac{1}{(2x+1)(2x+3)(2x+5)} = \frac{1}{8(2x+1)} - \frac{1}{4(2x+3)} + \frac{1}{8(2x+5)}.$$

Part (b).

$$\begin{aligned}\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} &= \sum_{r=1}^{n}\left(\frac{1}{8(2r+1)} - \frac{1}{4(2r+3)} + \frac{1}{8(2r+5)}\right)\\ &= \frac{1}{8}\left[\left(\sum_{r=1}^{n}\frac{1}{2r+1} - \sum_{r=1}^{n}\frac{1}{2r+3}\right) - \left(\sum_{r=1}^{n}\frac{1}{2r+3} - \sum_{r=1}^{n}\frac{1}{2r+5}\right)\right]\end{aligned}$$

Observe that the two terms in brackets clearly telescope, leaving us with

$$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{8}\left[\left(\frac{1}{3} - \frac{1}{2n+3}\right) - \left(\frac{1}{5} - \frac{1}{2n+5}\right)\right],$$

which simplifies to

$$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$$

as desired.

Part (c).

$$\begin{aligned}&\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{3\cdot5\cdot7\cdot9} + \ldots + \frac{1}{41\cdot43\cdot45}\\ &= \frac{1}{1\cdot3\cdot5} + \sum_{r=1}^{20}\frac{1}{(2r+1)(2r+3)(2r+5)}\\ &= \frac{1}{15} + \left(\frac{1}{60} - \frac{1}{4(2\cdot20+3)(2\cdot20+5)}\right)\\ &= \frac{161}{1935}.\end{aligned}$$

Self-Practice A4

Problem 1. Evaluate $\sum_{r=2}^{n}\left(2^{-r}+2nr+n^2\right)$, giving your answer in terms of n.

Solution. Splitting the sum, we get

$$\sum_{r=2}^{n}\left(2^{-r}+2nr+n^2\right)=\sum_{r=2}^{n}\left(\frac{1}{2}\right)^{r}+2n\sum_{r=2}^{n}r+n^2\sum_{r=2}^{n}1.$$

Hence,

$$\begin{aligned}\sum_{r=2}^{n}\left(2^{-r}+2nr+n^2\right)&=\left(\frac{1}{2}\right)^{2}\left(\frac{1-(1/2)^{n-1}}{1-1/2}\right)+2n\left(\frac{n(n+1)}{2}-1\right)+n^2\left(n-1\right)\\&=\left[\frac{1}{2}-\left(\frac{1}{2}\right)^{n}\right]+n^2\left(n+1\right)-2n+n^2\left(n-1\right)\\&=\frac{1}{2}-\left(\frac{1}{2}\right)^{n}+2n\left(n^2-1\right).\end{aligned}$$

* * * * *

Problem 2. A geometric sequence $\{a_n\}$ has first term a and common ratio r. The sequence of numbers $\{b_n\}$ satisfy the relation $b_n=\ln(a_n)$ for $n\in\mathbb{Z}^+$.

(a) Show that $\{b_n\}$ is an arithmetic sequence and determine the value of the common difference in terms of r.

(b) Find an expression for $\sum_{n=1}^{N+1}b_n$ in terms of a, a_{N+1} and N.

(c) Hence, obtain an expression for $a_1\times a_2\times\cdots\times a_{N+1}$ in terms of a, a_{N+1} and N.

Solution.

Part (a). Note that $a_n=ar^{n-1}$. Hence,

$$b_n=\ln a_n=\ln\left(ar^{n-1}\right)=\ln a+(n-1)\ln r.$$

Hence,

$$b_n-b_{n-1}=[\ln a+n\ln r]-[\ln a+(n-1)\ln r]=\ln r.$$

Thus, $\{b_n\}$ is an arithmetic progression with common difference $\ln r$.

Part (b). Since $\{b_n\}$ is in arithmetic progression, we have

$$\sum_{n=1}^{N+1}b_n=\frac{N+1}{2}\left(b_1+b_{N+1}\right)=\frac{N+1}{2}\left(\ln a_1+\ln a_{N+1}\right)=\frac{N+1}{2}\ln(aa_{N+1}).$$

Part (c). Since $b_n=\ln a_n$, we can write the sum as

$$\sum_{n=1}^{N+1}b_n=\sum_{n=1}^{N+1}\ln a_n=\ln\prod_{n=1}^{N+1}a_n.$$

Equating this with the above result yields

$$\prod_{n=1}^{N+1}a_n=\left(aa_{N+1}\right)^{(N+1)/2}.$$

Problem 3. It is given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

(a) Show that $\sum_{r=1}^{n}(2r-7)(r+1) = \frac{1}{6}n\left(4n^2-9n-55\right)$.

(b) Find $\sum_{r=1}^{n} 3^{-r}$ in terms of n, and find the least value of n such that

$$\sum_{r=1}^{n}(2r-7)(r+1) > \sum_{r=1}^{n} 3^{-r}.$$

(c) Express $\sum_{r=n+1}^{2n}(2r-7)(r+1)$ in terms of n.

(d) Hence, or otherwise, find the value of

$$43 \times 26 + 45 \times 27 + 47 \times 28 + \cdots + 87 \times 48 + 89 \times 49.$$

Solution.

Part (a). Note that $(2r-7)(r+1) = 2r^2 - 5r - 7$. Hence,

$$\begin{aligned}\sum_{r=1}^{n}(2r-7)(r+1) &= \sum_{r=1}^{n}\left(2r^2-5r-7\right)\\ &= 2\left(\frac{n(n+1)(2n+1)}{6}\right) - 5\left(\frac{n(n+1)}{2}\right) - 7n\\ &= \frac{n\left(4n^2-9n-55\right)}{6}.\end{aligned}$$

Part (b).

$$\sum_{r=1}^{n} 3^{-r} = \sum_{r=1}^{n}\left(\frac{1}{3}\right)^r = \left(\frac{1}{3}\right)\left(\frac{1-(1/3)^{n-1}}{1-1/3}\right) = \frac{1}{2}\left(1-\frac{1}{3^n}\right).$$

The inequality hence becomes

$$\frac{n\left(4n^2-9n-55\right)}{6} > \frac{1}{2}\left(1-\frac{1}{3^n}\right).$$

Using G.C., $n \geq 5.019$. Since n is an integer, the least n that satisfies the inequality is 6.

Part (c). We have

$$\begin{aligned}\sum_{r=n+1}^{2n}(2r-7)(r+1) &= \sum_{r=1}^{2n}(2r-7)(r+1) - \sum_{r=1}^{n}(2r-7)(r+1)\\ &= \frac{2n\left[4(2n)^2-9(2n)-55\right]}{6} - \frac{n\left(4n^2-9n-55\right)}{6}\\ &= \frac{n\left(28n^2-27n-55\right)}{6}.\end{aligned}$$

Part (d). We have

$$43 \times 26 + 45 \times 27 + 47 \times 28 + \cdots + 87 \times 48 + 89 \times 49 = \sum_{r=24+1}^{2(24)}(2r-7)(r+1)$$

$$= \frac{(24)\left[28(24)^2-27(24)-55\right]}{6} = 61700.$$

Problem 4. It is given that $\sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} = \frac{n(5n+7)}{4(n+1)(n+2)}$.

(a) Show that the series $\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)}$ converges and write down its sum to infinity.

(b) Find $\sum_{r=0}^{n-2} \frac{2r+5}{(r+2)(r+3)(r+4)}$.

Solution.

Part (a). Clearly,

$$\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)} = \lim_{n\to\infty} \frac{n(5n+7)}{4(n+1)(n+2)} = \lim_{n\to\infty} \frac{5+\frac{7}{n}}{4\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{5}{4}.$$

Thus, the series converges and its sum to infinity is $5/4$.

Part (b). Reindexing $r \mapsto r-2$,

$$\sum_{r=0}^{n-2} \frac{2r+5}{(r+2)(r+3)(r+4)} = \sum_{r=2}^{n} \frac{2r+1}{r(r+1)(r+2)}$$

$$= \sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} - \frac{2(1)+1}{1(1+1)(1+2)} = \frac{n(5n+7)}{4(n+1)(n+2)} - \frac{1}{2}.$$

Assignment A4

Problem 1. Find $\sum_{r=0}^{n}\left(n^2+1-3r\right)$ in terms of n, giving your answer in factorized form.

Solution.

$$\sum_{r=0}^{n}\left(n^2+1-3r\right)=(n+1)(n^2+1)-3\left[\frac{n(n+1)}{2}\right]=\frac{1}{2}(n+1)\left(2n^2-3n+2\right).$$

* * * * *

Problem 2. Given that $\sum_{k=1}^{n}k!\left(k^2+1\right)=(n+1)!\,n$, find $\sum_{k=1}^{n-1}(k+1)!\left(k^2+2k+2\right)$.

Solution. Reindexing $k+1\mapsto k$,

$$\sum_{k=1}^{n-1}(k+1)!\left(k^2+2k+2\right)=\sum_{k=2}^{n}k!\left(k^2+1\right).$$

Using the given result,

$$\sum_{k=2}^{n}k!\left(k^2+1\right)=\sum_{k=1}^{n}k!\left(k^2+1\right)-1!\left(1^2+1\right)=(n+1)!\,n-2.$$

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Problem 3. Given that $\sum_{r=1}^{n}=\frac{1}{6}n(n+1)(2n+1)$, find $\sum_{r=N+1}^{2N}\left(7^{r+1}+3r^2\right)$ in terms of N, simplifying your answer.

Solution. Note that

$$\sum_{r=N+1}^{2N}7^{r+1}=\frac{7^{(N+1)+1}(7^N-1)}{7-1}=\frac{7^{N+2}(7^N-1)}{6}.$$

Next, we split the sum of squares:

$$\sum_{r=N+1}^{2N}3r^2=3\left(\sum_{r=1}^{2N}r^2-\sum_{r=1}^{N}r^2\right).$$

Using the given result,

$$\sum_{r=N+1}^{2N}3r^2=3\left(\frac{(2N)(2N+1)(4N+1)}{6}-\frac{N(N+1)(2N+1)}{6}\right)=\frac{N(2N+1)(7N+1)}{2}.$$

Thus,

$$\sum_{r=N+1}^{2N}\left(7^{r+1}+3r^2\right)=\frac{7^{N+2}(7^N-1)}{6}+\frac{N(2N+1)(7N+1)}{2}.$$

Problem 4. Let $f(r) = \frac{3}{r-1}$.

(a) Show that $f(r+1) - f(r) = -\frac{3}{r(r-1)}$.

(b) Hence, find in terms of N, the sum of the series $S_N = \sum_{r=2}^{N} \frac{1}{r(r-1)}$.

(c) Explain why $\sum_{r=2}^{\infty} \frac{1}{r(r-1)}$ is a convergent series, and find the value of the sum to infinity.

(d) Using the result from part (b), find $\sum_{r=2}^{N} \frac{1}{r(r+1)}$.

Solution.

Part (a).

$$f(r+1) - f(r) = \frac{3}{(r+1)-1} - \frac{3}{r-1} = \frac{3(r-1) - 3r}{r(r-1)} = -\frac{3}{r(r-1)}.$$

Part (b). Observe that

$$S_N = \sum_{r=2}^{N} \frac{1}{r(r-1)} = -\frac{1}{3}\sum_{r=2}^{N} -\frac{3}{r(r-1)} = -\frac{1}{3}\left[\sum_{r=2}^{N} f(r+1) - \sum_{r=2}^{N} f(r)\right],$$

which clearly telescopes. Thus,

$$S_N = -\frac{f(N+1) - f(2)}{3} = -\frac{1}{3}\left(\frac{3}{N+1-1} - \frac{3}{2-1}\right) = 1 - \frac{1}{N}.$$

Part (c).

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1 - \frac{1}{N}\right) = 1 - 0 = 1.$$

Since 1 is a constant, $\sum_{r=2}^{\infty} \frac{1}{r(r-1)}$ is a convergent series.

Part (d). Reindexing $r \mapsto r-1$,

$$\sum_{r=2}^{N} \frac{1}{r(r+1)} = \sum_{r=3}^{N+1} \frac{1}{(r-1)r} = \sum_{r=2}^{N} \frac{1}{r(r-1)} - \frac{1}{2(2-1)} + \frac{1}{(N+1)N}.$$

Using the result from part (b),

$$\sum_{r=2}^{N} \frac{1}{r(r+1)} = \left(1 - \frac{1}{N}\right) - \frac{1}{2(2-1)} + \frac{1}{(N+1)N} = \frac{1}{2} - \frac{1}{N+1}.$$

A5 Recurrence Relations

Tutorial A5

Problem 1. Solve these recurrence relations together with the initial conditions.

(a) $u_n = 2u_{n-1}$, for $n \geq 1$, $u_0 = 3$

(b) $u_n = 3u_{n-1} + 7$, for $n \geq 1$, $u_0 = 5$

Solution.
Part (a). $u_n = 2^n \cdot u_0 = 3 \cdot 2^n$.
Part (b). Let k be a constant such that $u_n + k = 3(u_{n-1} + k)$. Then $k = \frac{7}{2}$. Hence,

$$u_n + \frac{7}{2} = 3\left(u_{n-1} + \frac{7}{2}\right) \implies u_n + \frac{7}{2} = 3^n\left(u_0 + \frac{7}{2}\right) \implies u_n = \frac{17}{2} \cdot 3^n - \frac{7}{2}.$$

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Problem 2. Solve these recurrence relations together with the initial conditions.

(a) $u_n = 5u_{n-1} - 6u_{n-2}$, for $n \geq 2$, $u_0 = 1$, $u_1 = 0$

(b) $u_n = 4u_{n-2}$, for $n \geq 2$, $u_0 = 0$, $u_1 = 4$

(c) $u_n = 4u_{n-1} - 4u_{n-2}$, for $n \geq 2$, $u_0 = 6$, $u_1 = 8$

(d) $u_n = -6u_{n-1} - 9u_{n-2}$, for $n \geq 2$, $u_0 = 3$, $u_1 = -3$

(e) $u_n = 2u_{n-1} - 2u_{n-2}$, for $n \geq 2$, $u_0 = 2$, $u_1 = 6$

Solution.
Part (a). Note that the characteristic equation of u_n, $x^2 - 5x + 6 = 0$, has roots 2 and 3. Thus,

$$u_n = A \cdot 2^n + B \cdot 3^n.$$

From $u_0 = 1$ and $u_1 = 0$, we have the equations $A + B = 1$ and $2A + 3B = 0$. Solving, we see that $A = 3$ and $B = 2$, whence

$$u_n = 3 \cdot 2^n + 2 \cdot 3^n.$$

Part (b). Note that the characteristic equation of u_n, $x^2 - 4 = 0$, has roots -2 and 2. Thus,

$$u_n = A(-2)^n + B \cdot 2^n.$$

From $u_0 = 0$ and $u_1 = 4$, we get $A + B = 0$ and $-2A + 2B = 4$. Solving, we see that $A = -1$ and $B = 1$, whence

$$u_n = -(-2)^n + 2^n.$$

Part (c). Note that the characteristic equation of u_n, $x^2 - 4x + 4 = 0$, has only one root, 2. Thus,

$$u_n = (A + Bn)2^n.$$

From $u_0 = 6$ and $u_1 = 8$, we obtain $A = 6$ and $A + B = 4$, whence $B = -2$. Thus,

$$u_n = (6 - 2n)2^n.$$

Part (d). Note that the characteristic equation of u_n, $x^2 + 6x + 9 = 0$, has only one root, -3. Thus,

$$u_n = (A + Bn)(-3)^n.$$

From $u_0 = 3$ and $u_1 = -3$, we get $A = 3$ and $A + B = 1$, whence $B = -2$. Thus,

$$u_n = (3 - 2n)2^n.$$

Part (e). Consider the characteristic equation of u_n, $x^2 - 2x + 2 = 0$. By the quadratic formula, this has roots $x = 1 \pm i = \sqrt{2}\exp\left(\pm\frac{i\pi}{4}\right)$. Hence,

$$u_n = A \cdot 2^{\frac{1}{2}n}\cos\left(\frac{n\pi}{4}n\right) + B \cdot 2^{\frac{1}{2}n}\sin\left(\frac{n\pi}{4}\right).$$

From $u_0 = 2$, we obtain $A = 2$. From $u_0 = 6$, we obtain $A + B = 6$, whence $B = 4$. Thus,

$$u_n = 2^{\frac{1}{2}n+1}\cos\left(\frac{n\pi}{4}\right) + 2^{\frac{1}{2}n+2}\sin\left(\frac{n\pi}{4}\right).$$

* * * * *

Problem 3.

(a) A sequence is defined by the formula $b_n = \frac{n!n!}{(2n)!} \cdot 2^n$, where $n \in \mathbb{Z}^+$. Show that the sequence satisfies the recurrence relation $b_{n+1} = \frac{n+1}{2n+1} b_n$.

(b) A sequence is defined recursively by the formula

$$u_{n+1} = 2u_n + 3, \qquad n \in \mathbb{Z}_0^+,\ u_0 = a$$

Show that $u_n = 2^n a + 3\left(2^n - 1\right)$.

Solution.

Part (a).

$$b_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot 2^{n+1} = \frac{2(n+1)^2}{(2n+1)(2n+2)}\left[\frac{n!n!}{(2n)!} \cdot 2^n\right] = \frac{n+1}{2n+1} b_n.$$

Part (b). Let k be a constant such that $u_{n+1} + k = 2(u_n + k)$. Then $k = 3$. Hence,

$$u_{n+1} + 3 = 2(u_n + 3) \implies u_n + 3 = 2^n(u_0 + 3) \implies u_n = 2^n(a+3) - 3 = 2^n a + 3\left(2^n - 1\right).$$

Problem 4. The volume of water, in litres, in a storage tank decreases by 10% by the end of each day. However, 90 litres of water is also pumped into the tank at the end of each day. The volume of water in the tank at the end of n days is denoted by x_n and x_0 is the initial volume of water in the tank.

(a) Write down a recurrence relation to represent the above situation.

(b) Show that $x_n = 0.9^n(x_0 - 900) + 900$.

(c) Deduce the amount of water in the tank when n becomes very large.

Solution.

Part (a). $x_{n+1} = 0.9x_n + 90$, $n \in \mathbb{N}$

Part (b). Let k be a constant such that $x_{n+1} + k = 0.9(x_n + k)$. Then $k = -900$. Hence,

$$x_{n+1} - 900 = 0.9(x_n - 900) \implies x_n - 900 = 0.9^n(x_0 - 900) \implies x_n = 0.9^n(x_0 - 900) + 900.$$

Part (c). As $n \to \infty$, $0.9^n \to 0$. Hence, the amount of water in the tank will converge to 900 litres.

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Problem 5. A deposit of $100,000 is made to an investment fund at the beginning of a year. On the last day of each year, two dividends are awarded and reinvested into the fund. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

(a) Find a recurrence relation $\{P_n\}$ where P_n is the amount at the start of the nth year if no money is ever withdrawn.

(b) How much is in the account after n years if no money is ever withdrawn?

Solution.

Part (a).

$$P_{n+2} = P_{n+1} + 0.2P_{n+1} + 0.45P_n = 1.2P_{n+1} + 0.45P_n.$$

Part (b). Note that the characteristic equation of P_n, $x^2 - 1.2x - 0.45 = 0$, has roots $-\frac{3}{10}$ and $\frac{3}{2}$. Thus,

$$P_n = A\left(-\frac{3}{10}\right)^n + B\left(\frac{3}{2}\right)^n.$$

From $P_0 = 0$ and $P_1 = 100000$, we have $A + B = 0$ and $-\frac{3}{10}A + \frac{3}{2}B = 100000$. Solving, we have $A = -\frac{500000}{9}$ and $B = \frac{500000}{9}$. Thus,

$$P_n = \frac{500000}{9}\left[\left(\frac{3}{2}\right)^n - \left(-\frac{3}{10}\right)^n\right].$$

Hence, there will be $\$\left\{\frac{500000}{9}\left[\left(\frac{3}{2}\right)^n - \left(-\frac{3}{10}\right)^n\right]\right\}$ in the account after n years if no money is ever withdrawn

Problem 6. A pair of rabbits does not breed until they are two months old. After they are two months old, each pair of rabbit produces another pair each month.

(a) Find a recurrence relation $\{f_n\}$ where f_n is the total number of pairs of rabbits, assuming that no rabbits ever die.

(b) What is the number of pairs of rabbits at the end of the nth month, assuming that no rabbits ever die?

Solution.

Part (a). $f_{n+2} = f_{n+1} + f_n$, $n \geq 2$, $f_0 = 0, f_1 = 1$

Part (b). Consider the characteristic equation of f_n, $x^2 - x - 1 = 0$. By the quadratic formula, the roots of the characteristic equation are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Hence,

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

From $f_0 = 0$, we get $A + B = 0$. From $f_1 = 1$, we get $A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$. Solving, we get $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. Hence,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

* * * * *

Problem 7. For $n \in \{2^j : j \in \mathbb{Z}, j \geq 1\}$, it is given that $T_n = 3T_{n/2} + 17$, where $T_1 = 4$. By considering the substitution $n = 2^i$ and another suitable substitution, show that the recurrence relation can be expressed in the form

$$t_i = 3t_{i-1} + 17, \qquad i \in \mathbb{Z}^+$$

Hence, find an expression for T_n in terms of n.

Solution. Let $n = 2^i \iff i = \log_2 n$. The given recurrence relation transforms to

$$T_{2^i} = 3T_{2^{i-1}} + 17, T_{2^0} = 4.$$

Let $t_i = T_{2i}$. Then

$$t_i = 3t_{i-1} + 17, t_0 = 4.$$

Let k be a constant such that $t_i + k = 3(t_{i-1} + k)$. Then $k = \frac{17}{2}$. We thus obtain a formula for t_i:

$$t_i + \frac{17}{2} = 3\left(t_{i-1} + \frac{17}{2}\right) \implies t_i + \frac{17}{2} = 3^i\left(t_0 + \frac{17}{2}\right) \implies t_i = \frac{25}{2} \cdot 3^i - \frac{17}{2}.$$

Thus,

$$T_{2i} = \frac{25}{2} \cdot 3^i - \frac{17}{2} \implies T_n = \frac{25}{2} \cdot 3^{\log_2 n} - \frac{17}{2}.$$

Problem 8. Consider the sequence $\{a_n\}$ given by the recurrence relation

$$a_{n+1} = 2a_n + 5^n, \qquad n \geq 1.$$

(a) Given that $a_n = k\,(5^n)$ satisfies the recurrent relation, find the value of the constant k.

(b) Hence, by considering the sequence $\{b_n\}$ where $b_n = a_n - k(5^n)$, find the particular solution to the recurrence relation for which $a_1 = 2$.

Solution.

Part (a).

$$a_{n+1} = 2a_n + 5^n \implies k\left(5^{n+1}\right) = 2 \cdot k\,(5^n) + 5^n \implies 5k = 2k + 1 \implies k = \frac{1}{3}.$$

Part (b).

$$b_n = a_n - \frac{5^n}{3} = \left(2a_{n-1} - 5^{n-1}\right) - \frac{5^n}{3} = 2a_{n-1} - \frac{2}{3} \cdot 5^{n-1} = 2\left(a_{n-1} - \frac{5^{n-1}}{3}\right) = 2b_{n-1}.$$

Hence, $b_n = b_1 \cdot 2^{n-1}$. Note that $b_1 = a_1 - \frac{5}{3} = \frac{1}{3}$. Thus, $b_n = \frac{2^{n-1}}{3}$, which gives

$$b_n = a_n - \frac{5^n}{3} = \frac{2^{n-1}}{3} \implies a_n = \frac{2^n + 2 \cdot 5^n}{6}.$$

* * * * *

Problem 9. The sequence $\{X_n\}$ is given by

$$\sqrt{X_{n+2}} = \frac{X_{n+1}}{X_n^2}, \qquad n \geq 1.$$

By applying the natural logarithm to the recurrence relation, use a suitable substitution to find the general solution of the sequence, expressing your answer in trigonometric form.

Solution. Taking the natural logarithm of the recurrence relation and simplifying, we get

$$\ln X_{n+2} = 2\ln X_{n+1} - 4\ln X_n.$$

Let $L_n = \ln X_n \iff X_n = \exp(L_n)$. Then,

$$L_{n+2} = 2L_{n+1} - 4L_n.$$

Consider the characteristic equation of L_n, $x^2 - 2x + 4 = 0$. By the quadratic formula, this has roots $1 \pm \sqrt{3}\mathrm{i} = 2\exp\left(\pm\frac{i\pi}{3}\right)$. Thus, we can express L_n as

$$L_n = A \cdot 2^n \cos\frac{n\pi}{3} + B \cdot 2^n \sin\frac{n\pi}{3} = 2^n\left(A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right).$$

Thus, X_n has the general solution

$$X_n = \exp\left(2^n\left(A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right)\right).$$

Problem 10. The sequence $\{X_n\}$ is given by $X_1 = 2$, $X_2 = 15$ and

$$X_{n+2} = 5\left(1 + \frac{1}{n+2}\right) X_{n+1} - 6\left(1 + \frac{2}{n+1}\right) X_n, \qquad n \geq 1.$$

By dividing the recurrence relation throughout by $n + 3$, use a suitable substitution to determine X_n as a function of n.

Solution. Dividing the recurrence relation by $n + 3$, we obtain

$$\frac{X_{n+2}}{n+3} = 5\left(\frac{1}{n+3} + \frac{1}{(n+2)(n+3)}\right) X_{n+1} - 6\left(\frac{1}{n+3} + \frac{2}{(n+1)(n+3)}\right) X_n.$$

Note that $\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$ and $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$. Thus,

$$\frac{X_{n+2}}{n+3} = 5\left(\frac{X_{n+1}}{n+2}\right) - 6\left(\frac{X_n}{n+1}\right).$$

Let $Y_n = \frac{n+1}{X_n} \iff X_n = (n+1)Y_n$. Then,

$$Y_{n+2} = 5Y_{n+1} - 6Y_n.$$

Note that the characteristic equation of Y_n, $x^2 - 5x + 6 = 0$, has roots 2 and 3. Hence,

$$Y_n = A \cdot 2^n + B \cdot 3^n \implies X_n = (n+1)\left(A \cdot 2^n + B \cdot 3^n\right).$$

From $X_1 = 2$ and $X_2 = 15$, we have $2A + 3B = 1$ and $4A + 9B = 5$. Solving, we obtain $A = -1$ and $B = 1$. Thus,

$$X_n = (n+1)\left(3^n - 2^n\right).$$

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Problem 11. A logistics company set up an online platform providing delivery services to users on a monthly paid subscription basis. The company's sales manager models the number of subscribers that the company has at the end of each month. She notes that approximately 10% of the existing subscribers leave each month, and that there will be a constant number k of new subscribers in each subsequent month after the first.

Let T_n, $n \geq 1$, denote the number of subscribers the company has at the end of the nth month after the online platform was set up.

(a) Express T_{n+1} in terms of T_n.

The company has 250 subscribers at the end of the first month.

(b) Find T_n in terms of n and k.

(c) Find the least number of subscribers the company needs to attract in each subsequent month after the first if it aims to have at least 350 subscribers by the end of the 12th month.

Let $k = 50$ for the rest of the question.

The monthly running cost of the company is assumed to be fixed at \$4,000. The monthly subscription fee is \$10 per user which is charged at the end of each month.

(d) Given that the mth month is the first month in which the company's revenue up to and including that month is able to cover its cost up to and including that month, find the value of m.

(e) Using your answer to part (b), determine the long-term behaviour of the number of subscribers that the company has. Hence, explain whether this behaviour is appropriate in terms of long-term prospects for the company's success.

Solution.

Part (a). $T_{n+1} = 0.9T_n + k$

Part (b). Let m be a constant such that $T_{n+1} + m = 0.9\,(T_n + m)$. Then $m = -10k$. Hence,

$$T_{n+1} - 10k = 0.9\,(T_n - 10k) \implies T_n - 10k = 0.9^{n-1}\,(T_0 - 10k)\,.$$

Since $T_0 = 250$, we get

$$T_n = 0.9^{n-1}\,(250 - 10k) + 10k.$$

Part (c). Consider $T_{12} \geq 350$.

$$T_{12} \geq 350 \implies 0.9^{12-1}\,(250 - 10k) + 10k \geq 350.$$

Using G.C., $k \geq 39.6$. Hence, the company needs to attract at least 40 subscribers in each subsequent month.

Part (d). Since $k = 50$, $T_n = -250 \cdot 0.9^{n-1} + 500$. Let $\$S_m$ be the total revenue for the first m months.

$$\begin{aligned} S_m &= 10\sum_{n=1}^{m} T_n = 10\sum_{n=1}^{m}\left(-250 \cdot 0.9^{n-1} + 500\right) \\ &= 10\left[-250\left(\frac{1 - 0.9^m}{1 - 0.9}\right) + 500m\right] = 25000\,(0.9^m - 1) + 5000m. \end{aligned}$$

Note that the total cost for the first m months is $\$4000m$. Hence, the total profit for the first m months is given by $\$(S_m - 4000m)$. Hence, we consider $S_m - 4000m \geq 0$:

$$S_m - 4000m \geq 0 \implies 25000\,(0.9^m - 1) + 1000m \geq 0.$$

Using G.C., we obtain $m \geq 22.7$, whence the least value of m is 23.

Part (e). As $n \to \infty$, $0.9^{n-1} \to 0$. Hence, $T_n \to 500$. Hence, as n becomes very large, the profit per month approaches $500 \cdot 10 - 4000 = 1000$ dollars. Thus, this behaviour is appropriate as the business will remain profitable in the long run.

Self-Practice A5

Problem 1. Tom wants to buy a new Aphone11. To save up for his purchase, Tom takes up a part-time job that pays him \$400 per month which will be credited into his bank account on the 25th of each month, starting from January 2012. On the first day of every month of 2012, he withdraws half of the total amount of money from his bank account for food and transportation. Assuming that Tom has \$250 in this bank account on 31 December 2011,

(a) write down a recurrence relation for u_n, where u_n denotes the amount in his bank account on the last day of the nth month after December 2011, and

(b) show that $u_n = 800 - 550\,(0.5^n)$.

Given that a new Aphone 11 costs \$850,

(c) explain why Tom is unable to buy the Aphone11, and

(d) find the maximum percentage of the total amount of money in the bank that Tom should spend on transport and food every month in order to be able to buy the Aphone11 on the last day of December 2012.

Solution.

Part (a). We have

$$u_n = \frac{1}{2}u_{n-1} + 400, \quad u_0 = 250.$$

Part (b). Note that the complementary solution is

$$u_n^{(c)} = C\left(\frac{1}{2}\right)^n,$$

where C is an arbitrary constant. Let the particular solution be $u_n^{(p)} = k$. Then

$$k = \frac{1}{2}k + 400 \implies k = 800.$$

Hence,

$$u_n = u_n^{(c)} + u_n^{(p)} = C\left(\frac{1}{2}\right)^n + 800.$$

Using the condition $u_0 = 250$, we get

$$250 = C + 800 \implies C = -500,$$

whence

$$u_n = 800 - 500\left(\frac{1}{2}\right)^n.$$

Part (c). Clearly, $-500(1/2)^n < 0$ for all $n > 0$. Hence,

$$u_n = 800 - 500\left(\frac{1}{2}\right)^n < 800 < 850.$$

Thus, Tom is unable to buy the Aphone11.

Part (d). Let the desired percentage be $p\%$. Then

$$u_n = \left(1 - \frac{p}{100}\right) u_{n-1} + 400.$$

Let the particular solution be $u_n^{(p)} = k$. Then

$$k = \left(1 - \frac{p}{100}\right) k + 400 \implies k = \frac{40000}{p}.$$

We thus want

$$\frac{40000}{p} \geq 850 \implies p \leq \frac{800}{17} = 47.059.$$

Hence, the maximum percentage is 47%.

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Problem 2. A sequence of real numbers $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation

$$u_n = 2u_{n-1} + 1, \quad n \geq 1.$$

Given that $u_1 = 2$, show that $u_n = 2^n + 2^{n-1} - 1$. Hence, determine the behaviour of the sequence.

Solution. Note that the complementary solution is

$$u_n^{(c)} = C2^n,$$

where C is an arbitrary constant. Let the particular solution be $u_n^{(p)} = k$. Then

$$k = 2k + 1 \implies k = -1.$$

Hence,

$$u_n = u_n^{(c)} + u_n^{(p)} = C2^n - 1.$$

Using the condition $u_1 = 2$, we get

$$2 = 2C - 1 \implies C = \frac{3}{2},$$

whence

$$u_n = \frac{3}{2} \cdot 2^n - 1 = (2+1)2^{n-1} - 1 = 2^n + 2^{n-1} - 1.$$

Clearly, u_n is increasing and diverges to infinity.

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Problem 3. Solve these recurrence relations together with the initial conditions.

(a) $u_n = 7u_{n-1} - 10u_{n-2}$ for $n \geq 2$, $u_0 = 2$, $u_1 = 1$.

(b) $u_n = \frac{1}{4}u_{n-2}$ for $n \geq 2$, $u_0 = 1$, $u_1 = 0$.

(c) $u_n = -4u_{n-1} - 4u_{n-2}$ for $n \geq 2$, $u_0 = 0$, $u_1 = 1$.

(d) $u_{n+2} = -4u_{n+1} + 5u_n$ for $n \geq 0$, $u_0 = 2$, $u_1 = 8$.

Solution.

Part (a). Consider the characteristic equation $x^2 - 7x + 10 = 0$, which has distinct roots $x = 2$ and $x = 5$. Hence,

$$u_n = A\left(2^n\right) + B\left(5^n\right).$$

Using the conditions $u_0 = 2$ and $u_1 = 1$, we get the system

$$\begin{cases} A + B = 2 \\ 2A + 5B = 1 \end{cases},$$

whence $A = 3$ and $B = -1$. Thus,

$$u_n = 3\left(2^n\right) - 5^n.$$

Part (b). Consider the characteristic equation $x^2 = 1/4$, which has distinct roots $x = \pm 1/2$. Hence,

$$u_n = A\left(\frac{1}{2}\right)^n + B\left(-\frac{1}{2}\right)^n = \frac{1}{2^n}\left[A + (-1)^n B\right].$$

Using the conditions $u_0 = 1$ and $u_1 = 0$, we get the system

$$\begin{cases} A - B = 1 \\ A + B = 0 \end{cases},$$

whence $A = 1/2$ and $B = -1/2$. Thus,

$$u_n = \frac{1}{2^n}\left[\frac{1}{2} + (-1)^n\left(-\frac{1}{2}\right)\right] = \frac{1 + (-1)^{n-1}}{2^{n+1}}.$$

Part (c). Consider the characteristic equation $x^2 - 4x + 4 = 0$, which has the unique root $x = -2$. Hence,

$$u_n = (A + Bn)(-2)^n.$$

Using the conditions $u_0 = 0$ and $u_1 = 1$, we get the system

$$\begin{cases} A = 0 \\ 2A - 2B = 1 \end{cases},$$

whence $A = 0$ and $B = -1/2$. Thus,

$$u_n = \left(0 - \frac{n}{2}\right)(-2)^n = n(-2)^{n-1}.$$

Part (d). Consider the characteristic equation $x^2 + 4x - 5 = 0$, which has distinct roots $x = -5$ and $x = 1$. Hence,

$$u_n = A(-5)^n + B(1)^n = A(-5)^n + B.$$

Using the conditions $u_0 = 2$ and $u_1 = 8$, we get the system

$$\begin{cases} A + B = 2 \\ 5A + B = 8 \end{cases},$$

whence $A = -1$ and $B = 3$. Thus,

$$u_n = 3 - (-5)^n.$$

Problem 4 (🌶). Find the unit digit of the number $(3+\sqrt{5})^{2016}+(3-\sqrt{5})^{2016}$.

Solution. Let u_n be a sequence such that

$$u_n = \left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n.$$

We aim to find a recurrence relation for u_n. First, observe that $3+\sqrt{5}$ and $3-\sqrt{5}$ are roots to the characteristic polynomial $P(x)$ of u_n:

$$P(x) = \left[x-\left(3+\sqrt{5}\right)\right]\left[x-\left(3-\sqrt{5}\right)\right] = x^2-6x+4.$$

Thus, u_n satisfies the recurrence relation

$$u_n = 6u_{n-1} - 4u_{n-2}.$$

Since we are interested in the unit digit of u_{2016}, we consider $u_n \pmod{10}$:

$$u_n = 6u_{n-1} - 4u_{n-2} \equiv 6u_{n-1} + 6u_{n-2} = 6\left(u_{n-1}+u_{n-2}\right) \pmod{10}.$$

Since $u_0 = 2$ and $u_1 = 6$, we construct the following table:

n	$u_n \pmod{10}$
0	2
1	6
2	8
3	4
4	2
5	6

Observe that the pattern repeats every four terms: 2, 6, 8, 4, 2, 6, 8, 4, 2, ... Thus,

$$u_n \pmod{10} \equiv \begin{cases} 2, & n \equiv 0 \pmod{4} \\ 6, & n \equiv 1 \pmod{4} \\ 8, & n \equiv 2 \pmod{4} \\ 4, & n \equiv 3 \pmod{4} \end{cases}.$$

Since $2016 \equiv 0 \pmod{4}$, it follows that the unit digit of u_{2016} is 2.

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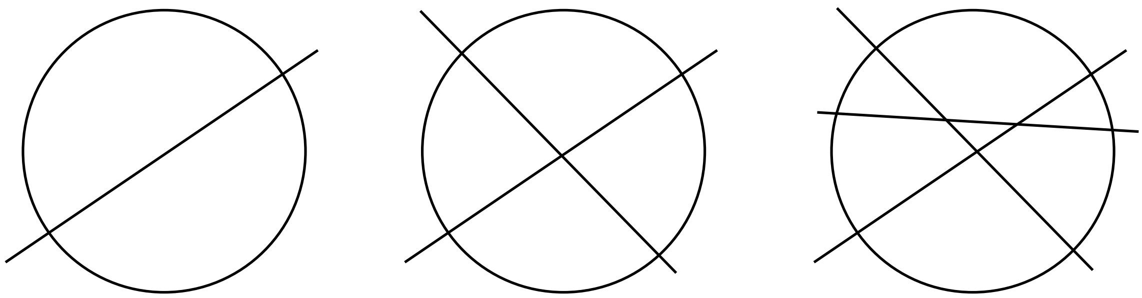
Problem 5 (🌶). A person attempts to cut a circular pizza into as many pieces as possible with a given number of straight cuts. In order to have as many slices as possible with each cut, no three cuts are concurrent, no two cuts are parallel, and the intersection of any two cuts should lie in the interior of the pizza.



$n=1$ $\quad$ $n=2$ $\quad$ $n=3$

Find the maximum number of slices of a circular pizza that a person can obtain by making n straight cuts with a knife.

Solution. Let u_n be the maximum number of slices obtainable from n cuts. From the above diagrams, we see that the nth slice can add at most n new slices. Hence,

$$u_n = u_{n-1} + n.$$

We can rewrite this as

$$u_n - u_{n-1} = n.$$

Summing over $k = 2, 3, \ldots, n$,

$$u_n - a_1 = \sum_{k=1}^{n} (u_k - u_{k-1}) = \sum_{k=2}^{n} k = \frac{n^2 + n}{2} - 1.$$

Since $a_1 = 2$, we have

$$u_n = \frac{n^2 + n}{2} + 1 = \frac{n^2 + n + 2}{2}.$$

* * * * *

Problem 6 (🌶). Solve the simultaneous recurrence relations:

$$a_n = 3a_{n-1} + 2b_{n-1}, \quad b_n = a_{n-1} + 2b_{n-1}$$

with $a_0 = 1$ and $b_0 = 2$.

Solution. Adding the two equations together, we see that $\{a_n + b_n\}$ is in geometric progression:

$$a_n + b_n = 4\left(a_{n-1} + b_{n-1}\right) = 4^n\left(a_0 + b_0\right) = 3 \cdot 4^n.$$

Substituting this into the first equation, we get

$$a_n - a_{n-1} = 2\left(a_{n-1} + b_{n-1}\right) = 6 \cdot 4^{n-1}.$$

Summing over $k = 1, 2, \ldots, n$,

$$a_n - a_0 = \sum_{k=1}^{n} (a_k - a_{k-1}) = \sum_{k=1}^{n} 6 \cdot 4^{k-1} = 6\left(\frac{1 - 4^n}{1 - 4}\right) = 2\left(4^n - 1\right).$$

Thus,

$$a_n = a_0 + 2\left(4^n - 1\right) = 2^{2n+1} - 1$$

and

$$b_n = 3 \cdot 4^n - a_n = 3 \cdot 2^{2n} - \left(2^{2n+1} - 1\right) = 2^{2n} + 1.$$

Assignment A5

Problem 1. In an auction at a charity gala dinner, a group of wealthy businessmen are competing with each other to be the highest bidder. Each time one of them makes a bid amount, another counter-bids by 50% more, less a service charge of ten dollars (e.g. If A bids \$1000, then B will bid \$1490). Let u_n be the amount at the nth bid and u_1 be the initial amount.

(a) Write down a recurrence relation that describes the bidding process.

(b) Show that $u_n = \$(1.5^{n-1}(u_1 - 20) + 20)$.

(c) The target amount to be raised is \$1 234 567 and the bidding stops when the bid amount meets or crosses this target amount. Given that $u_1 = 111$,

  (i) state the least number of bids required to meet this amount.

  (ii) find the winning bid amount, correct to the nearest thousand dollars.

Solution.

Part (a). $u_{n+1} = 1.5u_n - 10$.

Part (b). Let k be the constant such that $u_{n+1} + k = 1.5(u_n + k)$. Then $k = -20$. Hence, $u_{n+1} - 20 = 1.5(u_n - 20)$.

$$u_{n+1} - 20 = 1.5(u_n - 20) \implies u_n - 20 = 1.5^{n-1}(u_1 - 20) \implies u_n = 1.5^{n-1}(u_1 - 20) + 20.$$

Part (c).

Part (c)(i). Let m be the least integer such that $u_m \geq 1234567$. Consider $u_m \geq 1234567$:

$$u_m \geq 1234567 \implies 1.5^{m-1}(111 - 20) + 20 \geq 1234567.$$

Using G.C., $m \geq 24.5$. Hence, it takes at least 25 bids to meet this amount.

Part (c)(ii). Since $u_{25} = 1.5^{25-1}(111 - 20) = 1532000$ (to the nearest thousand), the winning bid is \$1 532 000.

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Problem 2. Solve these recurrence relations together with the initial conditions.

(a) $u_{n+2} = -u_n + 2u_{n+1}$, for $n \geq 0$, $u_0 = 5$, $u_1 = -1$.

(b) $4u_n = 4u_{n-1} + u_{n-2}$, for $n \geq 2$, $u_0 = a$, $u_1 = b$, $a, b \in \mathbb{R}$.

Solution.

Part (a). Observe that the characteristic equation of u_n, $x^2 - 2x + 1 = 0$, has only one root, namely $x = 1$. Thus,

$$u_n = (A + Bn) \cdot 1^n = A + Bn.$$

Thus, u_n is in AP. Since $u_0 = 5$ and $u_1 = -1$, it follows that

$$u_n = 5 - 6n.$$

Part (b). Rewriting the given recurrence relation, we have $u_n = u_{n-1} + \frac{1}{4}u_{n-2}$. Thus, the characteristic equation is $x^2 - x - \frac{1}{4} = 0$, which has roots $\frac{1}{2}(1 \pm \sqrt{2})$. Thus,

$$u_n = A\left(\frac{1 + \sqrt{2}}{2}\right)^n + B\left(\frac{1 - \sqrt{2}}{2}\right)^n.$$

Since $u_0 = a$, we obviously have $A+B = a$. Since $u_1 = b$, we get $A\left(\frac{1+\sqrt{2}}{2}\right)+B\left(\frac{1-\sqrt{2}}{2}\right) = b$. Solving, we get

$$A = \frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b, \quad B = \frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b.$$

Thus,

$$u_n = \left(\frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b\right)\left(\frac{1+\sqrt{2}}{2}\right)^n + \left(\frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b\right)\left(\frac{1-\sqrt{2}}{2}\right)^n.$$

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Problem 3. A passcode is generated using the digits 1 to 5, with repetitions allowed. The passcodes are classified into two types. A Type A passcode has an even number of the digit 1, while a Type B passcode has an odd number of the digit 1. For example, a Type A passcode is 1231, and a Type B passcode is 1541213. Let a_n and b_n denote the number of n-digit Type A and Type B passcodes respectively.

(a) State the values of a_1 and a_2.

(b) By considering the relationship between a_n and b_n, show that

$$a_n = xa_{n-1} + y^{n-1}, \qquad n \geq 2$$

where x and y are constants to be determined.

(c) Using the substitution $c_n = za_n + y^n$, where z is a constant to be determined, find a first order linear recurrence relation for c_n. Hence, find the general term formula for a_n.

Solution.

Part (a). $a_1 = 4$, $a_2 = 17$.

Part (b). Let P be an n-digit passcode with Type T, where T is either A or B. Let Type T' be the other type.

By concatenating a digit from 1 to 5 to P, five $(n+1)$-digit passcodes can be created. Let P' denote a new passcode that is created via this process. If the digit 1 is concatenated, then P' is of Type T'. If the digit 1 is not concatenated, then P' is of Type T. There are 4 choices for such a case. This hence gives the recurrence relations

$$\begin{cases} a_n = 4a_{n-1} + b_{n-1} \\ b_n = 4b_{n-1} + a_{n-1} \end{cases}$$

Adding the two equations, we see that $a_n + b_n = 5(a_{n-1} + b_{n-1})$. Thus,

$$a_n + b_n = 5^{n-1}(a_1 + b_1) = 5^{n-1}(4+1) = 5^n.$$

Hence,

$$a_n = 4a_{n-1} + b_{n-1} = 3a_{n-1} + a_{n-1} + b_{n-1} = 3a_{n-1} + 5^{n-1},$$

whence $x = 3$ and $y = 5$.

Part (c). Observe that

$$\begin{aligned} c_n &= za_n + 5^n = z\left(3a_{n-1} + 5^{n-1}\right) + 5^n = 3\left(za_{n-1} + 5^{n-1}\right) + (2+z)5^{n-1} \\ &= 3c_{n-1} + (2+z)5^{n-1}. \end{aligned}$$

Let $z = -2$. Then,

$$c_n = 3c_{n-1} = 3^{n-1}c_1 = 3^{n-1}\left(-2a_1 + 5\right) = -3^n.$$

Note that $a_n = \frac{1}{z}\left(c_n - y^n\right)$. Thus,

$$a_n = \frac{-3^n - 5^n}{-2} = \frac{3^n + 5^n}{2}.$$

A6 Polar Coordinates

Tutorial A6

Problem 1.

(a) Find the rectangular coordinates of the following points.

(i) $(3, -\frac{\pi}{4})$

(ii) $(1, \pi)$

(iii) $(\frac{1}{2}, \frac{3}{2}\pi)$

(b) Find the polar coordinates of the following points.

(i) $(3, 3)$

(ii) $(-1, -\sqrt{3})$

(iii) $(2, 0)$

(iv) $(4, 2)$

Solution.

Part (a).

Part (a)(i). Note that $r = 3$ and $\theta = -\frac{\pi}{4}$. This gives

$$x = r\cos\theta = \frac{3}{\sqrt{2}}, \quad y = r\sin\theta = -\frac{3}{\sqrt{2}}.$$

Hence, the rectangular coordinate of the point is $(3/\sqrt{2}, -3\sqrt{2})$.

Part (a)(ii). Note that $r = 1$ and $\theta = \pi$. This gives

$$x = r\cos\theta = -1, \quad y = r\sin\theta = 0.$$

Hence, the rectangular coordinate of the point is $(-1, 0)$.

Part (a)(iii). Note that $r = \frac{1}{2}$ and $\theta = \frac{3}{2}\pi$. This gives

$$x = \rho\cos\theta = 0, \quad y = r\sin\theta = -\frac{1}{2}.$$

Hence, the rectangular coordinate of the point is $(0, -1/2)$.

Part (b).

Part (b)(i). Note that $x = 3$ and $y = -3$. This gives

$$r^2 = x^2 + y^2 \implies r = 3\sqrt{2}, \quad \tan\theta = \frac{y}{x} \implies \theta = -\frac{\pi}{4}.$$

Hence, the polar coordinate of the point is $(3\sqrt{2}, -\pi/4)$.

Part (b)(ii). Note that $x = -1$ and $y = -\sqrt{3}$. This gives

$$r^2 = x^2 + y^2 \implies r = 2, \quad \tan\theta = \frac{y}{x} \implies \theta = \frac{\pi}{3}.$$

Hence, the polar coordinate of the point is $(2, \pi/3)$.

Part (b)(iii). Note that $x = 2$ and $y = 0$. This gives

$$r^2 = x^2 + y^2 \implies r = 2, \quad \tan\theta = \frac{y}{x} \implies \theta = 0.$$

Hence, the polar coordinate of the point is $(2, 0)$.

Part (b)(iv). Note that $x = 4$ and $y = 2$. This gives

$$r^2 = x^2 + y^2 \implies r = 2\sqrt{5}, \quad \tan\theta = \frac{y}{x} \implies \theta = \arctan\frac{1}{2}.$$

Hence, the polar coordinate of the point is $(2\sqrt{5}, \arctan(1/2))$.

* * * * *

Problem 2. Rewrite the following equations in polar form.

(a) $2x^2 + 3y^2 = 4$

(b) $y = 2x^2$

Solution.

Part (a).

$$2x^2 + 3y^2 = 2(r\cos\theta)^2 + 3(r\sin\theta)^2 = 4 \implies r^2 = \frac{4}{2\cos^2\theta + 3\sin^2\theta} = \frac{4}{2 + \sin^2\theta}.$$

Part (b).

$$y = 2x^2 \implies \frac{y}{x} = 2x \implies \tan\theta = 2r\cos\theta \implies r = \frac{1}{2}\tan\theta\sec\theta.$$

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Problem 3. Rewrite the following equations in rectangular form.

(a) $r = \frac{1}{1-2\cos\theta}$

(b) $r = \sin\theta$

Solution.

Part (a).

$$r = \frac{1}{1 - 2\cos\theta} \implies r - 2r\cos\theta = 1 \implies r = 2x + 1 \implies r^2 = 4x^2 + 4x + 1$$
$$\implies x^2 + y^2 = 4x^2 + 4x + 1 \implies y^2 = 3x^2 + 4x + 1.$$

Part (b).

$$r = \sin\theta \implies r^2 = r\sin\theta \implies x^2 + y^2 = y.$$

Problem 4.

(a) Show that the curve with polar equation $r = 3a\cos\theta$, where a is a positive constant, is a circle. Write down its centre and radius.

(b) By finding the Cartesian equation, sketch the curve whose polar equation is $r = a\sec\left(\theta - \frac{\pi}{4}\right)$, where a is a positive constant.

Solution.

Part (a).

$$r = 3a\cos\theta \implies r^2 = 3ar\cos\theta \implies x^2 + y^2 = 3ax \implies x^2 - 3ax + y^2 = 0.$$

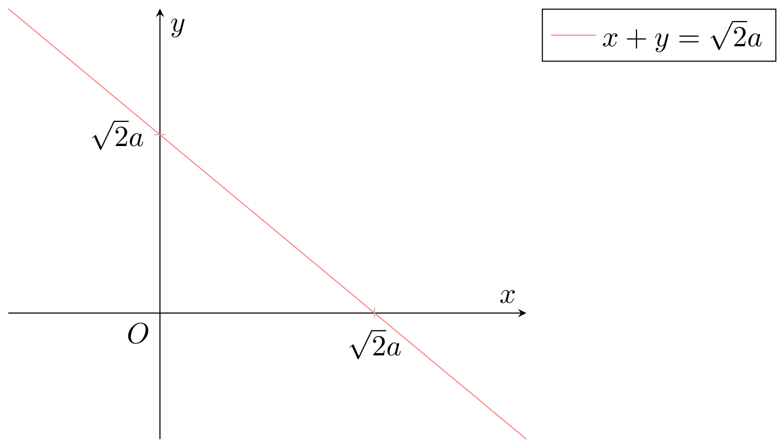
Completing the square, we get

$$\left(x - \frac{3a}{2}\right)^2 + y^2 \left(\frac{3a}{2}\right)^2.$$

Thus, the circle has centre $(3a/2, 0)$ and radius $3a/2$.

Part (b).

$$r = a\sec\left(\theta - \frac{\pi}{4}\right) \implies r\cos\left(\theta - \frac{\pi}{4}\right) = a \implies r\left(\cos\theta + \sin\theta\right) = \sqrt{2}a \implies x + y = \sqrt{2}a.$$



Problem 5. Sketch the following polar curves, where r is non-negative and $0 \le \theta \le 2\pi$.

(a) $r = 2$

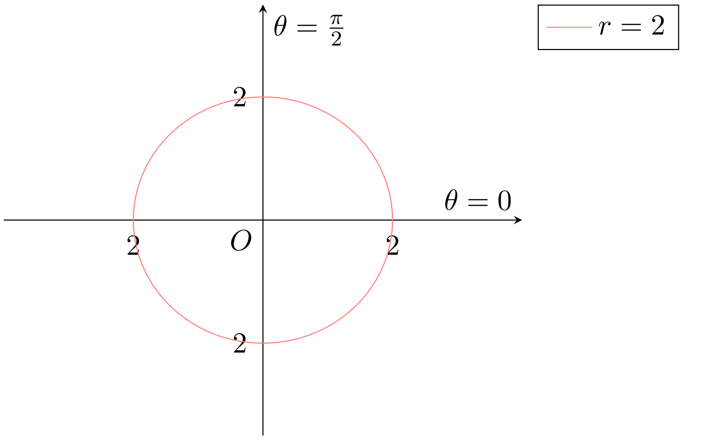
(b) $\theta = \frac{\pi}{4}$

(c) $r = \frac{1}{2}\theta$

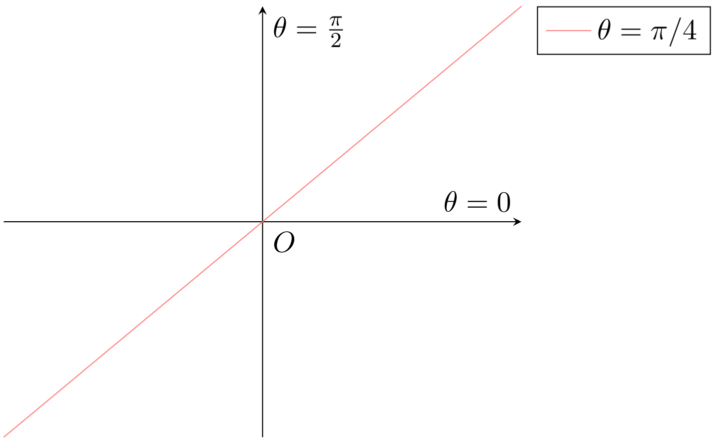
(d) $r = 2\csc\theta$

Solution.

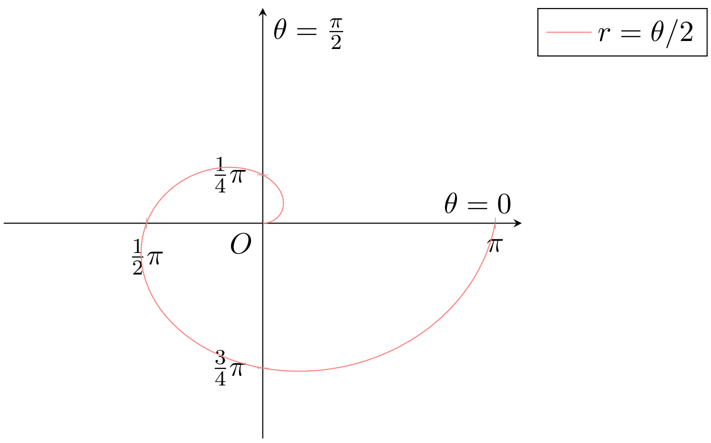
Part (a).



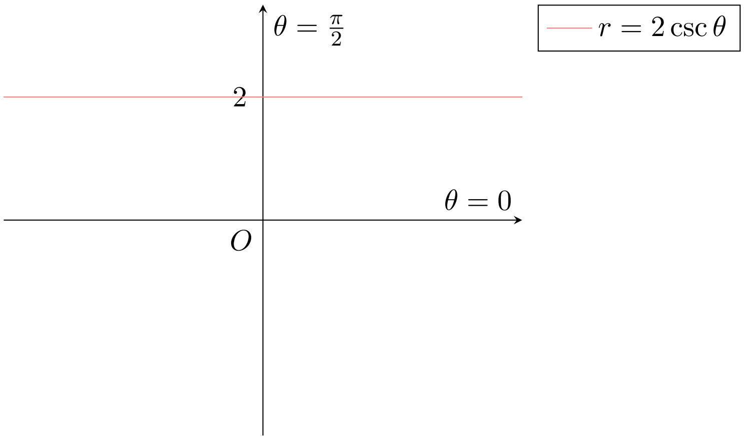
Part (b).



Part (c).



Part (d).



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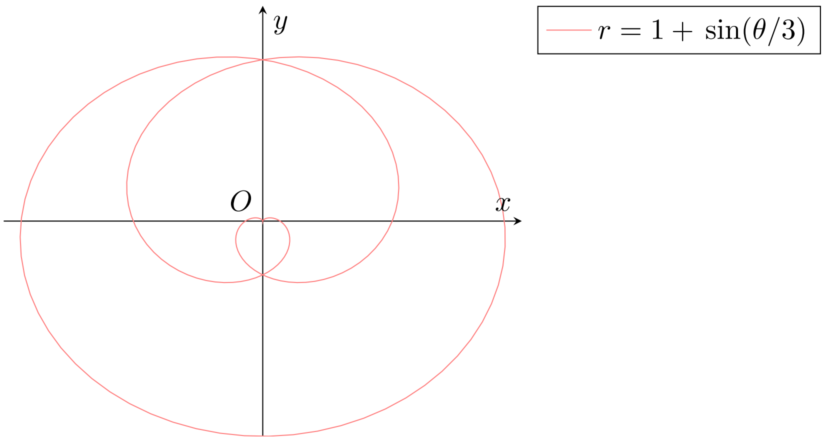
Problem 6. A sketch of the curve $r = 1 + \sin\frac{\theta}{3}$ is shown. Copy the diagram and indicate the x- and y-intercepts.



Solution. Observe that the curve is symmetric about the y-axis. Also observe that $\frac{\theta}{3} \in [0, 2\pi)$, hence we take $\theta \in [0, 6\pi)$.

For x-intercepts, $y = r\sin\theta = 0 \implies \theta = n\pi$, where $n \in \mathbb{Z}$. Due to the symmetry of the curve, we consider only $n = 0, 2, 4$.

Case 1. $n = 0 \implies r = 1 + \sin\frac{0}{3}\pi = 1$.

Case 2. $n = 2 \implies r = 1 + \sin\frac{2}{3}\pi = 1 + \frac{\sqrt{3}}{2}$.

Case 3. $n = 4 \implies r = 1 + \sin\frac{4}{3}\pi = 1 - \frac{\sqrt{3}}{2}$.

Hence, the curve intersects the x-axis at $x = 1, 1 + \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2}$. Correspondingly, the curve also intersects the x-axis at $x = -1, -1 - \frac{\sqrt{3}}{2}, -1 + \frac{\sqrt{3}}{2}$.

For y-intercepts, $x = r\cos\theta = 0 \implies \theta = (n + \frac{1}{2})\pi$, where $n \in \mathbb{Z}$. Due to the restriction on θ, we consider $n \in [0, 5)$.

Case 1. $n = 0, r = 1 + \sin\frac{1/2}{3}\pi = \frac{3}{2}$.

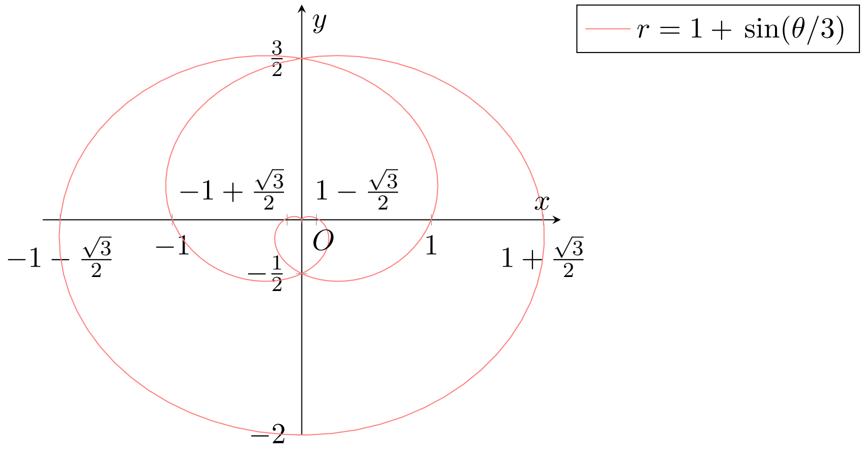
Case 2. $n = 1, r = 1 + \sin\frac{3/2}{3}\pi = 2$.

Case 3. $n = 2, r = 1 + \sin\frac{5/2}{3}\pi = \frac{3}{2}$.

Case 4. $n = 3, r = 1 + \sin\frac{7/2}{3}\pi = \frac{1}{2}$.

Case 5. $n = 4, r = 1 + \sin\frac{9/2}{3}\pi = 0$.

Hence, the curve intersects the y-axis at $y = -2, -\frac{1}{2}, \frac{3}{2}$.



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Problem 7.

(a) A graph has polar equation $r = \frac{2}{\cos\theta \sin\alpha - \sin\theta\cos\alpha}$, where α is a constant. Express the equation in Cartesian form. Hence, sketch the graph in the case $\alpha = \frac{\pi}{4}$, giving the Cartesian coordinates of the intersection with the axes.

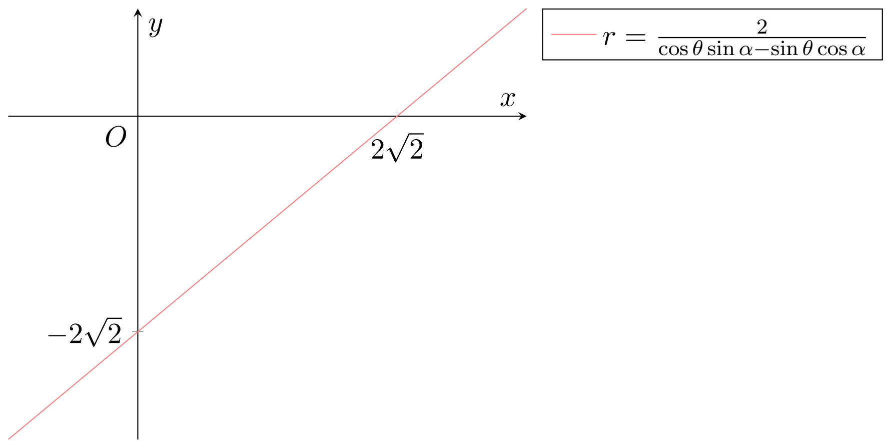
(b) A graph has Cartesian equation $(x^2 + y^2)^2 = 4x^2$. Express the equation in polar form. Hence, or otherwise, sketch the graph.

Solution.

Part (a).

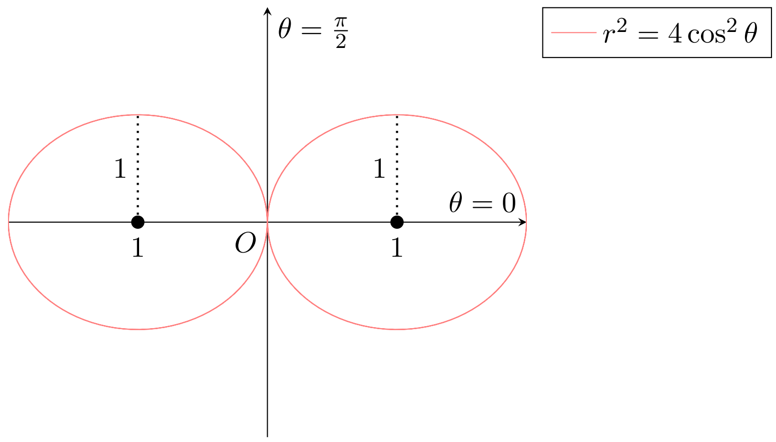
$$r = \frac{2}{\cos\theta\sin\alpha - \sin\theta\cos\alpha} \implies r\cos\theta\sin\alpha - r\sin\theta\cos\alpha = x\sin\alpha - y\cos\alpha = 2$$
$$\implies y = x\tan\alpha - 2\sec\alpha.$$

When $\alpha = \frac{\pi}{4}$, we have $y = x - 2\sqrt{2}$.



Part (b).

$$(x^2+y^2)^2 = 4x^2 \implies \left(r^2\right)^2 = 4(r\cos\theta)^2 \implies r^4 = 4r^2\cos^2\theta \implies r^2 = 4\cos^2\theta.$$



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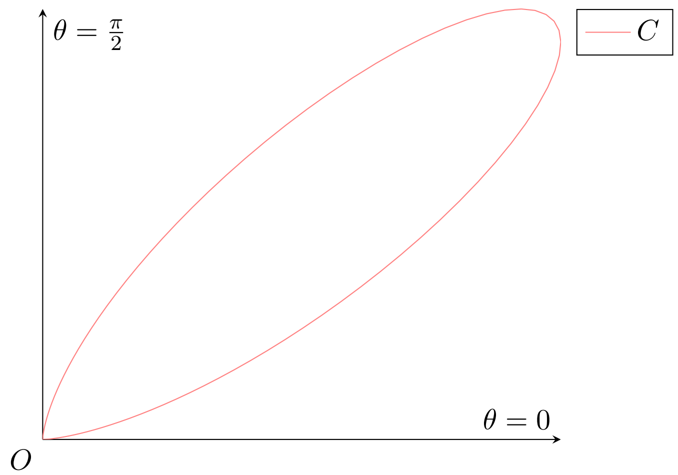
Problem 8. Find the polar equation of the curve C with equation $x^5 + y^5 = 5bx^2y^2$, where b is a positive constant. Sketch the part of the curve C where $0 \leq \theta \leq \pi/2$.

Solution.

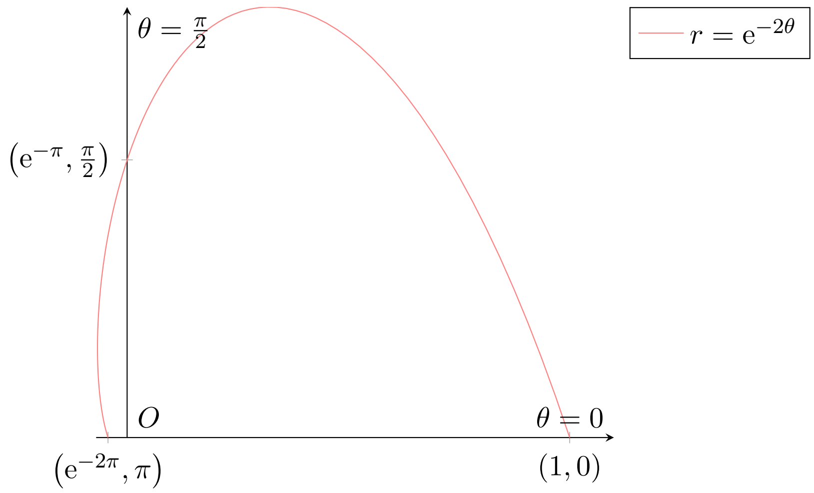
$$x^5 + y^5 = 5bx^2y^2 \implies (r\cos\theta)^5 + (r\sin\theta)^5 = 5b(r\cos\theta)^2(r\sin\theta)^2$$

$$\implies r\left(\cos^5\theta + \sin^5\theta\right) = 5b\cos^2\theta\sin^2\theta \implies r = \frac{5b\cos^2\theta\sin^2\theta}{\cos^5\theta + \sin^5\theta}.$$



Problem 9. The equation of a curve, in polar coordinates, is $r = e^{-2\theta}$, for $0 \leq \theta \leq \pi$. Sketch the curve, indicating clearly the polar coordinates of any axial intercepts.

Solution.



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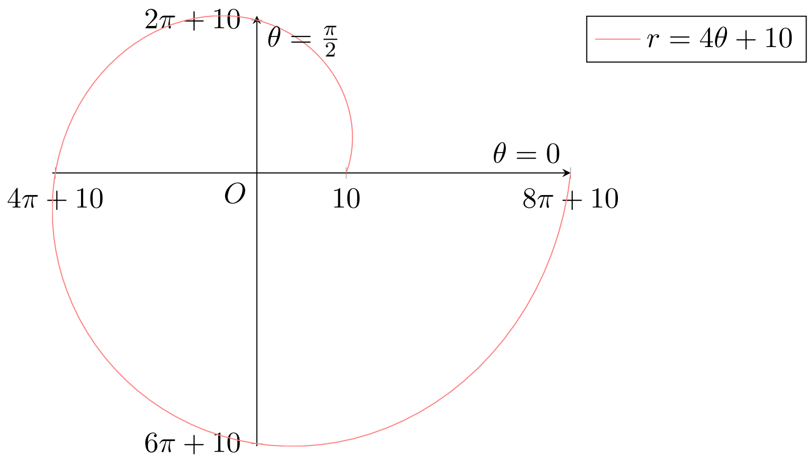
Problem 10. Suppose that a long thin rod with one end fixed at the pole of a polar coordinate system rotates counter-clockwise at the constant rate of 0.5 rad/sec. At time $t = 0$, a bug on the rod is 10 mm from the pole and is moving outward along the rod at a constant speed of 2 mm/sec. Find an equation of the form $r = f(\theta)$ for the part of motion of the bug, assuming that $\theta = 0$ when $t = 0$. Sketch the path of the bug on the polar coordinate system for $0 \leq t \leq 4\pi$.

Solution. Let $\theta(t)$ and $r(t)$ be functions of time, with $\theta(0) = 0$ and $r(0) = 10$. We know that $\mathrm{d}\theta/\mathrm{d}t = 0.5$ and $\mathrm{d}r/\mathrm{d}t = 2$. Hence,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^{-1} = 2 \cdot (0.5)^{-1} = 4.$$

Thus, $r = 4\theta + r(0) = 4\theta + 10$.

Since $\mathrm{d}\theta/\mathrm{d}t = 0.5$ and $\theta(0) = 0$, we have $\theta = 0.5t$. Hence, $0 \leq t \leq 4\pi \implies 0 \leq \theta \leq 2\pi$.



Problem 11. The equation, in polar coordinates, of a curve C is $r = ae^{\frac{1}{2}\theta}$, $0 \le \theta \le 2\pi$, where a is a positive constant. Write down, in terms of θ, the Cartesian coordinates, x and y, of a general point P on the curve. Show that the gradient at P is given by $\frac{dy}{dx} = \frac{\tan\theta+2}{1-2\tan\theta}$.

Hence, show that the tangent at P is inclined to $\overrightarrow{OP}$ at a constant angle α, where $\tan\alpha = 2$. Sketch the curve C.

Solution. Note that $x = r\cos\theta$ and $y = r\sin\theta$, whence $x = ae^{\frac{1}{2}\theta}\cos\theta$ and $y = ae^{\frac{1}{2}\theta}\sin\theta$. Hence, $P\left(ae^{\frac{1}{2}\theta}\cos\theta, ae^{\frac{1}{2}\theta}\sin\theta\right)$.

Observe that $\frac{dr}{d\theta} = \frac{1}{2}ae^{\frac{1}{2}\theta} = \frac{1}{2}r$. Hence,

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\frac{1}{2}r\sin\theta + r\cos\theta}{\frac{1}{2}r\cos\theta - r\sin\theta} = \frac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} = \frac{\tan\theta + 2}{1 - 2\tan\theta}.$$

Let $\mathbf{t} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ represent the direction of the tangent line. Then

$$\mathbf{t} = \begin{pmatrix} 1 \\ dy/dx \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\tan\theta+2}{1-2\tan\theta} \end{pmatrix} = \frac{1}{1-2\tan\theta}\begin{pmatrix} 1-2\tan\theta \\ \tan\theta + 2 \end{pmatrix}$$

and

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ae^{\frac{1}{2}\theta}\cos\theta \\ ae^{\frac{1}{2}\theta}\sin\theta \end{pmatrix} = ae^{\frac{1}{2}\theta}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

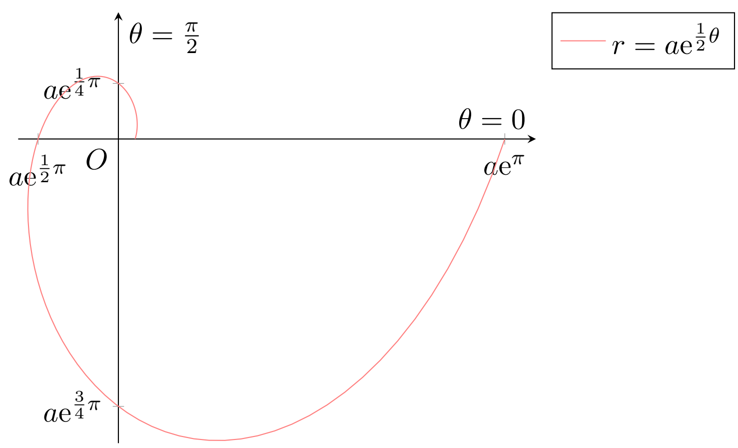
By the definition of the dot-product, we have $\mathbf{t}\cdot\overrightarrow{OP} = |\mathbf{t}|\left|\overrightarrow{OP}\right|\cos\alpha$, whence

$$\begin{aligned}\cos\alpha = \frac{\mathbf{t}\cdot\overrightarrow{OP}}{|\mathbf{t}|\left|\overrightarrow{OP}\right|} &= \frac{(1-2\tan\theta)\cos\theta + (\tan\theta+2)\sin\theta}{\sqrt{(1-2\tan\theta)^2 + (\tan\theta+2)^2}\cdot\sqrt{\cos^2\theta+\sin^2\theta}} \\ &= \frac{\cos\theta + \tan\theta\sin\theta}{\sqrt{5\tan^2\theta+5}} = \frac{\cos^2\theta+\sin^2\theta}{\sqrt{5\sin^2\theta+5\cos^2\theta}} = \frac{1}{\sqrt{5}}.\end{aligned}$$

Thus, $\alpha = \arccos\frac{1}{\sqrt{5}}$. Since $\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x}$,

$$\tan\alpha = \tan\left(\arccos\frac{1}{\sqrt{5}}\right) = \frac{\sqrt{1-(1/\sqrt{5})^2}}{1/\sqrt{5}} = 2.$$

Hence, the tangent at P is inclined to $\overrightarrow{OP}$ at a constant angle α, where $\tan\alpha = 2$.



Problem 12. The polar equation of a curve is given by $r = \mathrm{e}^{\theta}$ where $0 \leq \theta \leq \frac{\pi}{2}$. Cartesian axes are taken at the pole O. Express x and y in terms of θ and hence find the Cartesian equation of the tangent at $\left(\mathrm{e}^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$.

Solution. Recall that $x = r\cos\theta$ and $y = r\sin\theta$, whence $x = \mathrm{e}^{\theta}\cos\theta$ and $y = \mathrm{e}^{\theta}\sin\theta$. Thus, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \mathrm{e}^{\theta}(\cos\theta - \sin\theta)$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{\theta}(\cos\theta + \sin\theta)$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{\mathrm{e}^{\theta}(\cos\theta + \sin\theta)}{\mathrm{e}^{\theta}(\cos\theta - \sin\theta)} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}.$$

At $\left(\mathrm{e}^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$, we clearly have $x = 0$ and $y = \mathrm{e}^{\pi/2}$. Also, $\mathrm{d}y/\mathrm{d}x = -1$. By the point-slope formula, the equation of the tangent line at $\left(\mathrm{e}^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$ is given by $y = -x + \mathrm{e}^{\frac{\pi}{2}}$.

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Problem 13. A curve C has polar equation $r = a\cot\theta$, $0 < \theta \leq \pi$, where a is a positive constant.

(a) Show that $y = a$ is an asymptote of C.

(b) Find the tangent at the pole.

Hence, sketch C and find the Cartesian equation of C in the form $y^2(x^2 + y^2) = bx^2$, where b is a constant to be determined.

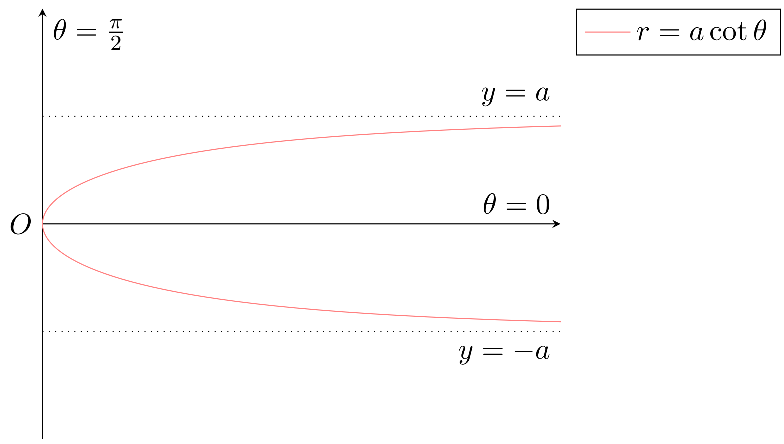
Solution.

Part (a). Note that

$$r = a\cot\theta \implies y = r\sin\theta = a\cos\theta.$$

As $\theta \to 0$, $r \to \infty$. Hence, there is an asymptote at $\theta = 0$. Since $\cos\theta = 1$ when $\theta = 0$, the line $y = a\cos\theta = a$ is an asymptote of C.

Part (b). For tangents at the pole, $r = 0 \implies \cot\theta = 0 \implies \theta = \frac{\pi}{2}$.



Note that

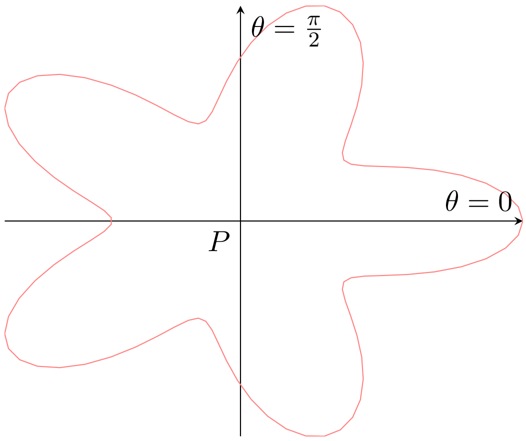
$$r = a\cot\theta = a\left(\frac{r\cos\theta}{r\sin\theta}\right) = a\left(\frac{x}{y}\right).$$

Thus,

$$x^2 + y^2 = r^2 = a^2\left(\frac{x^2}{y^2}\right) \implies y^2\left(x^2 + y^2\right) = a^2x^2,$$

whence $b = a^2$.

Problem 14.



Relative to the pole P and the initial line $\theta = 0$, the polar equation of the curve shown is either

i. $r = a + b \sin n\theta$, or

ii. $r = a + b \cos n\theta$

where a, b and n are positive constants. State, with a reason, whether the equation is (i) or (ii) and state the value of n.

The maximum value of r is $\frac{11}{2}$ and the minimum value of r is $\frac{5}{2}$. Find the values of a and b.

Solution. Since the curve is symmetrical about the horizontal half-line $\theta = 0$, the polar equation of the curve is a function of $\cos n\theta$ only. Hence, the polar equation of the curve is $r = a + b \cos n\theta$, with $n = 5$.

Observe that the maximum value of r is achieved when $\cos 5\theta = 1$, whence $r = a + b$. Thus, $a + b = \frac{11}{2}$. Also observe that the minimum value of r is achieved when $\cos 5\theta = -1$, whence $r = a - b$. Thus, $a - b = \frac{5}{2}$. Solving, we get $a = 4$ and $b = \frac{3}{2}$.

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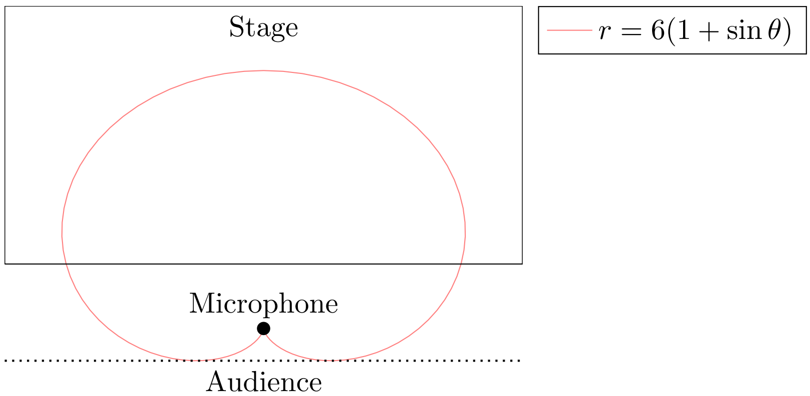
Problem 15.



Sound engineers often use a microphone with a cardioid acoustic pickup pattern to record live performances because it reduces pickup from the audience. Suppose a cardioid microphone is placed 3 metres from the front of the stage, and the boundary of the optimal pickup region is given by the cardioid with polar equation

$$r = 6(1 + \sin\theta)$$

where r is measured in metres and the microphone is at the pole.

Find the minimum distance from the front of the stage the first row of the audience can be seated such that the microphone does not pick up noise from the audience.

Solution. Note that $r = 6(1+\sin\theta) = 6(1+\frac{y}{r})$, whence $r^2 = 6r + 6y$. Thus,

$$r^2 - 6r - 6y = 0 \implies r = 3 \pm \sqrt{9+6y} \implies 9 + 6y = (r-3)^2.$$

Since $9 + 6y = (r-3)^2 \geq 0$, we have $y \geq -1.5$. Thus, the furthest distance the audience has to be from the stage is $|-1.5| + 3 = 4.5$ m.

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Problem 16. To design a flower pendant, a designer starts off with a curve C_1, given by the Cartesian equation

$$\left(x^2+y^2\right)^2 = a^2\left(3x^2 - y^2\right)$$

where a is a positive constant.

(a) Show that a corresponding polar equation of C_1 is $r^2 = a^2(1 + 2\cos 2\theta)$.

(b) Find the equations of the tangents to C_1 at the pole.

Another curve C_2 is obtained by rotating C_1 anti-clockwise about the origin by $\frac{\pi}{3}$ radians.

(c) State a polar equation of C_2.

(d) Sketch C_1 and C_2 on the same diagram, stating clearly the exact polar coordinates of the points of intersection of the curves with the axes. Find also the exact polar coordinates of the points of intersection with C_1 and C_2.

The curve C_3 is obtained by reflecting C_2 in the line $\theta = \frac{\pi}{2}$.

(e) State a polar equation of C_3.

(f) The designer wishes to enclose the 3 curves inside a circle given by the polar equation $r = r_1$. State the minimum value of r_1 in terms of a.

Solution.

Part (a). Observe that $\left(x^2+y^2\right)^2 = r^4$ and $3x^2 - y^2 = r^2\left(3\cos^2\theta - \sin^2\theta\right)$. Hence,

$$\left(x^2+y^2\right)^2 = a^2\left(3x^2-y^2\right) \implies r^2 = a^2\left(3\cos^2\theta - \sin^2\theta\right).$$

Note that

$$3\cos^2\theta - \sin^2\theta = 1 + 2\cos^2\theta - 2\sin^2\theta = 1 + 2\cos 2\theta.$$

Thus,

$$r^2 = a^2\left(1 + 2\cos 2\theta\right).$$

Part (b). For tangents at the pole,

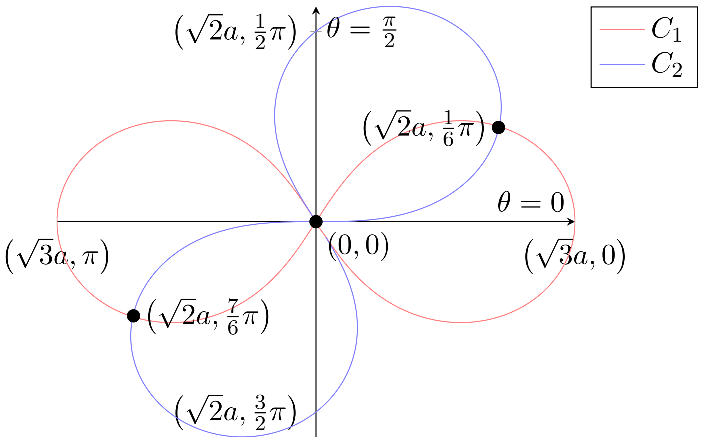
$$r = 0 \implies 1 + 2\cos 2\theta = 0 \implies \cos 2\theta = -\frac{1}{2}.$$

Since $0 \leq 2\theta \leq 2\pi$, we have $\theta = \pi/3, 2\pi/3$. For full lines, we also have $\theta = 4\pi/3$ and $\theta = 5\pi/3$.

Part (c).

$$r^2 = a^2 \left[1 + 2\cos\left(2\left(\theta - \frac{\pi}{3}\right)\right)\right] = a^2 \left[1 + 2\cos\left(2\theta - \frac{2}{3}\pi\right)\right].$$

Part (d).



Consider the horizontal intercepts of C_1. When $\theta = 0$, $r = \sqrt{3}a$. Hence, by symmetry, C_1 intercepts the horizontal axis at $\left(\sqrt{3}a, 0\right)$ and $\left(\sqrt{3}a, \pi\right)$.

Consider the vertical intercepts of C_2. When $\theta = \pi/2$, $r = \sqrt{2}a$. Hence, by symmetry, C_2 intercepts the vertical axis at $\left(\sqrt{2}a, \pi/2\right)$ and $\left(\sqrt{2}a, 3\pi/2\right)$.

Now consider the intersections between C_1 and C_2. By symmetry, it is obvious that the points of intersections must lie along the half-lines $\pi/6$ and $7\pi/6$, or along the half-lines $4\pi/6$ and $10\pi/6$. By symmetry, we consider only the half-lines $\pi/6$ and $4\pi/6$.

Case 1: $\theta = \pi/6$. Substituting $\theta = \pi/6$ into the equation of C_1, we obtain $r = \sqrt{2}a$. Hence, C_1 and C_2 intersect at $\left(\sqrt{2}a, \pi/6\right)$ and, by symmetry, at $\left(\sqrt{2}a, 7\pi/6\right)$.

Case 2.$\theta = 4\pi/6$ Substituting $\theta = 4\pi/6$ into the equation of C_1, we obtain $r = 0$. Hence, C_1 and C_2 intersect at $(0, 0)$.

Part (e). Reflecting about the line $\theta = \pi/2$ is equivalent to applying the map $\theta \mapsto \theta + \pi/3$ to C_1. Hence,

$$r^2 = a^2 \left[1 + 2\cos\left(2\left(\theta + \frac{1}{3}\pi\right)\right)\right] = a^2 \left[1 + 2\cos\left(2\theta + \frac{2}{3}\pi\right)\right].$$

Part (f). $r_1 = \sqrt{3}a$.

Self-Practice A6

Problem 1. A curve C has equation, in polar coordinates, $r = a\sqrt{\left(4+\sin^2\theta\right)\cos\theta}$, $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$, where a is a positive constant.

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\left(4+\sin^2\theta\right)\cos\theta\right] = -\left(2+3\sin^2\theta\right)\sin\theta$. Hence, state, with a reason, whether r increases or decreases as θ increases, for $0 < \theta \leq \frac{1}{2}\pi$.

(b) Sketch the curve C.

(c) Find the Cartesian equation of C in the form $\left(x^2+y^2\right)^m = a^2x\left(bx^2+cy^2\right)$, giving the numerical values of m, b and c.

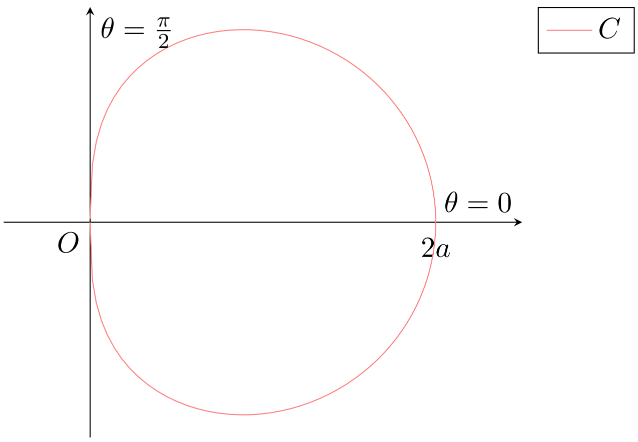
Solution.

Part (a).

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\left(4+\sin^2\theta\right)\cos\theta\right] &= -\left(4+\sin^2\theta\right)\sin\theta + 2\sin\theta\cos^2\theta \\ &= -\sin\theta\left(\sin^2\theta - 2\cos^2\theta + 4\right) \\ &= -\sin\theta\left[\sin^2\theta - 2\left(1-\sin^2\theta\right) + 4\right] \\ &= -\sin\theta\left(3\sin^2\theta + 2\right).\end{aligned}$$

For $t \in (0, \pi/2]$, we have $\sin\theta > 0$ and $3\sin^2\theta + 2 > 0$. Hence, r is decreasing.

Part (b).



Part (c). Squaring, we have

$$r^2 = a^2\left(4+\sin^2\theta\right)\cos\theta.$$

Recall that $x = r\cos\theta$ and $y = r\sin\theta$, so

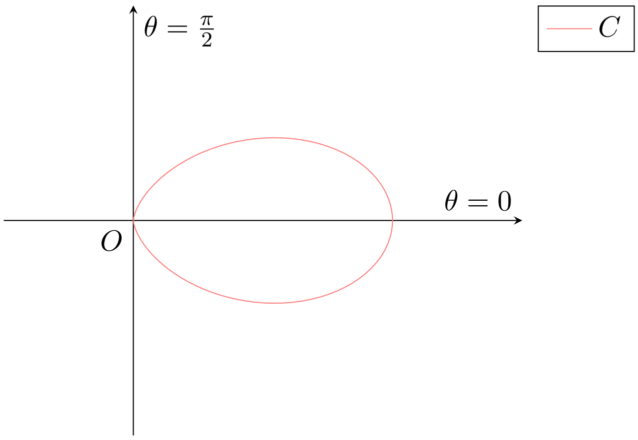
$$r^2 = a^2\left[4+\left(\frac{y}{r}\right)^2\right]\left(\frac{x}{r}\right) \implies r^5 = a^2x\left(4r^2+y^2\right).$$

Since $x^2+y^2 = r^2$, we get

$$\left(x^2+y^2\right)^{5/2} = a^2x\left(4x^2+5y^2\right),$$

whence $m = 5/2$, $b = 4$ and $c = 5$.

Problem 2. The diagram shows a sketch of the curve C with polar equation $r = a\cos^2\theta$, where a is a positive constant and $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$.



(a) Explain briefly about how you can tell from this form of the equation that C is symmetrical about the line $\theta = 0$ and that the tangent to C at the pole O is perpendicular to the line $\theta = 0$.

(b) Show that the equation of C in Cartesian coordinates may be expressed in the form $y^2 = a^{2/3}x^{4/3} - x^2$.

Solution.

Part (a). Observe that

$$a\cos^2\theta = a\cos^2(-\theta).$$

Hence, C is invariant under the transformation $\theta \mapsto -\theta$, whence it is symmetrical about the line $\theta = 0$.

For tangents to the pole, we have $r = 0$. Since $a > 0$, we require $\cos\theta = 0$, whence $\theta = \pm\pi/2$, which are clearly perpendicular to the line $\theta = 0$.

Part (b). We have

$$r = a\cos^2\theta = a\left(\frac{x}{r}\right)^2 \implies r^3 = ax^2.$$

Hence,

$$x^2 + y^2 = r^2 = \left(ax^2\right)^{2/3} \implies y^2 = a^{2/3}x^{4/3} - x^2.$$

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Problem 3. The equation of curve C is given in polar coordinates by $r = 1 + \sin 2\theta$, $0 \leq \theta \leq 2\pi$.

(a) Prove that C is symmetric about the pole.

(b) Sketch C and any tangents to C at the pole. Label any points of intersection with the axes, and show clearly the symmetries and curvature near the pole.

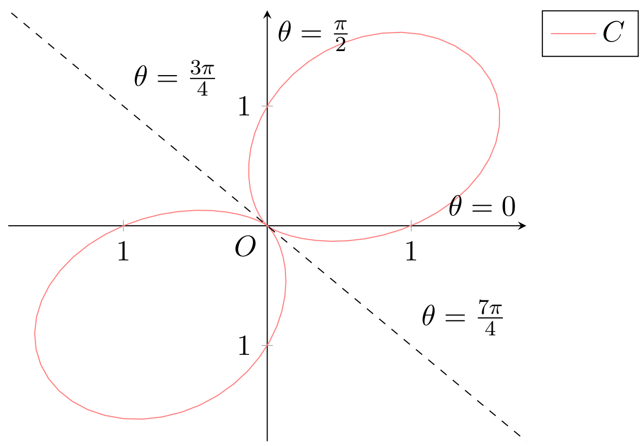
(c) Determine whether each loop of C is a circle. Justify your answer.

(d) Show that the Cartesian equation of C is $\left(x^2 + y^2\right)^3 = (x + y)^4$.

Solution.
Part (a). Observe that

$$1 + \sin 2\theta = 1 + \sin(2\theta + 2\pi) = 1 + \sin(2(\theta + \pi)).$$

Hence, C is invariant under the transformation $\theta \mapsto \theta + \pi$, whence C is symmetric about the pole.
Part (b).



Part (c). Consider the top-right loop. r attains a maximum of 2 when $\theta = \pi/4$. Suppose the loop is a circle (with radius 1). Then the centre should be $(1, \pi/4)$, which is $(1/\sqrt{2}, 1\sqrt{2})$ in Cartesian coordinates. The distance between $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1, 0)$ is given by

$$\sqrt{\left(\frac{1}{\sqrt{2}} - 1\right)^2 + \left(\frac{1}{\sqrt{2}} - 0\right)^2} = \sqrt{2 - \sqrt{2}} \neq 1.$$

Hence, the loop is not a circle.
Part (d). We have

$$r = 1 + \sin 2\theta = 1 + 2\cos\theta\sin\theta = 1 + 2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right).$$

Thus,

$$r^3 = r^2 + 2xy \implies \left(x^2 + y^2\right)^{3/2} = x^2 + y^2 + 2xy = (x + y)^2.$$

Squaring both sides yields the desired equation:

$$\left(x^2 + y^2\right)^3 = (x + y)^4.$$

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Problem 4 (🌶). Prove that at all points of intersection of the polar curves with equations $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$, the tangent lines are perpendicular.

Solution. Consider the gradient of C_1. Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta = -a\sin\theta - 2a\sin\theta\cos\theta = -a\left(\sin\theta + \sin 2\theta\right).$$

Next, we have

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t}\sin\theta + r\cos\theta = a\cos\theta - a\cos^2\theta + a\sin^2\theta = a\left(\cos\theta - \cos 2\theta\right).$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = -\left(\frac{\cos\theta - \cos 2\theta}{\sin\theta + \sin 2\theta}\right).$$

Consider the gradient of C_2. Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta = -b\sin\theta + 2b\cos\theta\sin\theta = b\left(\sin 2\theta - \sin\theta\right).$$

Next, we have

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t}\sin\theta + r\cos\theta = b\cos\theta - b\cos^2\theta + b\sin^2\theta = b\left(\cos\theta - \cos 2\theta\right).$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}.$$

Consider the product of the gradients:

$$-\left(\frac{\cos\theta - \cos 2\theta}{\sin\theta + \sin 2\theta}\right)\left(\frac{\cos\theta - \cos 2t}{\sin 2\theta - \sin\theta}\right) = -\frac{\cos^2\theta - \cos^2 2\theta}{\sin^2 2\theta - \sin^2\theta}.$$

Observe that

$$\cos^2\theta - \cos^2 2\theta = \cos^2\theta - \left(2\cos^2\theta - 1\right)^2 = -4\cos^4\theta + 5\cos^2\theta - 1.$$

Also observe that

$$\begin{aligned}\sin^2 2\theta - \sin^2\theta &= 4\sin^2\theta\cos^2\theta - \sin^2\theta\\ &= 4\left(1 - \cos^2\theta\right)\cos^2\theta - \left(1 - \cos^2\theta\right)\\ &= -4\cos^4\theta + 5\cos^2\theta - 1.\end{aligned}$$

Hence, the product of the gradients is

$$-\frac{\cos^2\theta - \cos^2 2\theta}{\sin^2 2\theta - \sin^2\theta} = -\frac{-4\cos^4\theta + 5\cos^2\theta - 1}{-4\cos^4\theta + 5\cos^2\theta - 1} = -1.$$

Thus, for any given θ, the tangents of C_1 and C_2 are perpendicular. This immediately implies that the tangent lines at all intersection points of C_1 and C_2 are perpendicular.

Assignment A6

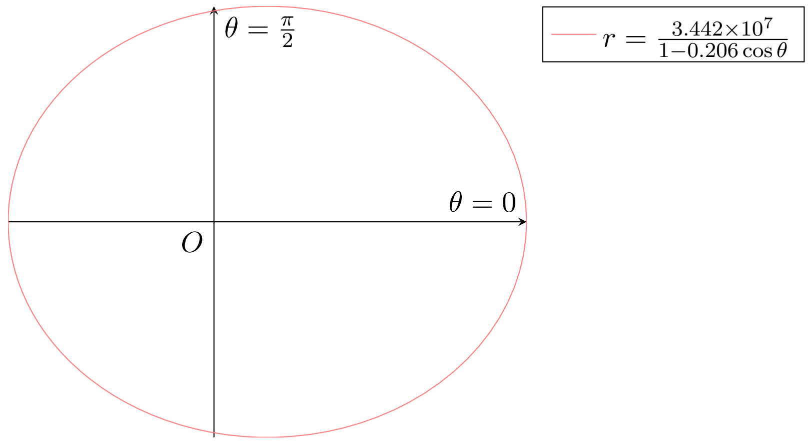
Problem 1. The planet Mercury travels around the sun in an elliptical orbit given approximately by

$$r = \frac{3.442 \times 10^7}{1 - 0.206\cos\theta},$$

where r is measured in miles and the sun is at the pole.

Sketch the orbit and find the distance from Mercury to the sun at the aphelion (the greatest distance from the sun) and at the perihelion (the shortest distance from the sun).

Solution.



Observe that r attains a maximum when $\cos\theta$ is also at its maximum. Since the maximum value of $\cos\theta$ is 1,

$$r = \frac{3.442 \times 10^7}{1 - 0.206(1)} = 4.34 \times 10^7 \text{ (3 s.f.)}.$$

Hence, the distance from Mercury to the sun at the aphelion is 4.34×10^7 miles.

Observe that r attains a minimum when $\cos\theta$ is also at its minimum. Since the minimum value of $\cos\theta$ is -1,

$$r = \frac{3.442 \times 10^7}{1 - 0.206(-1)} = 2.85 \times 10^7 \text{ (3 s.f.)}.$$

Hence, the distance from Mercury to the sun at the perihelion is 2.85×10^7 miles.

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Problem 2. A variable point P has polar coordinates (r, θ), and fixed points A and B have polar coordinates $(1, 0)$ and $(1, \pi)$ respectively. Given that P moves so that the product $PA \cdot PB = 2$, show that

$$r^2 = \cos 2\theta + \sqrt{3 + \cos^2 2\theta}.$$

(a) Given that $r \geq 0$ and $0 \leq \theta \leq 2\pi$, find the maximum and minimum values of r, and the values of θ at which they occur.

(b) Verify that the path taken by P is symmetric about the lines $\theta = 0$ and $\theta = \dfrac{\pi}{2}$, giving your reasons.

Solution. Note that A and B have Cartesian coordinates $(1, 0)$ and $(-1, 0)$ respectively. Let $P(x, y)$. Then

$$PA^2 = (x-1)^2 + y^2, \qquad PB^2 = (x+1)^2 + y^2.$$

Hence,

$$PA \cdot PB = \left((x-1)^2 + y^2\right)\left((x+1)^2 + y^2\right) = \left(x^2 + y^2\right)^2 - 2\left(x^2 - y^2\right) + 1.$$

Since $x^2 - y^2 = r^2\left(\cos^2\theta - \sin^2\theta\right) = r^2 \cos 2\theta$, the polar equation of the locus of P is

$$r^4 - 2r^2 \cos 2\theta + 1 = (PA \cdot PB)^2 = 4 \implies r^4 - 2r^2 \cos 2\theta - 3 = 0.$$

By the quadratic formula, we have

$$r^2 = \frac{2\cos 2\theta \pm \sqrt{4\cos^2 2\theta + 12}}{2} = \cos 2\theta \pm \sqrt{\cos^2 2\theta + 3}.$$

Since $\sqrt{\cos^2 2\theta + 3} > \cos 2\theta$ and $r^2 \geq 0$, we reject the negative case. Thus,

$$r^2 = \cos 2\theta + \sqrt{3 + \cos^2 2\theta}.$$

Part (a). Differentiating with respect to θ, we obtain

$$2r\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2\sin 2\theta \left(1 + \frac{1}{2\sqrt{3 + \cos^2 2\theta}}\right).$$

For stationary points, $\mathrm{d}r/\mathrm{d}\theta = 0$. Since $1 + 1/2\sqrt{3 + \cos^2 2\theta} > 0$, we must have $\sin 2\theta = 0$, whence $\theta = 0, \pi/2, \pi, 3\pi/2$. By symmetry, we only consider $\theta = 0$ and $\theta = \pi/2$.

Case 1. When $\theta = 0$, we have $r^2 = 3$, whence $r = \sqrt{3}$.

Case 2. When $\theta = \pi/2$, we have $r^2 = 1$, whence $r = 1$.

Thus, $\max r = \sqrt{3}$ and occurs when $\theta = 0, \pi$, while $\min r = 1$ and occurs when $\theta = \pi/2, 3\pi/2$.

Part (b). Recall that the path taken by P is given by

$$\left((x-1)^2 + y^2\right)\left((x+1)^2 + y^2\right) = 4.$$

Observe that the above equation is invariant under the transformations $x \mapsto -x$ and $y \mapsto -y$. Hence, the path is symmetric about both the x- and y-axes, i.e. the lines $\theta = 0$ and $\theta = \pi/2$.

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Problem 3.

(a) Explain why the curve with equation $x^3 + 2xy^2 - a^2y = 0$ where a is a positive constant lies entirely in the region $|x| \leq 2^{-\frac{3}{4}}a$.

(b) Show that the polar equation of this curve is $r^2 = \dfrac{a^2 \tan\theta}{2 - \cos^2\theta}$.

(c) Sketch the curve.

Solution.

Part (a). Consider the discriminant Δ of $x^3 + 2xy^2 - a^2y = 0$ with respect to y:

$$\Delta = \left(-a^2\right)^2 - 4\left(2x\right) = a^4 - 8x^4.$$

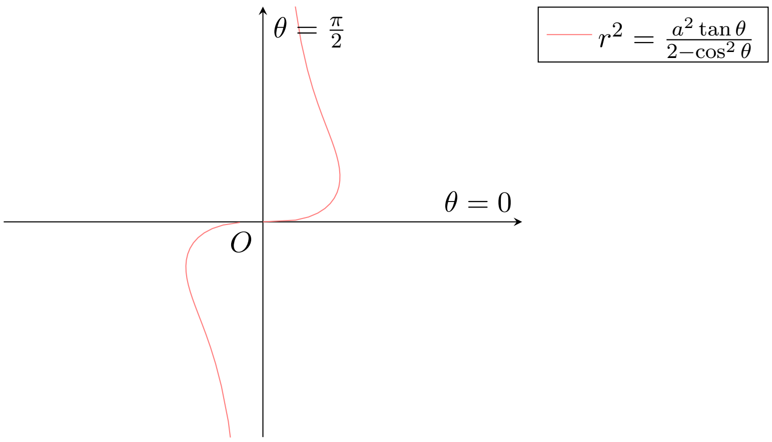
For points on the curve, we clearly have $\Delta \geq 0$. Thus,

$$a^3 - 8x^4 \geq 0 \implies x^4 \leq 2^{-3}a^4 \implies |x| \leq 2^{-3/4}a.$$

Part (b).

$$x^3 + 2xy^2 - a^2y = 0 \implies 2\left(x^2+y^2\right) - x^2 - a^2\frac{y}{x} = 0 \implies 2r^2 - r^2\cos^2\theta - a^2\tan\theta = 0$$
$$\implies r^2 = \frac{a^2\tan\theta}{2-\cos^2\theta}.$$

Part (c).



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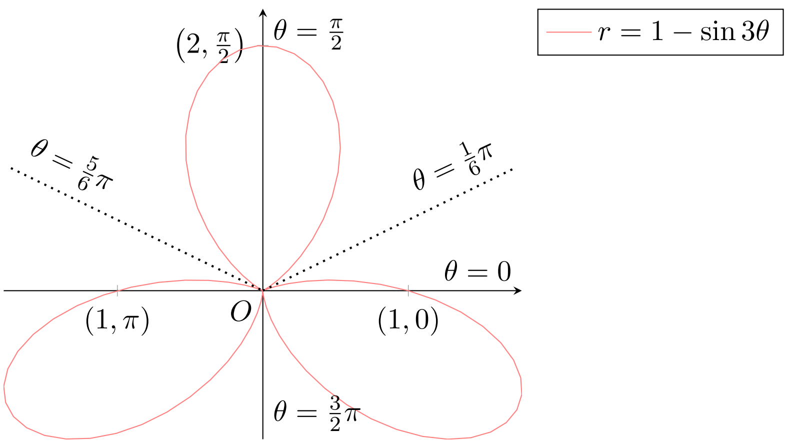
Problem 4. The curve C has polar equation $r = 1 - \sin 3\theta$, where $0 \leq \theta \leq 2\pi$.

(a) Sketch the curve C, showing the tangents at the pole and the intersections with the axes.

(b) Find the gradient of the curve at the point where $\theta = \dfrac{\pi}{3}$, giving your answer in the form $a + b\sqrt{3}$, where a and b are constants to be determined.

Solution.

Part (a).



When $\theta = 0$ or $\theta = \pi$, we have $r = 1$. Thus, C intersects the horizontal axis at $(1, 0)$ and $(1, \pi)$. When $\theta = \pi/2$, we have $r = 2$. Thus, C intersects the vertical axis at $(2, \pi/2)$. When $\theta = 3\pi/2$, we have $r = 0$. Thus, C passes through the pole.

For tangents at the pole, $r = 0 \implies \sin 3\theta = 1 \implies \theta = \pi/6, 5\pi/6, 3\pi/2$.

Part (b). Note that $\mathrm{d}r/\mathrm{d}\theta = -3\cos 3\theta$ evaluates to 3 when $\theta = \pi/3$. Thus,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}\right|_{\theta=\frac{\pi}{3}} = \frac{3\sqrt{3}+1}{3-\sqrt{3}} = \frac{12+10\sqrt{3}}{6} = 2 + \frac{5}{3}\sqrt{3}.$$

Hence, when $\theta = \pi/3$, the gradient of the curve is $2 + 5\sqrt{3}/2$.

A7 Vectors I - Basic Properties and Vector Algebra

Tutorial A7

Problem 1. The vector $\mathbf{v}$ is defined by $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$. Find the unit vector in the direction of $\mathbf{v}$ and hence find a vector of magnitude 25 which is parallel to $\mathbf{v}$.

Solution.

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3^2 + (-4)^2 + 1^2}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \quad 25\hat{\mathbf{v}} = \frac{25}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}.$$

* * * * *

Problem 2. With respect to an origin O, the position vectors of the points A, B, C and D are $4\mathbf{i} + 7\mathbf{j}$, $\mathbf{i} + 3\mathbf{j}$, $2\mathbf{i} + 4\mathbf{j}$ and $3\mathbf{i} + d\mathbf{j}$ respectively.

(a) Find the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

(b) Find the value of d if B, C and D are collinear. State the ratio $\frac{BC}{BD}$.

Solution.

Part (a). Note that

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

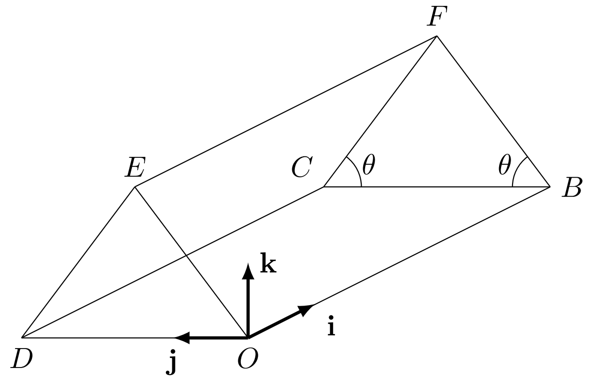
Part (b). If B, C and D are collinear, then $\overrightarrow{BC} = \lambda\overrightarrow{CD}$ for some $\lambda \in \mathbb{R}$.

$$\overrightarrow{BC} = \lambda\overrightarrow{CD} \implies \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda\left(\overrightarrow{OD} - \overrightarrow{OC}\right) = \lambda\left[\begin{pmatrix} 3 \\ d \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix}\right] = \begin{pmatrix} \lambda \\ \lambda(d-4) \end{pmatrix}.$$

Hence, $\lambda = 1$ and $\lambda(d - 4) = 1$, whence $d = 5$. Also, $\overrightarrow{BC} = \overrightarrow{CD}$. Thus,

$$\frac{BC}{BD} = \frac{BC}{BC + CD} = \frac{BC}{BC + BC} = \frac{1}{2}.$$

Problem 3. The diagram shows a roof, with horizontal rectangular base $OBCD$, where $OB = 10$ m and $BC = 6$ m. The triangular planes ODE and BCF are vertical and the ridge EF is horizontal to the base. The planes $OBFE$ and $DCFE$ are each inclined at an angle θ to the horizontal, where $\tan\theta = 4/3$. The point O is taken as the origin and vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, each of length 1 m, are taken along OB, OD and vertically upwards from O respectively.



Find the position vectors of the points B, C, D, E and F.

Solution. Note that $\overrightarrow{OB} = 10\mathbf{i}$ and $\overrightarrow{BC} = 6\mathbf{j}$. Thus, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = 10\mathbf{i} + 6\mathbf{j}$. Also, note that $\triangle ODE \cong \triangle BCF$. Hence, $\overrightarrow{OD} = \overrightarrow{BC} = 6\mathbf{j}$. Note that $\triangle ODE$ is isosceles. Let G be the mid-point of OD. Since $\tan\theta = 4/3$, we have

$$\frac{EG}{DG} = \frac{4}{3} \implies EG = \frac{4}{3}DG = \frac{2}{3}OD = \frac{2}{3} \cdot 6 = 4 \implies \overrightarrow{GE} = 4\mathbf{k}.$$

Hence,

$$\overrightarrow{OE} = \overrightarrow{OG} + \overrightarrow{GE} = \frac{1}{2}\overrightarrow{OD} + \overrightarrow{GE} = 3\mathbf{j} + 4\mathbf{k}.$$

Hence,

$$\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = \overrightarrow{OB} + \overrightarrow{OE} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

Thus,

$$\overrightarrow{OB} = 10\mathbf{i}, \quad \overrightarrow{OC} = 10\mathbf{i} + 6\mathbf{j}, \quad \overrightarrow{OD} = 6\mathbf{j}, \quad \overrightarrow{OE} = 3\mathbf{j} + 4\mathbf{k}, \quad \overrightarrow{OF} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

* * * * *

Problem 4. Find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ given that

(a) $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$

(b) $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

Solution.

Part (a). We have $\mathbf{u} = \langle 1,\ -1,\ 1\rangle$ and $\mathbf{v} = \langle 3,\ 2,\ 7\rangle$. Hence,

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (-1)(2) + (1)(7) = 8, \quad \mathbf{u} \times \mathbf{v} = \begin{pmatrix} (-1)(7) - (2)(1) \\ (1)(3) - (7)(1) \\ (1)(2) - (3)(-1) \end{pmatrix} = \begin{pmatrix} -9 \\ -4 \\ 5 \end{pmatrix}.$$

Let the angle between $\mathbf{u}$ and $\mathbf{v}$ be θ.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{8}{\sqrt{3}\sqrt{62}} \implies \theta = 54.1^\circ \text{ (1 d.p.)}.$$

Part (b). We have $\mathbf{u} = \langle 2, 0, -3 \rangle$ and $\mathbf{v} = \langle -1, 7, 2 \rangle$. Hence,

$$\mathbf{u} \cdot \mathbf{v} = (2)(-1) + (0)(7) + (-3)(2) = -8, \quad \mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} (0)(2) - (7)(-3) \\ (-3)(-1) - (2)(2) \\ (2)(7) - (-1)(0) \end{pmatrix} = \begin{pmatrix} 21 \\ -1 \\ 14 \end{pmatrix}.$$

Let the angle between $\mathbf{u}$ and $\mathbf{v}$ be θ.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{-8}{\sqrt{13}\sqrt{54}} \implies \theta = 107.6^\circ \text{ (1 d.p.)}.$$

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Problem 5. Find $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$ given that $\mathbf{u} = 2\mathbf{a} - \mathbf{b}$, $\mathbf{v} = -\mathbf{a} + 3\mathbf{b}$, where $|\mathbf{a}| = 2$, $|\mathbf{b}| = 1$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.

Solution.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (2\mathbf{a} - \mathbf{b}) \cdot (-\mathbf{a} + 3\mathbf{b}) = -2\mathbf{a} \cdot \mathbf{a} + 6\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - 3\mathbf{b} \cdot \mathbf{b} \\ &= -2\,|\mathbf{a}|^2 - 3\,|\mathbf{b}|^2 + 7\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta = -2(2)^2 - 3(1)^2 + 7(2)(1)\cos 60^\circ = -4. \\ |\mathbf{u} \times \mathbf{v}| &= |(2\mathbf{a} - \mathbf{b}) \times (-\mathbf{a} + 3\mathbf{b})| = |-2\mathbf{a} \times \mathbf{a} + 6\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - 3\mathbf{b} \times \mathbf{b}| \\ &= |5\mathbf{a} \times \mathbf{b}| = 5\,|\mathbf{a}|\,|\mathbf{b}| \sin\theta = 5(2)(1)\sin 60^\circ = 5\sqrt{3}. \end{aligned}$$

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Problem 6. If $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$, find

(a) a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$,

(b) a vector perpendicular to both $(3\mathbf{b} - 5\mathbf{c})$ and $(7\mathbf{b} + \mathbf{c})$.

Solution.

Part (a). Note that $\mathbf{a} \times \mathbf{b} = \langle 11, -4, -5 \rangle$. Hence, $\widehat{\mathbf{a} \times \mathbf{b}} = \frac{1}{\sqrt{162}} \langle 11, -4, -5 \rangle$.

Part (b). Observe that $(3\mathbf{b} - 5\mathbf{c}) \times (7\mathbf{b} + \mathbf{c}) = \lambda \mathbf{b} \times \mathbf{c}$ for some $\lambda \in \mathbb{R}$. It hence suffices to find $\mathbf{b} \times \mathbf{c}$, which works out to be $\langle -3, 6, 3 \rangle$.

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Problem 7. The position vectors of the points A, B and C are given by $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 11\mathbf{i} + \lambda\mathbf{j} + 14\mathbf{k}$ respectively. Find

(a) a unit vector parallel to $\overrightarrow{AB}$;

(b) the position vector of the point D such that $ABCD$ is a parallelogram, leaving your answer in terms of λ;

(c) the value of λ if A, B and C are collinear;

(d) the position vector of the point P on AB is $AP : PB = 2 : 1$.

Solution.

Part (a).

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}.$$

Note that $\left|\overrightarrow{AB}\right| = \sqrt{61}$. Hence, the required vector is $\frac{1}{\sqrt{61}} \langle 3, -4, 6 \rangle$.

Part (b). Since $ABCD$ is a parallelogram, we have that $\overrightarrow{AD} = \overrightarrow{BC}$. Thus,

$$\overrightarrow{OD} - \mathbf{a} = \mathbf{c} - \mathbf{b} \implies \overrightarrow{OD} = \mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} = \begin{pmatrix} 8 \\ \lambda + 4 \\ 8 \end{pmatrix}.$$

Part (c). Given that A, B and C are collinear, we have $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R}$. Hence,

$$\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} = k(\mathbf{c} - \mathbf{b}) = k\left[\begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}\right] = k\begin{pmatrix} 6 \\ \lambda + 1 \\ 12 \end{pmatrix}.$$

We hence see that $k = 1/2$, whence $\lambda = -9$.

Part (d). By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a} + 2\mathbf{b}}{2+1} = \frac{1}{3}\left[\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}.$$

* * * * *

Problem 8. $ABCD$ is a square, and M and N are the midpoints of BC and CD respectively. Express $\overrightarrow{AC}$ in terms of $\mathbf{p}$ and $\mathbf{q}$, where $\overrightarrow{AM} = \mathbf{p}$ and $\overrightarrow{AN} = \mathbf{q}$.

Solution. Let $ABCD$ be a square with side length $2k$ with A at the origin. Then $\mathbf{p} = \overrightarrow{AM} = \langle 2k, -k \rangle$ and $\mathbf{q} = \overrightarrow{AN} = \langle k, -2k \rangle$. Hence, $\mathbf{p} + \mathbf{q} = \langle 3k, -3k \rangle$. Thus, $\overrightarrow{AC} = \langle 2k, -2k \rangle = \frac{2}{3}\langle 3k, -3k \rangle = \frac{2}{3}(\mathbf{p} + \mathbf{q})$.

* * * * *

Problem 9. The points A, B have position vectors $\mathbf{a}$, $\mathbf{b}$ respectively, referred to an origin O, where $\mathbf{a}$ and $\mathbf{b}$ are not parallel to each other. The point C lies on AB between A and B and is such that $\frac{AC}{CB} = 2$, and D is the mid-point of OC. The line AD produced meets OB at E.

Find, in terms of $\mathbf{a}$ and $\mathbf{b}$,

(a) the position vector of C (referred to O),

(b) the vector $\overrightarrow{AD}$. Find the values of $\frac{OE}{EB}$ and $\frac{AE}{ED}$.

Solution.

Part (a). By the ratio theorem,

$$\overrightarrow{OC} = \frac{\mathbf{a} + 2\mathbf{b}}{2+1} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}.$$

Part (b). Since D is the midpoint of OC, we have $\overrightarrow{OD} = \frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}$. Hence,

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \left(\frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}\right) - \mathbf{a} = -\frac{5}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}.$$

Using Menelaus' theorem on $\triangle BCO$,

$$\frac{BA}{AC}\frac{CD}{DO}\frac{OE}{EB} = 1 \implies \frac{OE}{EB} = \frac{2}{3}.$$

Using Menelaus' theorem on $\triangle BEA$,

$$\frac{BO}{OE}\frac{ED}{DA}\frac{AC}{CB} = 1 \implies \frac{ED}{AD} = \frac{1}{5} \implies \frac{AE}{ED} = \frac{AD + DE}{ED} = 6.$$

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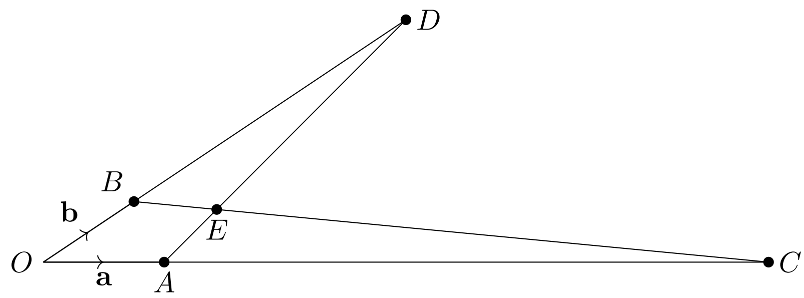
Problem 10.

(a) The angle between the vectors $(3\mathbf{i} - 2\mathbf{j})$ and $(6\mathbf{i} + d\mathbf{j} - \sqrt{7}\mathbf{k})$ is $\arccos \frac{6}{13}$. Show that $2d^2 - 117d + 333 = 0$.

(b) With reference to the origin O, the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{AC} = 5\mathbf{a}$, $\overrightarrow{BD} = 3\mathbf{b}$. The lines AD and BC cross at E.



(i) Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) The point F divides the line CD in the ratio $5:3$. Show that O, E and F are collinear, and find $OE : EF$.

Solution.

Part (a). Let $\mathbf{a} = \langle 3, -2, 0\rangle$ and $\mathbf{b} = \left\langle 6, d, -\sqrt{7}\right\rangle$. Note that $\mathbf{a}\cdot\mathbf{b} = 18 - 2d$. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \implies \frac{6}{13} = \frac{18-2d}{\sqrt{43+d^2}\sqrt{13}} \implies \frac{9}{13} = \frac{(9-d)^2}{43+d^2}$$
$$\implies 9(43+d^2) = 13(d^2 - 18d + 81) \implies 2d^2 - 117d + 333 = 0.$$

Part (b).

Part (b)(i). By Menelaus' theorem,

$$\frac{OC}{CA}\frac{AE}{ED}\frac{DB}{BO} = 1 \implies \frac{AE}{ED} = \frac{5}{18} \implies \overrightarrow{AE} = \frac{5}{23}\overrightarrow{AD} \implies \overrightarrow{OE} = \overrightarrow{OA} + \frac{5}{23}\overrightarrow{AD}.$$

Since $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 4\mathbf{b} - \mathbf{a}$. Thus,

$$\overrightarrow{OE} = \mathbf{a} + \frac{5}{23}(4\mathbf{b} - \mathbf{a}) = \frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}.$$

Part (b)(ii). By the ratio theorem,

$$\overrightarrow{OF} = \frac{3\mathbf{c} + 5\mathbf{d}}{5+3} = \frac{23}{8}\left(\frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}\right) = \frac{23}{8}\overrightarrow{OE}.$$

Thus, $OE : OF = 8 : 23$.

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Problem 11. Relative to the origin O, two points A and B have position vectors given by $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$ and $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$ respectively.

(a) The point P divides the line AB in the ratio $2 : 1$. Find the coordinates of P.

(b) Show that AB and OP are perpendicular.

(c) The vector $\mathbf{c}$ is a unit vector in the direction of $\overrightarrow{OP}$. Write $\mathbf{c}$ as a column vector and give the geometrical meaning of $|\mathbf{a} \cdot \mathbf{c}|$.

(d) Find $\mathbf{a} \times \mathbf{p}$, where $\mathbf{p}$ is the vector $\overrightarrow{OP}$, and give the geometrical meaning of $|\mathbf{a} \times \mathbf{p}|$. Hence, write down the area of triangle OAP.

Solution.

Part (a). We have $\mathbf{a} = \langle 14, 14, 14\rangle = 14\langle 1, 1, 1\rangle$ and $\mathbf{b} = \langle 11, -13, 2\rangle$. By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a} + 2\mathbf{b}}{2+1} = \frac{1}{3}\left[\begin{pmatrix}14\\14\\14\end{pmatrix} + 2\begin{pmatrix}11\\-13\\2\end{pmatrix}\right] = \begin{pmatrix}12\\-4\\6\end{pmatrix} = 2\begin{pmatrix}6\\-2\\3\end{pmatrix}.$$

Hence, $P(12, -4, 6)$

Part (b). Consider $\overrightarrow{AB} \cdot \overrightarrow{OP}$.

$$\overrightarrow{AB} \cdot \overrightarrow{OP} = \left[\begin{pmatrix}11\\-13\\2\end{pmatrix} - \begin{pmatrix}14\\14\\14\end{pmatrix}\right] \cdot \begin{pmatrix}12\\-4\\6\end{pmatrix} = -3\begin{pmatrix}1\\9\\4\end{pmatrix} \cdot 2\begin{pmatrix}6\\-2\\3\end{pmatrix} = 0.$$

Since $\overrightarrow{AB} \cdot \overrightarrow{OP} = 0$, AB and OP must be perpendicular.

Part (c). We have

$$\mathbf{c} = \frac{\overrightarrow{OP}}{\left|\overrightarrow{OP}\right|} = \frac{1}{\sqrt{6^2 + (-2)^2 + 3^2}}\begin{pmatrix}6\\-2\\3\end{pmatrix} = \frac{1}{7}\begin{pmatrix}6\\-2\\3\end{pmatrix}.$$

$|\mathbf{a} \cdot \mathbf{c}|$ is the length of the projection of $\mathbf{a}$ on $\overrightarrow{OP}$.

Part (d). We have

$$\mathbf{a} \times \mathbf{p} = 14\begin{pmatrix}1\\1\\1\end{pmatrix} \times 2\begin{pmatrix}6\\-2\\3\end{pmatrix} = 28\begin{pmatrix}1 \cdot 3 - (-2) \cdot 1\\ 1 \cdot 6 - 3 \cdot 1\\ 1 \cdot -2 - 6 \cdot 1\end{pmatrix} = 28\begin{pmatrix}5\\3\\-8\end{pmatrix}.$$

$|\mathbf{a} \times \mathbf{p}|$ is twice the area of $\triangle OAP$.

$$[\triangle OAP] = \frac{1}{2}|\mathbf{a} \times \mathbf{p}| = 14\sqrt{98} = 98\sqrt{2} \text{ units}^2.$$

Problem 12. The points A, B and C have position vectors given by $\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{j}-\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}-\mathbf{k}$ respectively.

(a) Find the area of the triangle ABC. Hence, find the sine of the angle BAC.

(b) Find a vector perpendicular to the plane ABC.

(c) Find the projection vector of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$.

(d) Find the distance of C to AB.

Solution.

Part (a). We have $\overrightarrow{OA} = \langle 1,\ -1,\ 1\rangle$, $\overrightarrow{OB} = \langle 0,\ 1,\ -1\rangle$ and $\overrightarrow{OC} = \langle 2,\ -1,\ -1\rangle$. Note that $\overrightarrow{AB} = \langle -1,\ 2,\ -2\rangle$ and $\overrightarrow{AC} = \langle 1,\ 0,\ -2\rangle$. Thus,

$$[\triangle ABC] = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|\begin{pmatrix}-4\\-4\\-2\end{pmatrix}\right| = \frac{1}{2}\cdot 6 = 3 \text{ units}^2.$$

We have

$$\sin BAC = \frac{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{6}{3\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

Part (b). $\langle 2,\ 2,\ 1\rangle$ is parallel to $\overrightarrow{AB}\times\overrightarrow{AC}$ and is hence perpendicular to the plane ABC.

Part (c). The projection vector of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$ is given by

$$\left(\overrightarrow{AC}\cdot\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}\right)\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{1}{3}\begin{pmatrix}-1\\2\\-2\end{pmatrix}.$$

Part (d). Observe that

$$\left|\overrightarrow{AC}\times\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}\right| = \frac{1}{3}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = 2.$$

Hence, the perpendicular distance between C and AB is 2 units.

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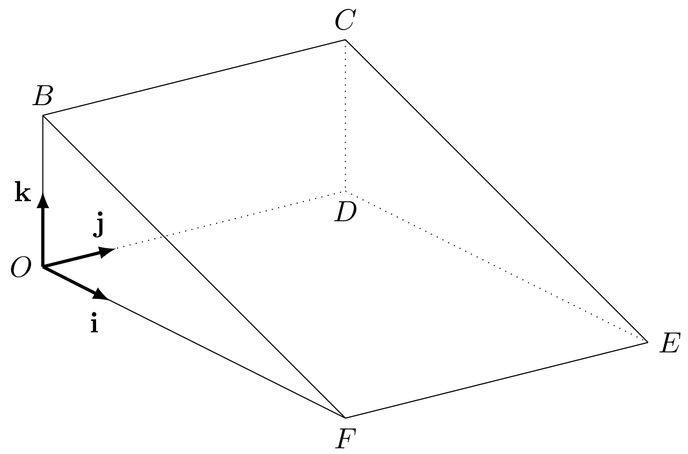
Problem 13.



The diagram shows a vehicle ramp $OBCDEF$ with horizontal rectangular base $ODEF$ and vertical rectangular face $OBCD$. Taking the point O as the origin, the perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to the edges OF, OD and OB respectively. The lengths of OF, OD and OB are $2h$ units, 3 units and h units respectively.

(a) Show that $\overrightarrow{OC} = 3\mathbf{j} + h\mathbf{k}$.

(b) The point P divides the segment CF in the ratio $2:1$. Find $\overrightarrow{OP}$ in terms of h.

For parts (c) and (d), let $h = 1$.

(c) Find the length of projection of $\overrightarrow{OP}$ onto $\overrightarrow{OC}$.

(d) Using the scalar product, find the angle that the rectangular face $BCEF$ makes with the horizontal base.

Solution.

Part (a). We have

$$\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DC} = \overrightarrow{OD} + \overrightarrow{OB} = 3\mathbf{j} + h\mathbf{k}.$$

Part (b). By the ratio theorem,

$$\overrightarrow{OP} = \frac{\overrightarrow{OC} + 2\overrightarrow{OF}}{2+1} = \frac{1}{3}\left[\begin{pmatrix}0\\3\\h\end{pmatrix} + 2\begin{pmatrix}2h\\0\\0\end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix}4h\\3\\h\end{pmatrix}.$$

Part (c). The length of projection of $\overrightarrow{OP}$ onto $\overrightarrow{OC}$ is given by

$$\left|\overrightarrow{OP} \cdot \frac{\overrightarrow{OC}}{\left|\overrightarrow{OC}\right|}\right| = \frac{1}{3\sqrt{10}}\left|\begin{pmatrix}4\\3\\1\end{pmatrix} \cdot \begin{pmatrix}0\\3\\1\end{pmatrix}\right| = \frac{\sqrt{10}}{3} \text{ units.}$$

Part (d). Note that $\overrightarrow{OF} = \langle 2, 0, 0\rangle$ and $\overrightarrow{BF} = \overrightarrow{OF} - \overrightarrow{OB} = \langle 2, 0, -1\rangle$. Let θ be the angle the rectangular face $BCEF$ makes with the horizontal base.

$$\cos\theta = \frac{\overrightarrow{OF} \cdot \overrightarrow{BF}}{\left|\overrightarrow{OF}\right|\left|\overrightarrow{BF}\right|} = \frac{4}{2\sqrt{5}} \implies \theta = 26.6^\circ \text{ (1 d.p.)}.$$

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Problem 14. The position vectors of the points A and B relative to the origin O are $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ respectively. The point P on AB is such that $AP : PB = \lambda : 1 - \lambda$. Show that $\overrightarrow{OP} = (1+\lambda)\mathbf{i} + (2 - 5\lambda)\mathbf{j} + (-2 + 8)\mathbf{k}$ where λ is a real parameter.

(a) Find the value of λ for which OP is perpendicular to AB.

(b) Find the value of λ for which angles $\angle AOP$ and $\angle POB$ are equal.

Solution. By the ratio theorem,

$$\overrightarrow{OP} = \frac{\lambda\overrightarrow{OB} + (1-\lambda)\overrightarrow{OA}}{\lambda + (1-\lambda)} = \lambda\begin{pmatrix}2\\-3\\6\end{pmatrix} + (1-\lambda)\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix}.$$

Part (a). Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 1, -5, 8 \rangle$. For OP to be perpendicular to AB, we must have $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$.

$$\overrightarrow{OP} \cdot \overrightarrow{AB} = 0 \implies \begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix} = 0 \implies -25 + 90\lambda = 0 \implies \lambda = \frac{5}{18}.$$

Part (b). Suppose $\angle AOP = \angle POB$. Then $\cos \angle AOP = \cos \angle POB$. Thus,

$$\frac{\overrightarrow{OP} \cdot \overrightarrow{OA}}{\left|\overrightarrow{OP}\right| \left|\overrightarrow{OA}\right|} = \frac{\overrightarrow{OP} \cdot \overrightarrow{OB}}{\left|\overrightarrow{OP}\right| \left|\overrightarrow{OB}\right|} \implies \overrightarrow{OP} \cdot \left(\frac{1}{3}\overrightarrow{OA} - \frac{1}{7}\overrightarrow{OB}\right) = 0 \implies \overrightarrow{OP} \cdot \left(7\overrightarrow{OA} - 3\overrightarrow{OB}\right) = 0.$$

This gives

$$\begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix} \cdot \left[7\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right] = \begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 23 \\ -32 \end{pmatrix} = 0.$$

Taking the dot product and simplifying, we see that $111 - 370\lambda = 0$, whence $\lambda = \frac{3}{10}$.

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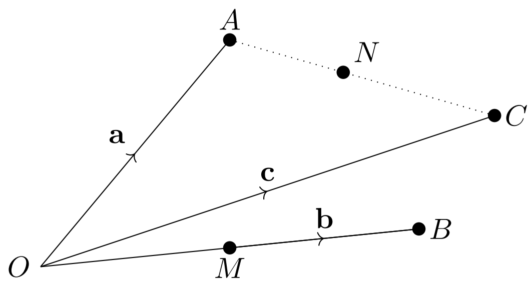
Problem 15.



The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$,

(a) Explain why $\mathbf{c}$ can be expressed as $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for constants λ and μ.

The point N is on AC such that $AN : NC = 3 : 4$.

(b) Write down the position vector of N in terms of $\mathbf{a}$ and $\mathbf{c}$.

(c) It is given that the area of triangle ONC is equal to the area of triangle OMC, where M is the mid-point of OB. By finding the areas of these triangles in terms of $\mathbf{a}$ and $\mathbf{b}$, find λ in terms of μ in the case where λ and μ are both positive.

Solution.

Part (a). Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are co-planar and $\mathbf{a}$ is not parallel to $\mathbf{b}$, $\mathbf{c}$ can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Part (b). By the ratio theorem,

$$\overrightarrow{ON} = \frac{4\mathbf{a} + 3\mathbf{c}}{3+4} = \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c}.$$

Part (c). Let $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$. The area of $\triangle ONC$ is given by

$$[\triangle ONC] = \frac{1}{2}\left|\overrightarrow{ON} \times \overrightarrow{OC}\right| = \frac{1}{2}\left|\left[\frac{4}{7}\mathbf{a} + \frac{3}{7}(\lambda\mathbf{a} + \mu\mathbf{b})\right] \times (\lambda\mathbf{a} + \mu\mathbf{b})\right| = \frac{2\mu}{7}\left|\mathbf{a} \times \mathbf{b}\right|.$$

Meanwhile, the area of $\triangle OMC$ is given by

$$[\triangle OMC] = \frac{1}{2}\left|\overrightarrow{OM} \times \overrightarrow{OC}\right| = \frac{1}{2}\left|\frac{1}{2}\mathbf{b} \times (\lambda\mathbf{a} + \mu\mathbf{b})\right| = \frac{\lambda}{4}\left|\mathbf{a} \times \mathbf{b}\right|.$$

Since the two areas are equal,

$$[\triangle ONC] = [\triangle OMC] \implies \frac{2\mu}{7}\left|\mathbf{a} \times \mathbf{b}\right| = \frac{\lambda}{4}\left|\mathbf{a} \times \mathbf{b}\right| \implies \lambda = \frac{8}{7}\mu.$$

Self-Practice A7

Problem 1. The position vector of points A, B and C relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $k\mathbf{a}$ respectively. The point P lies on AB and is such that $AP = 2PB$. The point Q lies on BC such that $CQ = 6QB$. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of P and Q. Given that OPQ is a straight line, find

(a) the value of k,

(b) the ratio of $OP : PQ$.

The position vector of a point R is $\frac{7}{3}\mathbf{a}$. Show that PR is parallel to BC.

Solution. By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a} + 2\mathbf{b}}{1+2} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$$

and

$$\overrightarrow{OQ} = \frac{k\mathbf{a} + 6\mathbf{b}}{6+1} = \frac{k}{7}\mathbf{a} + \frac{6}{7}\mathbf{b}.$$

Part (a). Since OPQ is a straight line, there exists some $\lambda \in \mathbb{R}$ such that

$$\overrightarrow{OQ} = \lambda\overrightarrow{OP} \implies \frac{k}{7}\mathbf{a} + \frac{6}{7}\mathbf{b} = \frac{\lambda}{3}\mathbf{a} + \frac{2\lambda}{3}\mathbf{b}.$$

Comparing coefficients of $\mathbf{b}$ terms, we have $\lambda = 9/7$, whence

$$\frac{k}{7} = \frac{9/7}{3} \implies k = 3.$$

Part (b). Note that $\overrightarrow{OQ} = \frac{9}{7}\overrightarrow{OP}$. Hence, $OP : PQ = 2 : 7$.

Note that

$$\overrightarrow{PR} = \frac{7}{3}\mathbf{a} - \left(\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}\right) = 2\mathbf{a} - \frac{2}{3}\mathbf{b}.$$

Hence,

$$\overrightarrow{BC} = 3\mathbf{a} - \mathbf{b} = \frac{3}{2}\left(2\mathbf{a} - \frac{2}{3}\mathbf{b}\right) = \frac{3}{2}\overrightarrow{PR}.$$

Hence, $PR \parallel BC$.

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Problem 2. The position vectors of the points P and R, relative to an origin O, are $\mathbf{p}$ and $\mathbf{r}$ respectively, where $\mathbf{p}$ and $\mathbf{r}$ are not parallel to each other. Q is a point such that $\overrightarrow{OQ} = 2\overrightarrow{OP}$ and S is a point such that $\overrightarrow{OS} = 2\overrightarrow{OR}$. T is the midpoint of QS.

Find, in terms of $\mathbf{p}$ and $\mathbf{r}$,

(a) $\overrightarrow{PR}$,

(b) $\overrightarrow{QT}$,

(c) $\overrightarrow{TR}$.

What shape is the quadrilateral $PRTQ$? Name another quadrilateral that has the same shape as $PRTQ$.

Solution. By the midpoint theorem,

$$\overrightarrow{OT} = \frac{\overrightarrow{OQ} + \overrightarrow{OS}}{2} = \mathbf{p} + \mathbf{r}.$$

Part (a).

$$\overrightarrow{PR} = \mathbf{r} - \mathbf{p}.$$

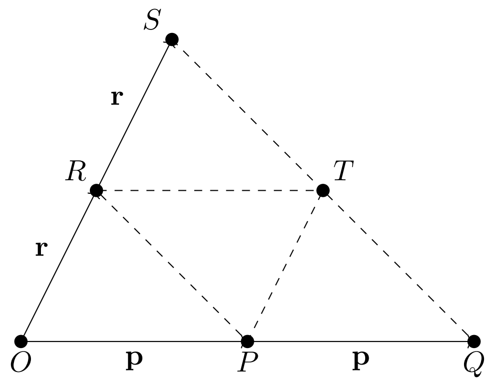
Part (b).

$$\overrightarrow{QT} = (\mathbf{p} + \mathbf{r}) - (2\mathbf{p}) = \mathbf{r} - \mathbf{p}.$$

Part (c).

$$\overrightarrow{TR} = \mathbf{r} - (\mathbf{r} + \mathbf{p}) = -\mathbf{p}.$$

Consider the following diagram:



Clearly, $PRTQ$ is a parallelogram. Likewise, $ORTP$ is also a parallelogram.

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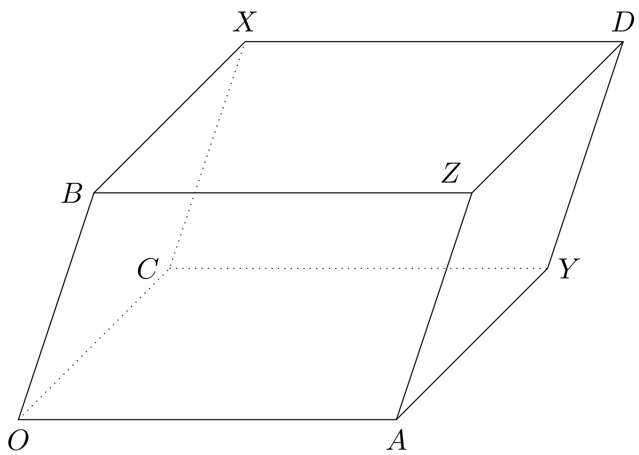
Problem 3. The position vectors of points A, B, C are given by $\overrightarrow{OA} = 5\mathbf{i}$, $\overrightarrow{OB} = \mathbf{i} + 3\mathbf{k}$, $\overrightarrow{OC} = \mathbf{i} + 4\mathbf{j}$. A parallelepiped has OA, OB and OC as three edges, and the remaining edges are X, Y, Z and D as shown in the diagram.



(a) Write down the position vectors of X, Y, Z and D in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and calculate the length of OD.

(b) Calculate the size of angle OZY.

(c) The point P divides CZ in the ratio $\lambda : 1$. Write down the position vector of P, and evaluate λ if $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{CZ}$.

Solution.

Part (a). We have

$$\begin{aligned}
\overrightarrow{OX} &= \quad\;\;\; \overrightarrow{OB} + \overrightarrow{OC} &&= 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}, \\
\overrightarrow{OY} &= \overrightarrow{OA} \quad\;\;\; + \overrightarrow{OC} &&= 6\mathbf{i} + 4\mathbf{j}, \\
\overrightarrow{OZ} &= \overrightarrow{OA} + \overrightarrow{OB} &&= 6\mathbf{i} + 3\mathbf{k}, \\
\overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} &&= 7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}.
\end{aligned}$$

Part (b). Note that $\overrightarrow{ZY} = \langle 0, 4, -3 \rangle$. Hence,

$$\cos \angle OZY = \frac{\overrightarrow{OZ} \cdot \overrightarrow{ZY}}{\left|\overrightarrow{OZ}\right| \left|\overrightarrow{ZY}\right|} = \frac{9}{\sqrt{45}\sqrt{25}} \implies \angle OZY = 74.4° \text{ (1 d.p.)}.$$

Part (c). By the ratio theorem,

$$\overrightarrow{OP} = \frac{\overrightarrow{OC} + \lambda \overrightarrow{OZ}}{1 + \lambda} = \frac{1}{1 + \lambda} \left[3\lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \right].$$

Note that $\overrightarrow{CZ} = \langle 5, -4, 3 \rangle$. Since $\overrightarrow{OP} \perp \overrightarrow{CZ}$, we have

$$\overrightarrow{OP} \cdot \overrightarrow{CZ} = 0.$$

Hence,

$$3\lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} = 39\lambda - 11 = 0,$$

whence $\lambda = 11/39$.

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Problem 4. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{c} = 2$. Given that $|\mathbf{a}| = 1$, $|\mathbf{b}| = 2$, $|\mathbf{c}| = 3$, find

(a) $|\mathbf{a} - \mathbf{b}|$;

(b) $|\mathbf{a} - \mathbf{b} - \mathbf{c}|$.

Solution.

Part (a). Observe that

$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}.$$

Since $\mathbf{a} \cdot \mathbf{b} = 0$, we get

$$|\mathbf{a} - \mathbf{b}|^2 = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 = 1^1 + 2^2 = 5.$$

Thus, $|\mathbf{a} - \mathbf{b}| = \sqrt{5}$.

Part (b). Observe that

$$|\mathbf{a} - \mathbf{b} - \mathbf{c}|^2 = (\mathbf{a} - \mathbf{b} - \mathbf{c}) \cdot (\mathbf{a} - \mathbf{b} - \mathbf{c}) = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) - 2\mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) + \mathbf{c} \cdot \mathbf{c}.$$

Since $\mathbf{a} \cdot \mathbf{c} = 2$ and $\mathbf{b} \cdot \mathbf{c} = 0$, we have

$$|\mathbf{a} - \mathbf{b} - \mathbf{c}|^2 = |\mathbf{a} - \mathbf{b}|^2 - 2(2) + |\mathbf{c}|^2 = 5 - 2(2) + 3^2 = 10.$$

Thus, $|\mathbf{a}-\mathbf{b}-\mathbf{c}| = \sqrt{10}$.

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Problem 5. The position vectors of the points M and N are given by

$$\overrightarrow{OM} = \lambda\mathbf{i} + (2\lambda-1)\mathbf{j} + \mathbf{k}, \qquad \overrightarrow{ON} = (1-\lambda)\mathbf{i} + 3\lambda\mathbf{j} - 2\mathbf{k},$$

where λ is a scalar. Find the values of λ for which $\overrightarrow{OM}$ and $\overrightarrow{ON}$ are perpendicular. When $\lambda = 1$, find the size of $\angle MNO$ to the nearest degree.

Solution. Since $\overrightarrow{OM} \perp \overrightarrow{ON}$, we have

$$\overrightarrow{OM} \cdot \overrightarrow{ON} = \begin{pmatrix} \lambda \\ 2\lambda-1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1-\lambda \\ 3\lambda \\ -2 \end{pmatrix} = 5\lambda^2 - 2\lambda - 2 = 0.$$

Solving the quadratic, we get

$$\lambda = \frac{1 \pm \sqrt{11}}{5}.$$

When $\lambda = 1$, we have

$$\overrightarrow{OM} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \overrightarrow{ON} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}, \quad \overrightarrow{MN} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}.$$

Hence,

$$\cos\angle MNO = \frac{\overrightarrow{ON} \cdot \overrightarrow{MN}}{\left|\overrightarrow{ON}\right| \left|\overrightarrow{MN}\right|} = \frac{12}{\sqrt{13}\sqrt{14}} \implies \angle MNO = 27°.$$

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Problem 6. The points A, B, C and D have position vectors $\mathbf{i}-2\mathbf{j}+5\mathbf{k}$, $\mathbf{i}+3\mathbf{j}$, $10\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $-2\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ respectively, with respect to an origin O. The point P on AB is such that $AP : PB = \lambda : 1-\lambda$ and point Q on CD is such that $CQ : QD = \mu : 1-\mu$. Find $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of λ and μ respectively.

Given that PQ is perpendicular to both AB and CD, show that $\overrightarrow{PQ} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Solution. By the ratio theorem,

$$\overrightarrow{OP} = \frac{(1-\lambda)\overrightarrow{OA} + \lambda\overrightarrow{OB}}{(1-\lambda)+\lambda} = \overrightarrow{OA} + \lambda\left(\overrightarrow{OB} - \overrightarrow{OA}\right) = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + 5\lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

and

$$\overrightarrow{OQ} = \frac{(1-\mu)\overrightarrow{OC} + \overrightarrow{OD}}{(1-\mu)+\mu} = \overrightarrow{OC} + \mu\left(\overrightarrow{OD} - \overrightarrow{OC}\right) = \begin{pmatrix} 10 \\ 1 \\ 2 \end{pmatrix} + 3\mu \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}.$$

Note that

$$\overrightarrow{PQ} = \left[\begin{pmatrix} 10 \\ 1 \\ 2 \end{pmatrix} + 3\mu \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + 5\lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right] = 3\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + 3\mu \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} - 5\lambda \begin{pmatrix} 0 \\ 1 \\ 01 \end{pmatrix}.$$

Since PQ is perpendicular to AB, we have

$$\overrightarrow{PQ} \cdot \overrightarrow{AB} = \left[3\begin{pmatrix}3\\1\\-1\end{pmatrix} + 3\mu\begin{pmatrix}-4\\1\\1\end{pmatrix} - 5\lambda\begin{pmatrix}0\\1\\01\end{pmatrix}\right] \cdot \left[5\begin{pmatrix}0\\1\\-1\end{pmatrix}\right] = 5(6-10\lambda) = 0.$$

Thus, $\lambda = 3/5$.

Since PQ is perpendicular to CD, we have

$$\overrightarrow{PQ} \cdot \overrightarrow{CD} = \left[3\begin{pmatrix}3\\1\\-1\end{pmatrix} + 3\mu\begin{pmatrix}-4\\1\\1\end{pmatrix} - 5\lambda\begin{pmatrix}0\\1\\01\end{pmatrix}\right] \cdot \left[3\begin{pmatrix}-4\\1\\1\end{pmatrix}\right] = 3(-36+54\mu) = 0.$$

Thus, $\mu = 2/3$.

Hence,

$$\overrightarrow{PQ} = 3\begin{pmatrix}3\\1\\-1\end{pmatrix} + 3\left(\frac{2}{3}\right)\begin{pmatrix}-4\\1\\1\end{pmatrix} - 5\left(\frac{3}{5}\right)\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}.$$

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Problem 7. The position vectors of the vertices A, B and C of a triangle are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. If O is the origin and not within the triangle, show that the area of triangle OAB is $\frac{1}{2}\left|\mathbf{a}\times\mathbf{b}\right|$, and deduce and expression for the area of the triangle ABC.

Hence, or otherwise, show that the perpendicular distance from B to AC is

$$\frac{\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|}{\left|\mathbf{c}-\mathbf{a}\right|}.$$

Solution. Let $\theta = \angle AOB$ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Clearly,

$$[\triangle OAB] = \frac{1}{2}(OA)(OB)\sin\theta = \frac{1}{2}\left|\mathbf{a}\times\mathbf{b}\right|.$$

Note that $AB = \left|\mathbf{b}-\mathbf{a}\right|$ and $AC = \left|\mathbf{c}-\mathbf{a}\right|$. Hence,

$$[\triangle ABC] = \frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\right|.$$

Expanding, we get

$$[\triangle ABC] = \frac{1}{2}\left|\mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}-\mathbf{a}\times\mathbf{c}\right| = \frac{1}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|.$$

Let the perpendicular distance from B to AC be h. Then

$$[\triangle ABC] = \frac{1}{2}h(AC) = \frac{1}{2}h\left|\mathbf{c}-\mathbf{a}\right|.$$

Hence,

$$h = \frac{2[\triangle ABC]}{\left|\mathbf{c}-\mathbf{a}\right|} = \frac{\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|}{\left|\mathbf{c}-\mathbf{a}\right|}.$$

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Problem 8 (🌶). The points A, B and C lie on a circle with centre O and diameter AC. It is given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Find $\overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Hence, show that AB is perpendicular to BC.

(b) Given that $\angle AOB = 30^\circ$, find $\overrightarrow{OF}$ where F is the foot of perpendicular of B to AC. Hence, find $\overrightarrow{OB'}$, where B' is the reflection of B in the line AC.

Solution.

Part (a). Since A, B and C lie on the same circle, $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$. Since AC is the diameter of the circle, $\mathbf{c}$ is in the opposite direction as $\mathbf{a}$. Hence, $\mathbf{c} = -\mathbf{a}$. Thus,

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -\mathbf{a} - \mathbf{b}.$$

Also note that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}.$$

Consider $\overrightarrow{AB} \cdot \overrightarrow{BC}$:

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = (\mathbf{b} - \mathbf{a}) \cdot -(\mathbf{a} + \mathbf{b}) = -(\mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a}) = -\left(|\mathbf{b}|^2 - |\mathbf{a}|^2\right) = 0.$$

Thus, AB is perpendicular to BC.

Part (b). Observe that

$$\frac{\sqrt{3}}{2} = \cos\angle AOB = \frac{OF}{OB} = \frac{\left|\overrightarrow{OF}\right|}{|\mathbf{a}|} \implies \left|\overrightarrow{OF}\right| = \frac{\sqrt{3}}{2}\,|\mathbf{a}|\,.$$

Since $\overrightarrow{OF}$ is in the same direction as $\overrightarrow{OA}$, we have

$$\overrightarrow{OF} = \frac{\sqrt{3}}{2}\mathbf{a}.$$

Note that

$$\overrightarrow{BF} = \frac{\sqrt{3}}{2}\mathbf{a} - \mathbf{b}.$$

By the midpoint theorem,

$$\overrightarrow{OF} = \frac{\overrightarrow{OB} + \overrightarrow{OB'}}{2} \implies \overrightarrow{OB'} = 2\overrightarrow{OF} - \overrightarrow{OB} = \sqrt{3}\mathbf{a} - \mathbf{b}.$$

Assignment A7

Problem 1. The points A and B have position vectors relative to the origin O, denoted by $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. The point P lies on AB such that $AP : PB = \lambda : 1$. The point Q lies on OP extended such that $OP = 2PQ$ and $\overrightarrow{BQ} = \overrightarrow{OA} + \mu\overrightarrow{OB}$. Find the values of the real constants λ and μ.

Solution. By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} \implies \overrightarrow{OQ} = \frac{3}{2}\overrightarrow{OP} = \frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda}.$$

However, we also have

$$\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{BQ} = \mathbf{a} + (1 + \mu)\mathbf{b}.$$

This gives the equality

$$\frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} = \mathbf{a} + (1 + \mu)\mathbf{b}.$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-parallel, we can compare the $\mathbf{a}$- and $\mathbf{b}$-components of both vectors separately. This gives us

$$\frac{3}{2} \cdot \frac{1}{1 + \lambda} = 1, \quad \frac{3}{2} \cdot \frac{\lambda}{1 + \lambda} = 1 + \mu,$$

which has the unique solution $\lambda = 1/2$ and $\mu = -1/2$.

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Problem 2. Given that $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{p} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$ where $\lambda \in \mathbb{R}$, find the possible value(s) of λ for which the angle between $\mathbf{p}$ and $\mathbf{k}$ is 45°.

Solution. Observe that

$$\mathbf{p} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b} = \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 - 3\lambda \\ -2 + 3\lambda \\ 6 - 6\lambda \end{pmatrix}.$$

Thus,

$$|\mathbf{p}|^2 = (4 - 3\lambda)^2 + (-2 + 3\lambda)^2 + (6 - 6\lambda)^2 = 54\lambda^2 - 108\lambda + 56.$$

Since the angle between $\mathbf{p}$ and $\mathbf{k}$ is 45°,

$$\cos 45° = \frac{\mathbf{p} \cdot \mathbf{k}}{|\mathbf{p}|\,|\mathbf{k}|} \implies \frac{1}{\sqrt{2}} = \frac{6 - 6\lambda}{|\mathbf{p}|} \implies \frac{|\mathbf{p}|^2}{2} = (6 - 6\lambda)^2.$$

We thus obtain the quadratic equation

$$\frac{54\lambda^2 - 108\lambda + 56}{2} = 36\lambda^2 - 72\lambda + 36 \implies 9\lambda^2 - 18\lambda + 8 = 0,$$

which has solutions $\lambda = 2/3$ and $\lambda = 4/3$. However, we must reject $\lambda = 4/3$ since $6 - 6\lambda = |\mathbf{p}|/\sqrt{2} > 0 \implies \lambda < 1$. Thus, $\lambda = 2/3$.

Problem 3.

(a) $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors such that $\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$. State the relation between the directions of $\mathbf{a}$ and $\mathbf{b}$, and find $|\mathbf{b}|$.

(b) $\mathbf{a}$ is a non-zero vector such that $\mathbf{a} = \sqrt{3}$ and $\mathbf{b}$ is a unit vector. Given that $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and the angle between them is $5\pi/6$, find the exact value of the length of projection of $\mathbf{a}$ on $\mathbf{b}$. By considering $(2\mathbf{a}+\mathbf{b}) \cdot (2\mathbf{a}+\mathbf{b})$, or otherwise, find the exact value of $|2\mathbf{a}+\mathbf{b}|$.

Solution.

Part (a). $\mathbf{a}$ and $\mathbf{b}$ either have the same or opposite direction. Let $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R}$.

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b} = (\mathbf{a} \cdot \lambda\mathbf{a})\lambda\mathbf{a} = \lambda^2 |\mathbf{a}|^2 \mathbf{a} \implies \lambda^2 |\mathbf{a}|^2 = 1 \implies |\mathbf{b}| = |\lambda |\mathbf{a}|| = 1.$$

Part (b). Note that $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \cos(5\pi/6) = -3/2$. Hence, the length of projection of $\mathbf{a}$ on $\mathbf{b}$ is $\left|\mathbf{a} \cdot \hat{\mathbf{b}}\right| = 3/2$ units.

Observe that

$$|2\mathbf{a}+\mathbf{b}|^2 = (2\mathbf{a}+\mathbf{b}) \cdot (2\mathbf{a}+\mathbf{b}) = 4|\mathbf{a}|^2 + 4(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2 = 7.$$

Thus, $|2\mathbf{a}+\mathbf{b}| = \sqrt{7}$.

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Problem 4. The points A, B, C, D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ given by $\mathbf{a} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{d} = 4\mathbf{i}-\mathbf{j}-\mathbf{k}$, respectively. The point P lies on AB produced such that $AP = 2AB$, and the point Q is the mid-point of AC.

(a) Show that PQ is perpendicular to AQ.

(b) Find the area of the triangle APQ.

(c) Find a vector perpendicular to the plane ABC.

(d) Find the cosine of the angle between $\overrightarrow{AD}$ and $\overrightarrow{BD}$.

Solution. Note that $\overrightarrow{AB} = \langle 0, 0, -1\rangle$, $\overrightarrow{AC} = \langle 2, 0, -2\rangle$ and $\overrightarrow{AD} = \langle 3, -3, -4\rangle$.

Part (a). Note that

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + 2\overrightarrow{AB} = \begin{pmatrix}1\\2\\1\end{pmatrix}$$

and

$$\overrightarrow{OQ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \begin{pmatrix}2\\2\\2\end{pmatrix}.$$

Thus,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = \begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

Since $\overrightarrow{PQ} \cdot \overrightarrow{AQ} = 0$, the two vectors are perpendicular, whence $PQ \perp AQ$.

Part (b). Note that $\overrightarrow{AP} = \langle 0,\, 0,\, -2\rangle$. Hence,

$$[\triangle APQ] = \frac{1}{2}\left|\overrightarrow{AP} \times \overrightarrow{AQ}\right| = \frac{1}{2}\left|\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right| = 1 \text{ units}^2.$$

Part (c). The vector $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0,\, -2,\, 0\rangle$ is perpendicular to the plane ABC.

Part (d). Let the angle between $\overrightarrow{AD}$ and $\overrightarrow{BD}$ be θ. Note that $\overrightarrow{BD} = -3\,\langle -1,\, 1,\, 1\rangle$. Hence,

$$\cos\theta = \frac{\overrightarrow{AD} \cdot \overrightarrow{BD}}{\left|\overrightarrow{AD}\right|\left|\overrightarrow{BD}\right|} = \frac{30}{\sqrt{34} \cdot 3\sqrt{3}} = \frac{10}{\sqrt{102}}.$$

A8 Vectors II - Lines

Tutorial A8

Problem 1. For each of the following, write down a vector equivalent of the line l and convert it to parametric and Cartesian forms.

(a) l passes through the point with position vector $-\mathbf{i}+\mathbf{k}$ and is parallel to the vector $\mathbf{i}+\mathbf{j}$.

(b) l passes through the points $P(1,-1,3)$ and $Q(2,1,-2)$.

(c) l passes through the origin and is parallel to the line $m : \mathbf{r} = \langle 1,\ -1,\ 3\rangle + \lambda\langle 1,\ 2,\ 3\rangle$, where $\lambda \in \mathbb{R}$.

(d) l is the x-axis.

(e) l passes through the point $C(4,-1,2)$ and is parallel to the z-axis.

Solution.

Part (a).

Form	Expression
Vector	$\mathbf{r} = \langle -1,\ 0,\ 1\rangle + \lambda\langle 1,\ 1,\ 0\rangle,\ \lambda \in \mathbb{R}$
Parametric	$x = \lambda - 1,\ y = \lambda,\ z = 1$
Cartesian	$x + 1 = y,\ z = 1$

Part (b).

Form	Expression
Vector	$\mathbf{r} = \langle 1,\ -1,\ 3\rangle + \lambda\langle 1,\ 2,\ -5\rangle,\ \lambda \in \mathbb{R}$
Parametric	$x = \lambda + 1,\ y = 2\lambda - 1,\ z = -5\lambda + 3$
Cartesian	$x - 1 = \frac{y+1}{2} = \frac{3-z}{5}$

Part (c).

Form	Expression
Vector	$\mathbf{r} = \lambda\langle 1,\ 2,\ 3\rangle,\ \lambda \in \mathbb{R}$
Parametric	$x = \lambda,\ y = 2\lambda,\ z = 3\lambda$
Cartesian	$x = \frac{y}{2} = \frac{z}{3}$

Part (d).

Form	Expression
Vector	$\mathbf{r} = \lambda\langle 1,\ 0,\ 0\rangle,\ \lambda \in \mathbb{R}$
Parametric	$x = \lambda,\ y = 0,\ z = 0$
Cartesian	$x \in \mathbb{R},\ y = 0,\ z = 0$

Part (e).

Form	Expression
Vector	$\mathbf{r} = \langle 4, -1, 2\rangle + \lambda \langle 0, 0, 1\rangle, \lambda \in \mathbb{R}$
Parametric	$x = 4,\ y = -1,\ z = \lambda + 2$
Cartesian	$x = 4,\ y = -1,\ z \in \mathbb{R}$

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Problem 2. For each of the following, determine if l_1 and l_2 are parallel, intersecting or skew. In the case of intersecting lines, find the position vector of the point of intersection. In addition, find the acute angle between the lines l_1 and l_2.

(a) $l_1 : x - 1 = -y = z - 2$ and $l_2 : \frac{x-2}{2} = -\frac{y+1}{2} = \frac{z-4}{2}$

(b) $l_1 : \mathbf{r} = \langle 1, 0, 0\rangle + \alpha \langle 4, -2, -3\rangle, \alpha \in \mathbb{R}$ and $l_2 : \mathbf{r} = \langle 0, 10, 1\rangle + \beta \langle 3, 8, 1\rangle$

(c) $l_1 : \mathbf{r} = (\mathbf{i} - 5\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}), \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(5\mathbf{i} - 4\mathbf{j} - \mathbf{k}), \mu \in \mathbb{R}$

Solution.

Part (a). Note that l_1 and l_2 have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda \begin{pmatrix}1\\-1\\1\end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix}2\\1\\4\end{pmatrix} + \mu \begin{pmatrix}2\\-2\\2\end{pmatrix}, \mu \in \mathbb{R}.$$

Since $\langle 2, -2, 2\rangle = 2\langle 1, -1, 1\rangle$, l_1 and l_2 are parallel ($\theta = 0$). Since $\langle 1, 0, 2\rangle \neq \langle 2, 1, 4\rangle + \mu \langle 2, -2, 2\rangle$ for all real μ, we have that l_1 and l_2 are distinct.

Part (b). Since $\langle 4, -2, 3\rangle \neq \beta \langle 3, 8, 1\rangle$ for all real β, it follows that l_1 and l_2 are not parallel.

Consider $l_1 = l_2$.

$$l_1 = l_2 \implies \begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha \begin{pmatrix}4\\-2\\-3\end{pmatrix} = \begin{pmatrix}0\\10\\1\end{pmatrix} + \beta \begin{pmatrix}3\\8\\1\end{pmatrix} \implies \alpha \begin{pmatrix}4\\-2\\-3\end{pmatrix} - \beta \begin{pmatrix}3\\8\\1\end{pmatrix} = \begin{pmatrix}-1\\10\\1\end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 4\alpha - 3\beta = -1 \\ -2\alpha - 8\beta = 10 \\ -3\alpha - \beta = 1 \end{cases}$$

There are no solutions to the above system. Hence, l_1 and l_2 do not intersect and are thus skew.

Let θ be the acute angle between l_1 and l_2.

$$\cos\theta = \frac{|\langle 4, -2, -3\rangle \cdot \langle 3, 8, 1\rangle|}{|\langle 4, -2, -3\rangle|\,|\langle 3, 8, 1\rangle|} = \frac{7}{\sqrt{2146}} \implies \theta = 81.3° \text{ (1 d.p.)}.$$

Part (c). Note that l_1 and l_2 have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix}1\\0\\-5\end{pmatrix} + \lambda \begin{pmatrix}1\\-1\\1\end{pmatrix} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix}1\\-1\\1\end{pmatrix} + \mu \begin{pmatrix}5\\-4\\-1\end{pmatrix}, \lambda, \mu \in \mathbb{R}.$$

Since $\langle 1,\, -1,\, 1\rangle \neq \mu\,\langle 5,\, -4,\, -1\rangle$ for all real μ, it follows that l_1 and l_2 are not parallel. Consider $l_1 = l_2$.

$$l_1 = l_2 \implies \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \implies \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} -5\mu + \lambda = 0 \\ 4\mu - \lambda = -1 \\ \mu + \lambda = 6 \end{cases}$$

The above system has the unique solution $\lambda = 5$ and $\mu = 1$. Hence, l_1 and l_2 intersect at $\langle 1,\, 0,\, -5\rangle + 5\,\langle 1,\, -1,\, 1\rangle = \langle 6,\, -5,\, 0\rangle$.

Let θ be the acute angle between l_1 and l_2.

$$\cos\theta = \frac{|\langle 1,\, -1,\, 1\rangle \cdot \langle 5,\, -4,\, -1\rangle|}{|\langle 1,\, -1,\, 1\rangle|\,|\langle 5,\, -4,\, -1\rangle|} = \frac{8}{3\sqrt{14}} \implies \theta = 44.5^\circ \text{ (1 d.p.)}.$$

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Problem 3.

(a) Find the shortest distance from the point $(1, 2, 3)$ to the line with equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$, $\lambda \in \mathbb{R}$.

(b) Find the length of projection of $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ onto the line with equation $\frac{x+5}{4} = \frac{y-5}{3} = 10 - 2z$.

(c) Find the projection of $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ onto the line with equation $\frac{x+5}{4} = \frac{y-5}{3} = 10 - 2z$.

Solution.

Part (a). Let $\overrightarrow{OP} = \langle 1,\, 2,\, 3\rangle$ and $\overrightarrow{OA} = \langle 3,\, 2,\, 4\rangle$. Note that $\overrightarrow{AP} = \langle -2,\, 0,\, -1\rangle$. The shortest distance between P and the line is thus

$$\text{Shortest distance} = \frac{|\langle -2,\, 0,\, -1\rangle \times \langle 1,\, 2,\, 2\rangle|}{|\langle 1,\, 2,\, 2\rangle|} = \frac{|\langle 2,\, -3,\, -4\rangle|}{3} = \frac{\sqrt{29}}{3} \text{ units.}$$

Part (b). Note that the line has vector form

$$\mathbf{r} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda' \begin{pmatrix} 4 \\ 3 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}.$$

The length of projection of $\langle 4,\, -5,\, 6\rangle$ onto the line is thus given by

$$\text{Length of projection} = \frac{|\langle 4,\, -5,\, 6\rangle \cdot \langle 8,\, 6,\, -1\rangle|}{|\langle 8,\, 6,\, -1\rangle|} = \frac{4}{\sqrt{101}} \text{ units.}$$

Part (c).

$$\text{Projection} = \left[\frac{\langle 4,\, -5,\, 6\rangle \cdot \langle 8,\, 6,\, -1\rangle}{|\langle 8,\, 6,\, -1\rangle|}\right] \cdot \frac{\langle 8,\, 6,\, -1\rangle}{|\langle 8,\, 6,\, -1\rangle|} = \frac{-4}{101}\begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix}$$

Problem 4. The points P and Q have coordinates $(0, -1, -1)$ and $(3, 0, 1)$ respectively, and the equations of the lines l_1 and l_2 are given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \mu \in \mathbb{R}.$$

(a) Show that P lies on l_1 but not on l_2.

(b) Determine if l_2 passes through Q.

(c) Find the coordinates of the foot of the perpendicular from P to l_2. Hence, or otherwise, find the perpendicular distance from P to l_2.

(d) Find the length of projection of $\overrightarrow{PQ}$ onto l_2.

Solution. We have that $\overrightarrow{OP} = \langle 0, -1, -1\rangle$ and $\overrightarrow{OQ} = \langle 3, 0, 1\rangle$.

Part (a). When $\lambda = -2$, we have $\langle 0, 1, -3\rangle - 2\langle 0, 1, -1\rangle = \langle 0, -1, -1\rangle = \overrightarrow{OP}$. Hence, P lies on l_1.

Observe that all points on l_2 have a z-coordinate of 1. Since P has a z-coordinate of -1, P does not lie on l_2.

Part (b). When $\mu = 3$, we have $\langle -3, 3, 1\rangle + 3\langle 2, -1, 0\rangle = \langle 3, 0, 1\rangle = \overrightarrow{OQ}$. Hence, l_2 passes through Q.

Part (c). Let the foot of the perpendicular from P to l_2 be F. Since F is on l_2, we have that $\overrightarrow{OF} = \langle -3, 3, 1\rangle + \mu\langle 2, -1, 0\rangle$ for some real μ. We also have that $\overrightarrow{PF} \cdot \langle 2, -1, 0\rangle = 0$. Note that

$$\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 + 2\mu \\ 4 - \mu \\ 2 \end{pmatrix}.$$

Hence,

$$\overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \implies \begin{pmatrix} -3 + 2\mu \\ 4 - \mu \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \implies -10 + 5\mu = 0 \implies \mu = 2.$$

Hence, $\overrightarrow{OF} = \langle -3, 3, 1\rangle + 2\langle 3, -1, 0\rangle = \langle 1, 1, 1\rangle$. Thus, $F(1, 1, 1)$. The perpendicular distance from P to l_2 is thus $\left|\overrightarrow{PF}\right| = |\langle 1, 2, 2\rangle| = 3$ units.

Part (d). Note that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. The length of projection of $\overrightarrow{PQ}$ onto l_2 is thus given by

$$\text{Length of projection} = \frac{|\langle 3, 1, 2\rangle \cdot \langle 2, -1, 0\rangle|}{|\langle 2, -1, 0\rangle|} = \frac{5}{\sqrt{5}} = \sqrt{5} \text{ units.}$$

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Problem 5. The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

respectively. Find the position vectors of the points P on l_1 and Q on l_2 such that O, P and Q are collinear, where O is the origin.

Solution. We have that $\overrightarrow{OP} = \langle 0,\ 1,\ 2\rangle + s\,\langle 1,\ 0,\ 3\rangle$ and $\overrightarrow{OQ} = \langle -2,\ 3,\ 1\rangle + t\,\langle 2,\ 1,\ 0\rangle$ for some $s, t \in \mathbb{R}$. For O, P and Q to be collinear, we need $\overrightarrow{OP} = \lambda\overrightarrow{OQ}$ for some $\lambda \in \mathbb{R}$:

$$\begin{pmatrix}0\\1\\2\end{pmatrix} + s\begin{pmatrix}1\\0\\3\end{pmatrix} = \lambda\left[\begin{pmatrix}-2\\3\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\0\end{pmatrix}\right] \implies \begin{pmatrix}s\\1\\2+3s\end{pmatrix} = \lambda\begin{pmatrix}-2+2t\\3+t\\1\end{pmatrix}.$$

This gives us the system:

$$\begin{cases} s = \lambda(-2+2t) \\ 1 = \lambda(3+t) \\ 2+3s = \lambda \end{cases}$$

Substituting the third equation into the first two gives the reduced system:

$$\begin{cases} s = (2+3s)(-2+2t) \\ 1 = (2+3s)(3+t) \end{cases}$$

Subtracting twice of the second equation from the first yields $s - 2 = -8(2+3s)$, whence $s = -14/25$. It quickly follows that $t = 1/8$. Hence,

$$\overrightarrow{OP} = \begin{pmatrix}0\\1\\2\end{pmatrix} - \frac{14}{25}\begin{pmatrix}1\\0\\3\end{pmatrix} = \frac{1}{25}\begin{pmatrix}-14\\25\\8\end{pmatrix}, \quad \overrightarrow{OQ} = \begin{pmatrix}-2\\3\\1\end{pmatrix} + \frac{1}{8}\begin{pmatrix}2\\1\\0\end{pmatrix} = \frac{1}{8}\begin{pmatrix}-14\\25\\8\end{pmatrix}.$$

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Problem 6. Relative to the origin O, the points A, B and C have position vectors $5\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}$, $-4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $-5\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}$ respectively.

(a) Find the Cartesian equation of the line AB.

(b) Find the length of projection of $\overrightarrow{AC}$ onto the line AB. Hence, find the perpendicular distance from C to the line AB.

(c) Find the position vector of the foot N of the perpendicular from C to the line AB.

(d) The point D is such that it is a reflection of point C about the line AB. Find the position vector of D.

Solution. We have that $\overrightarrow{OA} = \langle 5,\ 4,\ 10\rangle$, $\overrightarrow{OB} = \langle -4,\ 4,\ -2\rangle$ and $\overrightarrow{OC} = \langle -5,\ 9,\ 5\rangle$.

Part (a). Note that $\overrightarrow{AB} = \langle -9,\ 0,\ -12\rangle = -3\,\langle 3,\ 0,\ 4\rangle$. The line AB hence has the vector form

$$\mathbf{r} = \begin{pmatrix}5\\4\\10\end{pmatrix} + \lambda\begin{pmatrix}3\\0\\4\end{pmatrix},\ \lambda \in \mathbb{R}$$

and Cartesian form $\frac{x-5}{3} = \frac{z-10}{4}$, $y = 4$.

Part (b). Note that $\overrightarrow{AC} = \langle -10,\ 5,\ -5\rangle = -5\,\langle 2,\ -1,\ 1\rangle$. Hence, the length of projection of $\overrightarrow{AC}$ onto the line AB is given by

$$\text{Length of projection} = \frac{\left|\overrightarrow{AC}\cdot\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|} = \frac{1}{15}\left|5\begin{pmatrix}2\\-1\\1\end{pmatrix}\cdot 3\begin{pmatrix}3\\0\\4\end{pmatrix}\right| = 10 \text{ units.}$$

Since $\left|\overrightarrow{AC}\right| = 5\sqrt{6}$, the perpendicular distance from C to the line AB is $\sqrt{\left(5\sqrt{6}\right)^2 - 10^2} = 5\sqrt{2}$ units.

Part (c). Let $\overrightarrow{AN} = \lambda \langle -9,\, 0,\, -12 \rangle$ for some $\lambda \in \mathbb{R}$ such that $\left|\overrightarrow{AN}\right| = 10$.

$$\left|\overrightarrow{AN}\right| = 10 \implies 15\lambda = 10 \implies \lambda = \frac{2}{3}.$$

Hence, $\overrightarrow{AN} = \frac{2}{3} \langle -9,\, 0,\, -12 \rangle = \langle -6,\, 0,\, -8 \rangle$. Thus, $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \langle -1,\, 4,\, 2 \rangle$.

Part (d). Note that $\overrightarrow{NC} = \overrightarrow{OC} - \overrightarrow{ON} = \langle -4,\, 5,\, 3 \rangle$. Since D is the reflection of C about AB, we have that $\overrightarrow{ND} = -\overrightarrow{NC}$. Thus,

$$\overrightarrow{OD} = \overrightarrow{ON} + \overrightarrow{ND} = \overrightarrow{ON} - \overrightarrow{NC} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}.$$

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Problem 7. The points A and B have coordinates $(0, 9, c)$ and $(d, 5, -2)$ respectively, where c and d are constants. The line l has equation $\frac{x+3}{-1} = \frac{y-1}{4} = \frac{z-5}{3}$.

(a) Given that $d = 22/7$ and the line AB intersects l, find the value of c. Find also the coordinates of the foot of the perpendicular from A to l.

(b) Given instead that the lines AB and l are parallel, state the value of c and d and find the shortest distance between the lines AB and l.

Solution. We have that $\overrightarrow{OA} = \langle 0,\, 9,\, c \rangle$ and $\overrightarrow{OB} = \langle d,\, 5,\, -2 \rangle$. We also have that the line l is given by the vector $\mathbf{r} = \langle -3,\, 1,\, 5 \rangle + \lambda \langle -1,\, 4,\, 3 \rangle$ for $\lambda \in \mathbb{R}$.

Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle d,\, -4,\, -2-c \rangle$. Hence, the line AB is given by the vector $\mathbf{r}_{AB} = \langle d,\, 5,\, -2 \rangle + \mu \langle d,\, -4,\, -2-c \rangle$ for $\mu \in \mathbb{R}$.

Part (a). Consider the direction vectors of AB and l. Since $\langle 22/7,\, -4,\, -2-c \rangle \neq \lambda \langle -1,\, 4,\, 3 \rangle$ for all real λ and c, the lines AB and l are not parallel. Hence, AB and l intersect at only one point. Thus, there must be a unique solution to $\mathbf{r} = \mathbf{r}_{AB}$.

$$\begin{aligned} \mathbf{r} = \mathbf{r}_{AB} &\implies \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 22/7 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 22/7 \\ -4 \\ -2-c \end{pmatrix} \\ &\implies \lambda \begin{pmatrix} -7 \\ 28 \\ 21 \end{pmatrix} - \mu \begin{pmatrix} 22 \\ -28 \\ -14-7c \end{pmatrix} = \begin{pmatrix} 43 \\ 28 \\ -49 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} -\lambda - \quad 22\mu = 43 \\ 4\lambda + \quad 28\mu = 28 \\ 3\lambda + (14+7c)\,\mu = -49 \end{cases}$$

Solving the first two equations gives $\lambda = 91/3$ and $\mu = -10/3$. It follows from the third equation that $c = 4$.

Let F be the foot of the perpendicular from A to l. We have that $\overrightarrow{OF} = \langle -3,\, 1,\, 5 \rangle + \lambda \langle -1,\, 4,\, 3 \rangle$ for some $\lambda \in \mathbb{R}$. We also have that $\overrightarrow{AF} \cdot \langle -1,\, 4,\, 3 \rangle = 0$. Note that

$$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix} -3-\lambda \\ -8+4\lambda \\ 1+3\lambda \end{pmatrix}.$$

Hence,

$$\overrightarrow{AF} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3-\lambda \\ -8+4\lambda \\ 1+3\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \implies -26+26\lambda = 0 \implies \lambda = 1.$$

Hence, $\overrightarrow{OF} = \langle -3,\ 1,\ 5\rangle + \langle -1,\ 4,\ 3\rangle = \langle -4,\ 5,\ 8\rangle$. The foot of the perpendicular from A to l hence has coordinates $(-4, 5, 8)$.

Part (b). Given that AB is parallel to l, one of their direction vectors must be a scalar multiple of the other. Hence, for some real λ, $\langle -1,\ 4,\ 3\rangle = \lambda \langle d,\ -4,\ -2-c\rangle$. It is obvious that $\lambda = -1$, whence $c = 1$ and $d = 1$.

Note that the direction vector of l and AB is $\langle -1,\ 4,\ 3\rangle$. Also note that l passes through $(-3, 1, 5)$ and AB passes through $(1, 5, -2)$. Since $\langle 1,\ 5,\ -2\rangle - \langle -3,\ 1,\ 5\rangle = \langle 4,\ 4,\ -7\rangle$, the shortest distance between AB and l is

$$\frac{|\langle -1,\ 4,\ 3\rangle \times \langle 4,\ 4,\ -7\rangle|}{|\langle -1,\ 4,\ 3\rangle|} = \frac{1}{\sqrt{26}} \left| \begin{pmatrix} -40 \\ -5 \\ -20 \end{pmatrix} \right| = \frac{45}{\sqrt{26}} \text{ units.}$$

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Problem 8. The equation of the line L is $\mathbf{r} = \langle 1,\ 3,\ 7\rangle + t\,\langle 2,\ -1,\ 5\rangle$, $t \in \mathbb{R}$. The points A and B have position vectors $\langle 9,\ 3,\ 26\rangle$ and $\langle 13,\ 9,\ \alpha\rangle$ respectively. The line L intersects the line through A and B at P.

(a) Find α and the acute angle between line L and AB.

The point C has position vector $\langle 2,\ 5,\ 1\rangle$ and the foot of the perpendicular from C to L is Q.

(b) Find the position vector of Q. Hence, find the shortest distance from C to L.

(c) Find the position vector of the point of reflection of the point C about the line L. Hence, find the reflection of the line passing through C and the point $(1, 3, 7)$ about the line L.

Solution.

Part (a). Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 4,\ 6,\ \alpha - 26\rangle$. The line AB is thus given by $\mathbf{r}_{AB} = \langle 9,\ 3,\ 26\rangle + u\,\langle 4,\ 6,\ \alpha-26\rangle$ for $u \in \mathbb{R}$. Note that AB is not parallel to L. Hence, $\overrightarrow{OP}$ is the only solution to the equation $\mathbf{r} = \mathbf{r}_{AB}$.

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix} + u \begin{pmatrix} 4 \\ 6 \\ \alpha-26 \end{pmatrix} \implies t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - u \begin{pmatrix} 4 \\ 6 \\ \alpha-26 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 19 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 2t - \quad 4u = 8 \\ -t - \quad 6u = 0 \\ 5t - (\alpha-26)\,u = 19 \end{cases}$$

Solving the first two equations gives $t = 3$ and $u = -\frac{1}{2}$. It follows from the third equation that $\alpha = 34$.

Let the acute angle between L and AB be θ.

$$\cos\theta = \frac{|\langle 2,\,-1,\,5\rangle \cdot \langle 4,\,6,\,8\rangle|}{|\langle 2,\,-1,\,5\rangle|\,|\langle 4,\,6,\,8\rangle|} = \frac{42}{\sqrt{30}\sqrt{116}} \implies \theta = 44.6^\circ \text{ (1 d.p.)}.$$

Part (b). Since Q is on L, we have that $\overrightarrow{OQ} = \langle 1,\,3,\,7\rangle + t\,\langle 2,\,-1,\,5\rangle$ for some real t. Further, since $\overrightarrow{CQ} \perp L$, we have that $\overrightarrow{CQ} \cdot \langle 2,\,-1,\,5\rangle = 0$. Note that

$$\overrightarrow{CQ} = \overrightarrow{OQ} - \overrightarrow{OC} = \begin{pmatrix} -1+2t \\ -2-t \\ 6+5t \end{pmatrix}.$$

Thus,

$$\overrightarrow{CQ} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 0 \implies \begin{pmatrix} -1+2t \\ -2-t \\ 6+5t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 0 \implies 30 + 30t = 0 \implies t = 1.$$

Hence, $\overrightarrow{OQ} = \langle 1,\,3,\,7\rangle + \langle 2,\,-1,\,5\rangle = \langle -1,\,4,\,2\rangle$. The shortest distance from C to L is thus

$$\left|\overrightarrow{CQ}\right| = \left|\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}\right| = \left|\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}\right| = \sqrt{11} \text{ units.}$$

Part (c). Let C' be the reflection of C about L. Note that

$$\overrightarrow{OC'} = \overrightarrow{OQ} - \overrightarrow{QC} = \overrightarrow{OQ} + \overrightarrow{CQ} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}.$$

Note that $(1,3,7)$ is on L and is hence invariant under a reflection about L. Let the reflection about L of the line passing through C and $(1,3,7)$ be L'. Since $\langle -4,\,3,\,3\rangle - \langle 1,\,3,\,7\rangle = \langle -5,\,0,\,-4\rangle \parallel \langle 5,\,0,\,4\rangle$, L' hence has direction vector $\langle 5,\,0,\,4\rangle$. Thus, L' is given by $\mathbf{r}' = \langle 1,\,3,\,7\rangle + \lambda\,\langle 5,\,0,\,4\rangle$ for $\lambda \in \mathbb{R}$.

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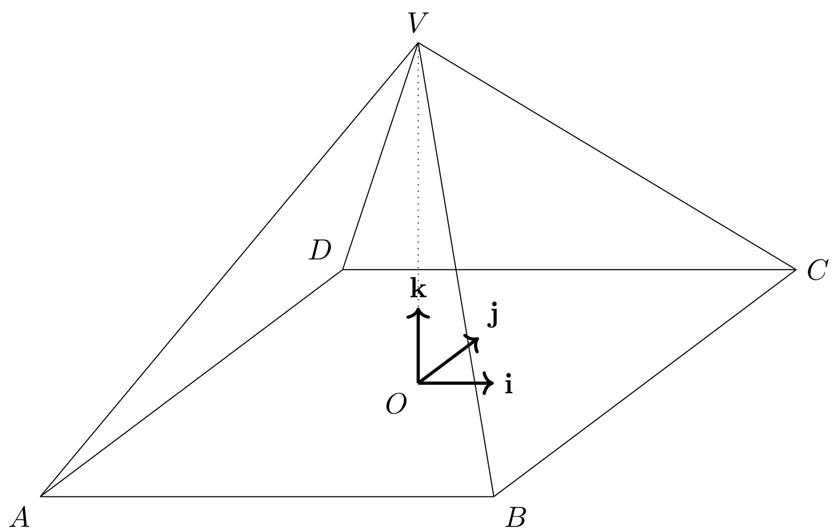
Problem 9.



In the diagram, O is the origin of the square base $ABCD$ of a right pyramid with vertex V. The perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to AB, AD and OV respectively. The length of AB is 4 units and the length of OV is $2h$ units. P, Q, M and N are the mid-points of AB, BC, CV and VA respectively. The point O is taken as the origin for position vectors.

Show that the equation of the line PM may be expressed as $\mathbf{r} = \langle 0, -2, 0\rangle + t\langle 1, 3, h\rangle$, where t is a parameter.

(a) Find an equation for the line QN.

(b) Show that the lines PM and QN intersect and that the position vector $\overrightarrow{OX}$ of their point of intersection is $\mathbf{r} = \frac{1}{2}\langle 1, -1, h\rangle$.

(c) Given that OX is perpendicular to VB, find the value of h and calculate the acute angle between PM and QN, giving your answer correct to the nearest $0.1°$.

Solution. We are given that $\overrightarrow{OP} = \langle 0, -2, 0\rangle$, $\overrightarrow{OC} = \langle 2, 2, 0\rangle$ and $\overrightarrow{OV} = \langle 0, 0, 2h\rangle$. Hence, $\overrightarrow{CV} = \overrightarrow{OV} - \overrightarrow{OC} = \langle -2, -2, 2h\rangle$. Thus, $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CV} = \langle -1, -1, h\rangle$. Since $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM} = \langle 1, 1, h\rangle$, we have that $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP} = \langle 1, 3, h\rangle$. Thus, PM is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}, \ t \in \mathbb{R}.$$

Part (a). Since $\overrightarrow{OM} = \langle 1, 1, h\rangle$, by symmetry, $\overrightarrow{ON} = \langle -1, -1, h\rangle$. Given that $\overrightarrow{OQ} = \langle 2, 0, 0\rangle$, we have that $\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ} = \langle -3, -1, h\rangle$. Thus, QN is given by

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix}, \ u \in \mathbb{R}.$$

Part (b). Consider $PM = QN$.

$$PM = QN \implies \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} \implies t\begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} - u\begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} t + 3u = 2 \\ 3t + u = 2 \\ ht - hu = 0 \end{cases}$$

From the first two equations, we see that $t = \frac{1}{2}$ and $u = \frac{1}{2}$, which is consistent with the third equation. Hence, $\overrightarrow{OX} = \langle 0, -2, 0\rangle + \frac{1}{2}\langle 1, 3, h\rangle = \frac{1}{2}\langle 1, -1, h\rangle$.

Part (c). Note that $\overrightarrow{OB} = \langle 2, -2, 6\rangle$, whence $\overrightarrow{VB} = \overrightarrow{OB} - \overrightarrow{OV} = \langle 2, -2, -2h\rangle$. Since OX is perpendicular to VB, we have that $\overrightarrow{OX} \cdot \overrightarrow{VB} = 0$.

$$\overrightarrow{OX} \cdot \overrightarrow{VB} = 0 \implies \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix} \cdot 2\begin{pmatrix} 1 \\ -1 \\ -h \end{pmatrix} = 0 \implies h^2 = 2.$$

We hence have that $h = \sqrt{2}$. Note that we reject $h = -\sqrt{2}$ since $h > 0$.

Let the acute angle between PM and QN be θ.

$$\cos\theta = \frac{\left|\overrightarrow{PM} \cdot \overrightarrow{QN}\right|}{\left|\overrightarrow{PM}\right|\left|\overrightarrow{QN}\right|} = \frac{1}{\sqrt{12}\sqrt{12}}\left|\begin{pmatrix} 1 \\ 3 \\ \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ \sqrt{2} \end{pmatrix}\right| = \frac{1}{3} \implies \theta = 70.5° \text{ (1 d.p.)}.$$

Self-Practice A8

Problem 1. The points A and B have positions vectors $\langle 8,\, 3,\, 2\rangle$ and $\langle -2,\, 3,\, 4\rangle$ respectively.

(a) Show that $AB = 2\sqrt{26}$.

(b) Find the Cartesian equation for the line AB.

(c) The line l has equation $\mathbf{r} = \langle -2,\, 3,\, 4\rangle + t\,\langle 2,\, 6,\, 5\rangle$. Find the length of the projection of AB onto l.

(d) Calculate the acute angle between AB and l, giving your answer correct to the nearest degree.

(e) Find the position vector of the foot N of the perpendicular from A to l. Hence, find the position vector of the image of A in the line l.

Solution.

Part (a). Note that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}.$$

Hence,

$$AB = \left|\overrightarrow{AB}\right| = 2\sqrt{(-5)^2 + 0^2 + 1^2} = 2\sqrt{26} \text{ units.}$$

Part (b). The vector equation of the line AB is

$$\mathbf{r} = \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Hence, the Cartesian equation is

$$\frac{x-8}{-5} = z - 2,\ y = 3.$$

Part (c). The length of projection of AB onto l is given by

$$\frac{|2\,\langle -5,\, 0,\, 1\rangle \cdot \langle 2,\, 6,\, 5\rangle|}{|\langle 2,\, 6,\, 5\rangle|} = \frac{10}{\sqrt{65}} \text{ units.}$$

Part (d). Let the acute angle be θ.

$$\cos\theta = \frac{|\langle -5,\, 0,\, 1\rangle \cdot \langle 2,\, 6,\, 5\rangle|}{|\langle -5,\, 0,\, 1\rangle|\,|\langle 2,\, 6,\, 5\rangle|} = \frac{5}{\sqrt{65}\sqrt{26}} \implies \theta = 83^\circ.$$

Part (e). Since N is on l, there exists some $t \in \mathbb{R}$ such that

$$\overrightarrow{ON} = \frac{-2}{3}4 + t \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix}.$$

Hence,

$$\overrightarrow{AN} = \left[\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix}\right] - \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix}.$$

Since AN is perpendicular to l, we have

$$\overrightarrow{AN} \cdot \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix} = \left[2 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix} = -10 + 65t = 0.$$

Hence, $t = 2/13$, whence

$$\overrightarrow{ON} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} + \frac{2}{13} \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} -22 \\ 51 \\ 62 \end{pmatrix}.$$

Let the image of A in l be A'. By the midpoint theorem,

$$\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}.$$

Hence,

$$\overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA} = \frac{2}{13} \begin{pmatrix} -22 \\ 51 \\ 62 \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} -148 \\ 63 \\ 98 \end{pmatrix}.$$

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Problem 2. The position vectors of the points A and B are $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + p\mathbf{k}$ respectively, where p is a constant. The point C is such that $OABC$ is a rectangle, where O is the origin.

(a) Show that $p = 2$.

(b) Write down the position vector of C.

(c) Find a vector equation of the line BC.

The equation of line l is given by $\frac{x-1}{3} = \frac{y-1}{3}$, $z = 1$.

(e) Show that the lines BC and l are skew.

Solution.

Part (a). Note that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ p \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ p-3 \end{pmatrix}.$$

Since $OABC$ is a rectangle, $\overrightarrow{OA} \perp \overrightarrow{AB}$. Hence,

$$\overrightarrow{OA} \cdot \overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ p-3 \end{pmatrix} = 3 + 3(p-3) = 0 \implies p = 2.$$

Part (b). Since $OABC$ is a rectangle,

$$\overrightarrow{OC} = \overrightarrow{AB} = \langle 1,\, 1,\, -1 \rangle.$$

Part (c). Since $OABC$ is a rectangle,

$$\overrightarrow{BC} = \overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Thus, the vector equation of line BC is

$$l_{BC} : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Part (d). Note that the vector equation of l is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}.$$

Consider $l \cap l_{BC}$:

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \implies \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \mu \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

This gives the system

$$\begin{cases} \lambda - 3\mu = 3 \\ 2\lambda - 3\mu = 3 \\ 3\lambda = 2 \end{cases},$$

which has no solution. Since the direction vectors of l and l_{BC} are not parallel (i.e. $\langle 1,\ 2,\ 3 \rangle \nparallel \langle 3,\ 3,\ 0 \rangle$), the two lines are skew.

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Problem 3. The lines l_1 and l_2 have equations $\mathbf{r} = \langle 3,\ 1,\ 2 \rangle + \lambda \langle b,\ 1,\ -1 \rangle$, where $b > 1$, and $\mathbf{r} = \langle 4,\ 0,\ 1 \rangle + \mu \langle -1,\ -1,\ 1 \rangle$ respectively.

(a) Given that the acute angle between l_1 and l_2 is $30°$, find the value of b, giving your answer correct to 2 decimal places.

For the rest of the question, use $b = 3$.

(b) Find the coordinates of the points A and B where l_1 and l_2 meet the xy-plane respectively.

(c) The point C has position vector $2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$. Find whether C is closer to l_1 or l_2.

Solution.

Part (a).

$$\frac{\sqrt{3}}{2} = \cos 30° = \frac{|\langle b,\ 1,\ -1 \rangle \cdot \langle -1,\ -1,\ 1 \rangle|}{|\langle b,\ 1,\ -1 \rangle|\,|\langle -1,\ -1,\ 1 \rangle|} = \frac{|-b-2|}{\sqrt{b^2+2}\sqrt{3}}.$$

Since $b > 1$, we clearly have $|-b-2| = b+2$. Thus,

$$\frac{b+2}{\sqrt{b^2+2}} = \frac{3}{2}.$$

Using G.C., we have $b = 0.13$ or $b = 3.07$. Since $b > 1$, we take $b = 3.07$.

Part (b). Note that the xy-plane has equation $z = 0$. Consider the intersection between l_1 and the xy-plane. Clearly, we need $\lambda = 2$, whence

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 0 \end{pmatrix},$$

and $A(9, 3, 0)$.

Consider the intersection between l_2 and the xy-plane. Clearly, we need $\mu = -1$, whence

$$\overrightarrow{OB} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix},$$

and $B(5, 1, 0)$.

Part (c). The perpendicular distance between C and l_1 is given by

$$\frac{|[\langle 2,\ 7,\ 3\rangle - \langle 3,\ 1,\ 2\rangle] \times \langle 3,\ 1,\ -1\rangle|}{|\langle 3,\ 1,\ -1\rangle|} = \frac{|\langle -7,\ 2,\ 19\rangle|}{\sqrt{11}} = \frac{\sqrt{414}}{\sqrt{11}} = 6.13 \text{ units.}$$

The perpendicular distance between C and l_2 is given by

$$\frac{|[\langle 2,\ 7,\ 3\rangle - \langle 4,\ 0,\ 1\rangle]| \times \langle -1,\ -1,\ 1\rangle}{|\langle -1,\ -1,\ 1\rangle|} = \frac{|\langle 9,\ 0,\ 9\rangle|}{\sqrt{3}} = \frac{\sqrt{162}}{\sqrt{3}} = 7.35 \text{ units.}$$

Thus, C is closer to l_1.

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Problem 4. Relative to an origin O, points C and D have position vectors $\langle 7,\ 3,\ 2\rangle$ and $\langle 10,\ a,\ b\rangle$ respectively, where a and b are constants.

(a) The straight line through C and D has equation $\mathbf{r} = \langle 7,\ 3,\ 2\rangle + t\,\langle 1,\ 3,\ 0\rangle$, $t \in \mathbb{R}$. Find the values of a and b.

(b) Find the position vector of the point P on the line CD such that $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{CD}$.

(c) Find the position vector of the point Q on the line CD such that the angle between $\overrightarrow{OQ}$ and $\overrightarrow{OC}$ is equal to the angle between $\overrightarrow{OQ}$ and $\overrightarrow{OD}$.

Solution.

Part (a). Note that

$$\overrightarrow{CD} = \begin{pmatrix} 10 \\ a \\ b \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ a-3 \\ b-2 \end{pmatrix}.$$

Since $\overrightarrow{CD}$ is parallel to $\langle 1,\ 3,\ 0\rangle$, we have

$$\begin{pmatrix} 3 \\ a-3 \\ b-2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 0 \end{pmatrix},$$

whence $a = 12$ and $b = 2$.

Part (b). Since P is on CD, there exists some $t \in \mathbb{R}$ such that

$$\overrightarrow{OP} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.$$

Since $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{CD}$, we have

$$\overrightarrow{OP} \cdot \overrightarrow{CD} = \left[\begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right] \cdot 3 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 16 + 10t = 0,$$

whence $t = -8/5$ and

$$\overrightarrow{OP} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} - \frac{8}{5} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 27 \\ -9 \\ 10 \end{pmatrix}.$$

Part (c). By the angle bisector theorem,

$$\frac{OC}{CQ} = \frac{OD}{DQ} \implies CQ : QD = OC : OD.$$

Since

$$OC = \left| \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \right| = \sqrt{62} \quad \text{and} \quad OD = \left| \begin{pmatrix} 10 \\ 12 \\ 2 \end{pmatrix} \right| = \sqrt{248},$$

we have

$$CQ : QD = \sqrt{62} : \sqrt{248} = 1 : 2.$$

By the ratio theorem,

$$\overrightarrow{OQ} = \frac{\overrightarrow{OD} + 2\overrightarrow{OC}}{1+2} = \frac{1}{3} \left[\begin{pmatrix} 10 \\ 12 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix}.$$

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Problem 5. Relative to an origin O, points A and B have position vectors $\langle 3, 4, 1 \rangle$ and $\langle -1, 2, 0 \rangle$ respectively. The line l has vector equation $\mathbf{r} = \langle 6, a, 0 \rangle + t \langle 1, 3, a \rangle$, where t is a real parameter and a is a constant. The line m passes through the point A and is parallel to the line OB.

(a) Find the position vector of the point P on m such that OP is perpendicular to m.

(b) Show that the two lines l and m have no common point.

(c) If the acute angle between the line l and the z-axis is $60°$, find the exact values of the constant a.

Solution.

Part (a). Note that the line m has vector equation

$$m : \mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}.$$

Since P is on m, there exists some $s \in \mathbb{R}$ such that

$$\overrightarrow{OP} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$$

Since $\overrightarrow{OP}$ is perpendicular to m, we have

$$\overrightarrow{OP} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \left[\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = 5 + 5s = 0,$$

whence $s = -1$ and

$$\overrightarrow{OP} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}.$$

Part (b). Consider $l \cap m$:

$$\begin{pmatrix} 6 \\ a \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ a \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$$

Comparing z-coordinates, we have

$$ta = 1 \implies t = \frac{1}{a}.$$

Substituting this into the equation, we get

$$\begin{pmatrix} 6 \\ a \\ 0 \end{pmatrix} + \frac{1}{a} \begin{pmatrix} 1 \\ 3 \\ a \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$$

This yields the system

$$\begin{aligned} 6 + \frac{1}{a} &= 3 - s \\ a + \frac{3}{a} &= 4 + 2s \end{aligned}$$

Adding the second equation to twice the first yields

$$2\left(6 + \frac{1}{a}\right) + \left(a + \frac{3}{a}\right) = 2(3 - s) + (4 + 2s) \implies a + \frac{5}{a} + 2 = 0.$$

Multiplying through by a gives the quadratic

$$a^2 + 2a + 5 = (a+1)^2 + 4 = 0,$$

which clearly has no real solution. Hence, $l \cap m$ has no solution, whence the two lines do not have any common point

Part (c). Note that the z-axis is parallel to the vector $\langle 0, 0, 1 \rangle$. Thus,

$$\frac{1}{2} = \cos 60^\circ = \frac{|\langle 1, 3, a \rangle \cdot \langle 0, 0, 1 \rangle|}{|\langle 1, 3, a \rangle| \, |\langle 0, 0, 1 \rangle|} = \frac{|a|}{\sqrt{10 + a^2}\sqrt{1}}.$$

Squaring, we get

$$\frac{1}{4} = \frac{a^2}{10+a^2} \implies 10 + a^2 = 4a^2 \implies a^2 = \frac{10}{3} \implies a = \pm\sqrt{\frac{10}{3}}.$$

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Problem 6. The lines l_1 and l_2 have vector equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

respectively, where λ and μ are real parameters.

(a) Find the acute angle between the two lines l_1 and l_2, giving your answer to the nearest $0.1°$.

(b) Show that l_1 passes through the point P with position vector $\langle 1, -4, 2\rangle$. Hence, show that the distance between point P and any point on the line l_2 is given by $\sqrt{6\mu^2 - 12\mu + 20}$. Deduce the shortest distance between point P and the line l_2.

Solution.

Part (a). Let the acute angle be θ. Then

$$\cos\theta = \frac{|\langle 0,\, 2,\, 1\rangle \cdot \langle 1,\, -2,\, 1\rangle|}{|\langle 0,\, 2,\, 1\rangle|\, \langle 1,\, -2,\, 1\rangle} = \frac{3}{\sqrt{5}\sqrt{6}} \implies \theta = 56.8°.$$

Part (b). Take $\lambda = -1$. Then

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}.$$

Hence, l_1 passes through $P(1, -4, 2)$.

Note that l_2 has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+\mu \\ -2\mu \\ 4+\mu \end{pmatrix}.$$

Hence,

$$\mathbf{r} - \overrightarrow{OP} = \begin{pmatrix} 1+\mu \\ -2\mu \\ 4+\mu \end{pmatrix} - \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} \mu \\ 4-2\mu \\ 2+\mu \end{pmatrix}.$$

Thus, the distance between P and any point on l_2 is given by

$$\begin{aligned} \sqrt{\mu^2 + (4-2\mu)^2 + (2+\mu)^2} &= \sqrt{\mu^2 + (4\mu^2 - 16\mu + 16) + (\mu^2 + 4\mu + 4)} \\ &= \sqrt{6\mu^2 - 12\mu + 20} \text{ units.} \end{aligned}$$

Since $6\mu^2 - 12\mu + 20 = 6(\mu+1)^2 + 13$, the shortest distance is $\sqrt{14}$ units.

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Problem 7 (🌶). The coordinates of the points A, B and C are given by $A(0, 2, 4)$, $B(4, 6, 11)$ and $C(8, 1, 0)$.

(a) Show that the triangle with vertices A, B and C is an isosceles right-angled triangle.

(b) Find the position vector of point D in the same plane as A, B and C such that BCD is an equilateral triangle.

Solution.

Part (a). Observe that

$$\overrightarrow{AB} = \begin{pmatrix}4\\6\\11\end{pmatrix} - \begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}4\\4\\7\end{pmatrix} \implies AB = \sqrt{4^2+4^2+7^2} = 9$$

and

$$\overrightarrow{CA} = \begin{pmatrix}0\\2\\4\end{pmatrix} - \begin{pmatrix}8\\1\\0\end{pmatrix} = \begin{pmatrix}-8\\1\\4\end{pmatrix} \implies AC = \sqrt{(-8)^2+1^2+4^2} = 9.$$

Since $AB = AC$, triangle ABC is isosceles.

Consider $\overrightarrow{AB} \cdot \overrightarrow{CA}$:

$$\overrightarrow{AB} \cdot \overrightarrow{CA} = \begin{pmatrix}4\\4\\7\end{pmatrix} \cdot \begin{pmatrix}-8\\1\\4\end{pmatrix} = -32+4+28 = 0.$$

Thus, $\overrightarrow{AB} \perp \overrightarrow{CA}$, whence triangle ABC is a right-angled triangle.

Hence, triangle ABC is an isosceles right-angled triangle.

Part (b). Let N be the foot of perpendicular of A on BC. Since $\triangle ABC$ is isosceles, with $AB = AC$, by symmetry, N is the midpoint of BC:

$$\overrightarrow{ON} = \frac{\overrightarrow{OB}+\overrightarrow{OC}}{2} = \frac{1}{2}\begin{pmatrix}12\\7\\11\end{pmatrix}.$$

Consider point D. Since $\triangle BCD$ is equilateral, it must also be isosceles, with $DB = DC$. Hence, D lies on AN (extended). Also, we have $ND/BC = \sin 60° = \sqrt{3}/2$.

Since

$$\overrightarrow{AN} = \frac{1}{2}\begin{pmatrix}12\\7\\11\end{pmatrix} - \begin{pmatrix}0\\2\\4\end{pmatrix} = \frac{3}{2}\begin{pmatrix}4\\1\\1\end{pmatrix},$$

the line AN has vector equation

$$\mathbf{r} = \begin{pmatrix}0\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}4\\1\\1\end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Hence, there exists some $\lambda \in \mathbb{R}$ such that

$$\overrightarrow{OD} = \begin{pmatrix}0\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}4\\1\\1\end{pmatrix}.$$

Thus,

$$\overrightarrow{ND} = \left[\begin{pmatrix}0\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}4\\1\\1\end{pmatrix}\right] - \frac{1}{2}\begin{pmatrix}12\\7\\11\end{pmatrix} = \left(\lambda - \frac{3}{2}\right)\begin{pmatrix}4\\1\\1\end{pmatrix}.$$

Note that $\overrightarrow{BC} = \langle 4,\ -5,\ -11 \rangle$. Hence,

$$\frac{ND}{BC} = \frac{|\lambda - 3/2|\sqrt{4^2+1^2+1^2}}{\sqrt{4^2+(-5)^2+(-11)^2}} = \frac{|\lambda - 3/2|\sqrt{18}}{\sqrt{162}} = \frac{\sqrt{3}}{2}.$$

Rearranging, we get

$$\left|\lambda - \frac{3}{2}\right| = \frac{\sqrt{3}\sqrt{162}}{2\sqrt{18}} = \frac{3\sqrt{3}}{2} \implies \lambda = \frac{3 \pm 3\sqrt{3}}{2}.$$

Thus,

$$\overrightarrow{OD} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \frac{3 \pm 3\sqrt{3}}{2} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

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Problem 8 (🌶). The equations of the lines l_1 and l_2 are given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ \lambda \in \mathbb{R} \quad \text{and} \quad l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \mu \in \mathbb{R}.$$

(a) The point P with coordinates $(2, 2, 3)$ lies on the line l_1. Find the reflection of P in the line l_2.

(b) The line l_3 is the reflection of the line l_1 in the line l_2. Find an equation for the line l_3.

(c) The line l_4 is such that it is parallel to l_1 and its distance between the two lines is $\sqrt{13/14}$. Find two possible vector equations of l_4.

Solution.

Part (a). Let N be the foot of perpendicular of P on l_2. Since N lies on l_2, there exists some $\mu \in \mathbb{R}$ such that

$$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Thus,

$$\overrightarrow{PN} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right] - \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Since PN is perpendicular to l_2,

$$\overrightarrow{PN} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \left[-\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2 + 2\mu = 0,$$

whence $\mu = -1$ and

$$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let P be the reflection of P in l_2. By the midpoint theorem,

$$\overrightarrow{ON} = \frac{\overrightarrow{OP} + \overrightarrow{OP'}}{2} \implies \overrightarrow{OP'} = 2\overrightarrow{ON} - \overrightarrow{OP} = 2\begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}2\\2\\3\end{pmatrix} = -\begin{pmatrix}2\\2\\1\end{pmatrix}.$$

Part (b). Note that l_1 and l_2 have a common point $(1, 0, 0)$. Under reflection, this point is an invariant. Hence, l_3 must also contain the point $(1, 0, 0)$. Additionally, l_3 must contain P', the reflection of P in l_2. Since

$$-\begin{pmatrix}2\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = -\begin{pmatrix}3\\2\\1\end{pmatrix} \parallel \begin{pmatrix}3\\2\\1\end{pmatrix},$$

l_3 has vector equation

$$\lambda_3 : \mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \nu\begin{pmatrix}3\\2\\1\end{pmatrix}, \quad \nu \in \mathbb{R}.$$

Part (c). Clearly, l_4 is given by

$$l_4 : \mathbf{r} = \begin{pmatrix}a\\b\\c\end{pmatrix} \xi \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \xi \in \mathbb{R}.$$

The perpendicular distance between l_1 and l_4 is hence given by

$$\frac{|[\langle a,\, b,\, c\rangle - \langle 1,\, 0,\, 0\rangle\, 0] \times \langle 1,\, 2,\, 3\rangle|}{|\langle 1,\, 2,\, 3\rangle|} = \frac{|\langle 3b - 2c,\, c - 3a + 3,\, 2a - 2 - b\rangle|}{\sqrt{14}} = \frac{\sqrt{13}}{\sqrt{14}}.$$

Hence,

$$\left|\begin{pmatrix}3b - 2c\\c - 3a + 3\\2a - 2 - b\end{pmatrix}\right| = \sqrt{13}.$$

This immediately gives

$$(3b - 2c)^2 + (c - 3a + 3)^2 + (2a - 2 - b)^2 = 13.$$

Taking $a = 0$, $b = 0$, this reduces to

$$(-2c)^2 + (c + 3)^2 + (-2)^2 = 13 \implies 5c^2 + 6c = 0 \implies c = 0 \text{ or } -\frac{6}{5}.$$

Thus,

$$l_4 : \mathbf{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \xi\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \xi \in \mathbb{R}$$

or

$$l_4 : \mathbf{r} = \begin{pmatrix}0\\0\\-6/5\end{pmatrix} + \xi\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \xi \in \mathbb{R}.$$

Assignment A8

Problem 1. Find the position vector of the foot of the perpendicular from the point with position vector $\mathbf{c}$ to the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$. Leave your answers in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Solution. Let the foot of the perpendicular be F. We have that $\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{b}$ for some real λ, and $\overrightarrow{CF} \cdot \mathbf{b} = 0$. Note that $\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{b} - \mathbf{c}$. Thus,

$$\overrightarrow{CF} \cdot \mathbf{b} = 0 \implies (\mathbf{a} + \lambda\mathbf{b} - \mathbf{c}) \cdot \mathbf{b} = 0 \implies \lambda\,|\mathbf{b}|^2 + (\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} = 0 \implies \lambda = \frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2}.$$

Thus,

$$\overrightarrow{OF} = \mathbf{a} + \left(\frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}.$$

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Problem 2. The point O is the origin, and points A, B, C have position vectors given by $\overrightarrow{OA} = 6\mathbf{i}$, $\overrightarrow{OB} = 3\mathbf{j}$, $\overrightarrow{OC} = 4\mathbf{k}$. The point P is on the line AB between A and B, and is such that $AP = 2PB$. The point Q has position vector given by $\overrightarrow{OQ} = q\mathbf{i}$, where q is a scalar.

(a) Express, in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the vector $\overrightarrow{CP}$.

(b) Show that the line BQ has equation $\mathbf{r} = 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j})$, where t is a parameter. Give an equation of the line CP in a similar form.

(c) Find the value of q for which the lines CP and BQ are perpendicular.

(d) Find the sine of the acute angle between the lines CP and BQ in terms of q.

Solution. We have that $\overrightarrow{OA} = \langle 6,\ 0,\ 0\rangle$, $\overrightarrow{OB} = \langle 0,\ 3,\ 0\rangle$ and $\overrightarrow{OC} = \langle 0,\ 0,\ 4\rangle$.
Part (a). By the ratio theorem,

$$\overrightarrow{OP} = \frac{2\,\overrightarrow{OB} + \overrightarrow{OA}}{1+2} = \frac{1}{3}\left[2\begin{pmatrix}0\\3\\0\end{pmatrix} + \begin{pmatrix}6\\0\\0\end{pmatrix}\right] = \begin{pmatrix}2\\2\\0\end{pmatrix} \implies \overrightarrow{CP} = \overrightarrow{OP} - \overrightarrow{OC} = \begin{pmatrix}2\\2\\-4\end{pmatrix}.$$

Hence, $\overrightarrow{CP} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.
Part (b). Note that $\overrightarrow{BQ} = \overrightarrow{OQ} - \overrightarrow{OB} = \langle q,\ -3,\ 0\rangle$. Thus, BQ is given by

$$\mathbf{r} = \begin{pmatrix}0\\3\\0\end{pmatrix} + t\begin{pmatrix}q\\-3\\0\end{pmatrix},\ t \in \mathbb{R} \iff \mathbf{r} = 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j}),\ t \in \mathbb{R}.$$

Note that $\overrightarrow{CP} = \langle 2,\ 2,\ -4\rangle = 2\,\langle 1,\ 1,\ -2\rangle$. Hence, CP is given by

$$\mathbf{r} = \begin{pmatrix}0\\0\\4\end{pmatrix} + u\begin{pmatrix}1\\1\\-2\end{pmatrix},\ u \in \mathbb{R} \iff \mathbf{r} = 4\mathbf{k} + u(\mathbf{i} + \mathbf{j} - 2\mathbf{k}),\ u \in \mathbb{R}.$$

Part (c). Since CP is perpendicular to BQ, we have $\overrightarrow{CP} \cdot \overrightarrow{BQ} = 0$. Thus,

$$\overrightarrow{CP} \cdot \overrightarrow{BQ} = 0 \implies 2\begin{pmatrix}1\\1\\-2\end{pmatrix} \cdot \begin{pmatrix}q\\-3\\0\end{pmatrix} = 0 \implies q - 3 + 0 = 0 \implies q = 3.$$

Part (d). Let θ be the acute angle between CP and BQ.

$$\sin\theta = \frac{|\langle 1,\, 1,\, -2\rangle \times \langle q,\, -3,\, 0\rangle|}{|\langle 1,\, 1,\, -2\rangle|\,|\langle q,\, -3,\, 0\rangle|} = \frac{|\langle -6,\, 2q,\, 3-q\rangle|}{\sqrt{6}\sqrt{q^2+9}} = \sqrt{\frac{5q^2-6q+45}{6q^2+54}}.$$

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Problem 3. Line l_1 passes through the point A with position vector $3\mathbf{i}-2\mathbf{k}$ and is parallel to $-2\mathbf{i}+4\mathbf{j}-\mathbf{j}$. Line l_2 has Cartesian equation given by $\frac{x-1}{2} = y = z+3$.

(a) Show that the two lines intersect and find the coordinates of their point of intersection.

(b) Find the acute angle between the two lines l_1 and l_2. Hence, or otherwise, find the shortest distance from point A to line l_2.

(c) Find the position vector of the foot N of the perpendicular from A to the line l_2. The point B lies on the line AN produced and is such that N is the mid-point of AB. Find the position vector of B.

Solution. We have

$$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}, \quad l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},\ \mu \in \mathbb{R}.$$

Part (a). Consider $l_1 = l_2$.

$$l_1 = l_2 \implies \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \implies \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 2\lambda + 2\mu = 2 \\ -4\lambda + \mu = 0 \\ \lambda + \mu = 1 \end{cases}$$

which has the unique solution $\mu = 4/5$ and $\lambda = 1/5$. Thus, the intersection point P has position vector $\langle 3,\, 0,\, -2\rangle + \frac{1}{5}\langle -2,\, 4,\, -1\rangle = \frac{1}{5}\langle 13,\, 4,\, -11\rangle$ and thus has coordinates $(13/5, 4/5, -11/5)$.

Part (b). Let θ be the acute angle between l_1 and l_2.

$$\cos\theta = \frac{|\langle -2,\, 4,\, -1\rangle \cdot \langle 2,\, 1,\, 1\rangle|}{|\langle -2,\, 4,\, -1\rangle|\,|\langle 2,\, 1,\, 1\rangle|} = \frac{1}{\sqrt{126}} \implies \theta = 84.9^\circ \text{ (1 d.p.)}.$$

Note that

$$AP = \sqrt{\left(\frac{17}{5}-3\right)^2 + \left(-\frac{4}{5}-0\right)^2 + \left(-\frac{9}{5}-(-2)\right)^2} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}.$$

Since $\sin\theta = \frac{AN}{AP}$, we have that $AN = AP\sin\theta$. Note that

$$\sin\theta = \sin\arccos\frac{1}{\sqrt{126}} = \frac{\sqrt{\left(\sqrt{126}\right)^2-1}}{\sqrt{126}} = \frac{\sqrt{125}}{\sqrt{126}} = \frac{5\sqrt{5}}{\sqrt{6}\sqrt{21}}.$$

Thus,

$$AN = \frac{\sqrt{21}}{5} \cdot \frac{5\sqrt{5}}{\sqrt{6}\sqrt{21}} = \sqrt{\frac{5}{6}}.$$

The shortest distance between A and l_2 is hence $\sqrt{\frac{5}{6}}$ units.

Part (c). Since N is on l_2, we have that $\overrightarrow{ON} = \langle 1, 0, -3 \rangle + \mu \langle 2, 1, 1 \rangle$ for some real μ. Additionally, since $\overrightarrow{AN} \perp l_2$, we have $\overrightarrow{AN} \cdot \langle 2, 1, 1 \rangle = 0$. Note that

$$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 + 2\mu \\ \mu \\ -1 + \mu \end{pmatrix}.$$

Thus,

$$\overrightarrow{AN} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \implies \begin{pmatrix} -2 + 2\mu \\ \mu \\ -1 + \mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \implies -5 + 6\mu = 0 \implies \mu = \frac{5}{6}.$$

Hence,

$$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix}.$$

Note that $\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$. Hence,

$$\overrightarrow{OB} = 2\overrightarrow{ON} - \overrightarrow{OA} = \frac{2}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 7 \\ 5 \\ -7 \end{pmatrix}.$$

A9 Vectors III - Planes

Tutorial A9

Problem 1. A student claims that a unique plane can always be defined based on the given information. True or False? (Whenever a line is mentioned, assume the vector equation is known.)

	Statement	**T/F**
(a)	Any 2 vectors parallel to the plane and a point lying on the plane.	False
(b)	Any 3 distinct points lying on the plane.	False
(c)	A vector perpendicular to the plane and a point lying on the plane.	True
(d)	A line l perpendicular to the plane and a particular point on l lying on the plane.	True
(e)	A line l lying on the plane.	False
(f)	A line l and a point not on l, both lying on the plane.	True
(g)	A pair of distinct, intersecting lines, both lying on the plane.	True
(h)	A pair of distinct, parallel lines, both lying on the plane.	True
(i)	A pair of skew lines both parallel to the plane.	False
(j)	2 intersecting lines both parallel to the plane.	False

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Problem 2. Find the equations of the following planes in parametric, scalar product and Cartesian form:

(a) The plane passes through the point with position vector $7\mathbf{i} + 2\mathbf{j} - 3\mathbf{j}$ and is parallel to $\mathbf{i} + 3\mathbf{j}$ and $4\mathbf{j} - 2\mathbf{k}$.

(b) The plane passes through the points $A(2, 0, 1)$, $B(1, -1, 2)$ and $C(1, 3, 1)$.

(c) The plane passes through the point with position vector $7\mathbf{i}$ and is parallel to the plane $\mathbf{r} = (2 - p + q)\mathbf{i} + (p + 3q)\mathbf{j} + (-2 - 3q)\mathbf{k}$, $p, q \in \mathbb{R}$.

(d) The plane contains the line $l : \mathbf{r} = (-2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, $\lambda \in \mathbb{R}$ and is perpendicular to the plane $\pi : \mathbf{r} \cdot (7\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) = 2$.

Solution.

Part (a). Parametric. Note that $\langle 0,\ 4,\ -2\rangle \parallel \langle 0,\ 2,\ -1\rangle$. Hence, the plane has parametric form

$$\mathbf{r} = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix},\ \lambda, \mu \in \mathbb{R}.$$

Scalar Product. Note that $\mathbf{n} = \langle 1,\ 3,\ 0\rangle \times \langle 0,\ 2,\ -1\rangle = \langle -3,\ 1,\ 2\rangle \implies d = \langle 7,\ 2,\ -3\rangle \cdot \langle -3,\ 1,\ 2\rangle = -25$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -2.$$

Cartesian. Let $\mathbf{r} = \langle x,\, y,\, z\rangle$. From the scalar product form, we have

$$-3x + y + 2z = -25.$$

Part (b). Parametric. Since the plane passes through the points A, B and C, it is parallel to both $\overrightarrow{AB} = -\langle 1,\, 1,\, -1\rangle$ and $\overrightarrow{AC} = \langle -1,\, 3,\, 0\rangle$. Hence, the plane has parametric form

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}.$$

Scalar Product. Note that $\mathbf{n} = \langle 1,\, 1,\, -1\rangle \times \langle -1,\, 3,\, 0\rangle = \langle 3,\, 1,\, 4\rangle \implies d = \langle 2,\, 0,\, 1\rangle \cdot \langle 3,\, 1,\, 4\rangle = 10$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 10.$$

Cartesian. Let $\mathbf{r} = \langle x,\, y,\, z\rangle$. From the scalar product form, we have

$$3x + y + 4z = 10.$$

Part (c). Parametric. Note that the plane is parallel to $\mathbf{r} = \langle 2,\, 0,\, -1\rangle + p\,\langle -1,\, 1,\, 0\rangle + q\,\langle 1,\, 3,\, -3\rangle$ and passes through $(7, 0, 0)$. Hence, the plane has parametric form

$$\mathbf{r} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}.$$

Scalar Product. Note that $\langle -1,\, 1,\, 0\rangle \times \langle 1,\, 3,\, -3\rangle = \langle -3,\, -3,\, -4\rangle \parallel \langle 3,\, 3,\, 4\rangle$. We hence take $\mathbf{n} = \langle 3,\, 3,\, 4\rangle$, whence $d = \langle 7,\, 0,\, 0\rangle \cdot \langle 3,\, 3,\, 4\rangle = 21$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = 21.$$

Cartesian. Let $\mathbf{r} = \langle x,\, y,\, z\rangle$. From the scalar product form, we have

$$3x + 3y + 4z = 21.$$

Part (d). Parametric. Since the plane contains the line with equation $\mathbf{r} = \langle -2,\, 5,\, -3\rangle + \lambda\,\langle 2,\, 1,\, 2\rangle,\ \lambda \in \mathbb{R}$, the plane passes through $(-2, 5, -3)$ and is parallel to the vector $\langle 2,\, 1,\, 2\rangle$. Furthermore, since the plane is perpendicular to the plane with normal $\langle 7,\, 4,\, 5\rangle$, it must be parallel to said vector. Thus, the plane has the following parametric form:

$$\mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}.$$

Scalar Product. Note that $\mathbf{n} = \langle 2,\, 1,\, 2\rangle \times \langle 7,\, 4,\, 5\rangle = \langle -3,\, 4,\, 1\rangle \implies d = \langle -2,\, 5,\, -3\rangle \cdot \langle -3,\, 4,\, 1\rangle = 23$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} = 23.$$

Cartesian. Let $\mathbf{r} = \langle x,\, y,\, z\rangle$. From the scalar product form, we have

$$-3x + 4y + z = 23.$$

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Problem 3. The line l passes through the points A and B with coordinates $(1, 2, 4)$ and $(-2, 3, 1)$ respectively. The plane p has equation $3x - y + 2z = 17$. Find

(a) the coordinates of the point of intersection of l and p,

(b) the acute angle between l and p,

(c) the perpendicular distance from A to p, and

(d) the position vector of the foot of the perpendicular from B to p.

The line m passes through the point C with position vector $6\mathbf{i} + \mathbf{j}$ and is parallel to $2\mathbf{j} + \mathbf{k}$.

(e) Determine whether m lies in p.

Solution. Note that $\overrightarrow{OA} = \langle 1,\, 2,\, 4\rangle$ and $\overrightarrow{OB} = \langle -2,\, 3,\, 1\rangle$, whence $\overrightarrow{AB} = -\langle 3,\, -1,\, 3\rangle$. Thus, the line l has vector equation

$$\mathbf{r} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\3\end{pmatrix},\ \lambda \in \mathbb{R}.$$

Also note that the equation of the plane p can be written as

$$\mathbf{r} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17.$$

Part (a). Let the point of intersection of l and p be P. Consider $l = p$.

$$l = p \implies \left[\begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\3\end{pmatrix}\right] \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17 \implies 9 + 16\lambda = 17 \implies \lambda = \frac{1}{2}.$$

Thus, $\overrightarrow{OP} = \langle 1,\, 2,\, 4\rangle + \frac{1}{2}\langle 3,\, -1,\, 3\rangle = \langle 5/2,\, 3/2,\, 11/2\rangle$, whence $P(5/2, 3/2, 11/2)$.

Part (b). Let θ be the acute angle between l and p.

$$\sin\theta = \frac{|\langle 3,\, -1,\, 3\rangle \cdot \langle 3,\, -1,\, 2\rangle|}{|\langle 3,\, -1,\, 3\rangle|\,|\langle 3,\, -1,\, 2\rangle|} = \frac{16}{\sqrt{266}} \implies \theta = 78.8^\circ \text{ (1 d.p.)}.$$

Part (c). Note that $\overrightarrow{AP} = -\frac{1}{2}\langle 3,\, -1,\, 3\rangle$. The perpendicular distance from A to p is hence

$$\left|\overrightarrow{AP} \cdot \hat{\mathbf{n}}\right| = \frac{\left|-\frac{1}{2}\langle 3,\, -1,\, 3\rangle \cdot \langle 3,\, -1,\, 2\rangle\right|}{|\langle 3,\, -1,\, 2\rangle|} = \frac{8}{\sqrt{14}} \text{ units.}$$

Part (d). Let F be the foot of the perpendicular from B to p. Since F is on p, we have $\overrightarrow{OF} \cdot \langle 3,\, -1,\, 2\rangle = 17$. Furthermore, since BF is perpendicular to p, we have $\overrightarrow{BF} = \lambda\mathbf{n} =$

$\lambda \langle 3, -1, 2\rangle$ for some $\lambda \in \mathbb{R}$. We hence have $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = \langle -2, 3, 1\rangle + \lambda \langle 3, -1, 2\rangle$. Thus,

$$\left[\begin{pmatrix}-2\\3\\1\end{pmatrix} + \lambda \begin{pmatrix}3\\-1\\2\end{pmatrix}\right] \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17 \implies -7 + 14\lambda = 17 \implies \lambda = \frac{12}{7}.$$

Hence, $\overrightarrow{OF} = \langle -2, 3, 1\rangle + \frac{12}{7}\langle 3, -1, 2\rangle = \frac{1}{7}\langle 22, 9, 31\rangle$.

Part (e). Note that m has the vector equation

$$\mathbf{r} = \begin{pmatrix}6\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}0\\2\\1\end{pmatrix}, \lambda \in \mathbb{R}.$$

Consider $m \cdot \mathbf{n}$:

$$m \cdot \mathbf{n} = \left[\begin{pmatrix}6\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}0\\2\\1\end{pmatrix}\right] \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17.$$

Since $m \cdot \mathbf{n} = 17$ for all $\lambda \in \mathbb{R}$, it follows that m lies in p.

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Problem 4. A plane contains distinct points P, Q, R and S, of which no 3 points are collinear. What can be said about the relationship between the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$?

Solution. Each of the three vectors can be expressed as a unique linear combination of the other two.

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Problem 5.

(a) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter.

(b) Interpret geometrically the vector equation $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{n}$ is a constant unit vector and d is a constant scalar, stating what d represents.

(c) Given that $\mathbf{b} \cdot \mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} \cdot \mathbf{n} = d$ to find $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ and d. Interpret the solution geometrically.

Solution.

Part (a). The vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ represents a line with direction vector $\mathbf{b}$ that passes through the point with position vector $\mathbf{a}$.

Part (b). The vector equation $\mathbf{r} \cdot \mathbf{n} = d$ represents a plane perpendicular to $\mathbf{n}$ that has a perpendicular distance of d units from the origin. Here, a negative value of d corresponds to a plane d units from the origin in the opposite direction of $\mathbf{n}$.

Part (c).

$$\begin{aligned}\mathbf{r} \cdot \mathbf{n} = d &\implies (\mathbf{a} + t\mathbf{b}) \cdot \mathbf{n} = d \implies \mathbf{a} \cdot \mathbf{n} + t\mathbf{b} \cdot \mathbf{n} = d \\ &\implies t = \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \implies \mathbf{r} = \mathbf{a} + \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}.\end{aligned}$$

$\mathbf{a} + \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}$ is the position vector of the point of intersection of the line and plane.

Problem 6. The planes p_1 and p_2 have equations $\mathbf{r}\cdot\langle 2,\,-2,\,1\rangle = 1$ and $\mathbf{r}\cdot\langle -6,\,3,\,2\rangle = -1$ respectively, and meet in the line l.

(a) Find the acute angle between p_1 and p_2.

(b) Find a vector equation for l.

(c) The point $A(4, 3, c)$ is equidistant from the planes p_1 and p_2. Calculate the two possible values of c.

Solution.

Part (a). Let θ the acute angle between p_1 and p_2.

$$\cos\theta = \frac{|\langle 2,\,-2,\,1\rangle\cdot\langle -6,\,3,\,2\rangle|}{|\langle 2,\,-2,\,1\rangle|\,|\langle -6,\,3,\,2\rangle|} = \frac{16}{21} \implies \theta = 40.4^\circ \text{ (1 d.p.).}$$

Part (b). Observe that p_1 has the Cartesian equation $2x - 2y + z = 1$ and p_2 has the Cartesian equation $-6x + 3y + 2z = -1$. Consider $p_1 = p_2$. Solving both Cartesian equations simultaneously gives the solution

$$x = -\frac{1}{6} + \frac{7}{6}t, \quad y = -\frac{2}{3} + \frac{5}{3}t, \quad z = t$$

for all $t \in \mathbb{R}$. The line l thus has vector equation

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 7 \\ 10 \\ 6 \end{pmatrix},\ t \in \mathbb{R}.$$

Part (c). Let Q be the point with position vector $-\frac{1}{6}\langle 1,\,4,\,0\rangle$. Then $\overrightarrow{AQ} = -\frac{1}{6}\langle 25,\,22,\,6c\rangle$. Since Q lies on l, it lies on both p_1 and p_2. Since A is equidistant to p_1 and p_2, the perpendicular distances from A to p_1 and p_2 are equal.

The perpendicular distance from A to p_1 is given by:

$$\frac{\left|\overrightarrow{AQ}\cdot\langle 2,\,-2,\,1\rangle\right|}{|\langle 2,\,-2,\,1\rangle|} = \frac{1}{3}\left|-\frac{1}{6}\begin{pmatrix} 25 \\ 22 \\ 6c \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right| = \frac{1}{3}\left|1 + c\right|.$$

Meanwhile, the perpendicular distance from A to p_2 is given by:

$$\frac{\left|\overrightarrow{AQ}\cdot\langle -6,\,3,\,2\rangle\right|}{|\langle -6,\,3,\,2\rangle|} = \frac{1}{7}\left|-\frac{1}{6}\begin{pmatrix} 25 \\ 22 \\ 6c \end{pmatrix}\cdot\begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}\right| = \frac{1}{7}\left|-14 + 2c\right|.$$

Equating the two gives

$$\frac{1}{3}\left|1 + c\right| = \frac{1}{7}\left|-14 + 2c\right| \implies \left|7 + 7c\right| = \left|-42 + 6c\right|.$$

This splits into the following two cases:

Case 1. $(7 + 7c)(-42 + 6c) > 0 \implies 7 + 7c = -42 + 6c \implies c = -49$.

Case 2. $(7 + 7c)(-42 + 6c) < 0 \implies 7 + 7c = -(-42 + 6c) \implies c = -35/13$.

Problem 7. A plane Π has equation $\mathbf{r} \cdot (2\mathbf{i} + 3\mathbf{j}) = -6$.

(a) Find, in vector form, an equation for the line passing through the point P with position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and normal to the plane Π.

(b) Find the position vector of the foot Q of the perpendicular from P to the plane Π and hence find the position vector of the image of P after the reflection in the plane Π.

(c) Find the sine of the acute angle between OQ and the plane Π.

The plane Π' has equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 5$.

(d) Find the position vector of the point A where the planes Π, Π' and the plane with equation $\mathbf{r} \cdot \mathbf{i} = 0$ meet.

(e) Hence, or otherwise, find also the vector equation of the line of intersection of planes Π and Π'.

Solution.

Part (a). Let l be the required line. Since l is normal to Π, it is parallel to the normal vector of Π, $\langle 2, 3, 0\rangle$. Thus, l has vector equation

$$l : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

Part (b). Since Q is on Π, $\overrightarrow{OQ} \cdot \langle 2, 3, 0\rangle = -6$. Furthermore, observe that Q is also on the line l. Thus, $\overrightarrow{OQ} = \langle 2, 1, 4\rangle + \lambda \langle 2, 3, 0\rangle$ for some $\lambda \in \mathbb{R}$. Hence,

$$\overrightarrow{OQ} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = -6 \implies \left[\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = -6 \implies 7 + 13\lambda = -6 \implies \lambda = -1.$$

Thus, $\overrightarrow{OQ} = \langle 2, 1, 4\rangle - \langle 2, 3, 0\rangle = \langle 0, -2, 4\rangle$.

Let the reflection of P in Π be P'. Then

$$\overrightarrow{PQ} = \overrightarrow{QP'} \implies \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{OP'} - \overrightarrow{OQ} \implies \overrightarrow{OP'} = 2\overrightarrow{OQ} - \overrightarrow{OP}.$$

Hence, $\overrightarrow{OP'} = 2\langle 0, -2, 4\rangle - \langle 2, 1, 4\rangle = \langle -2, -5, 4\rangle$.

Part (c). Let θ be the acute angle between OQ and Π.

$$\sin\theta = \frac{|\langle 0, -2, 4\rangle \cdot \langle 2, 3, 0\rangle|}{|\langle 0, -2, 4\rangle| \, |\langle 2, 3, 0\rangle|} = \frac{3}{\sqrt{65}}.$$

Part (d). Let $\overrightarrow{OA} = \langle x, y, z\rangle$. We thus have the following system:

$$\begin{cases} \langle x, y, z\rangle \cdot \langle 2, 3, 0\rangle = -6 & \implies 2x + 3y = -6 \\ \langle x, y, z\rangle \cdot \langle 1, 1, 1\rangle = 5 & \implies x + y + z = 5 \\ \langle x, y, z\rangle \cdot \langle 1, 0, 0\rangle = 0 & \implies x = 0 \end{cases}$$

Solving, we obtain $x = 0$, $y = -2$ and $z = 7$, whence $\overrightarrow{OA} = \langle 0, -2, 7\rangle$.

Part (e). Let the line of intersection of Π and Π' be l'. Observe that A is on Π and Π' and thus lies on l'. Hence,

$$l' : \mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix} + \lambda \mathbf{b},\ \lambda \in \mathbb{R}.$$

Since l' lies on both Π and Π', $\mathbf{b}$ is perpendicular to the normals of both planes, i.e. $\langle 2, 3, 0\rangle$ and $\langle 1, 1, 1\rangle$. Thus, $\mathbf{b} = \langle 2, 3, 0\rangle \times \langle 1, 1, 1\rangle = \langle 3, -2, -1\rangle$ and

$$l' : \mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}.$$

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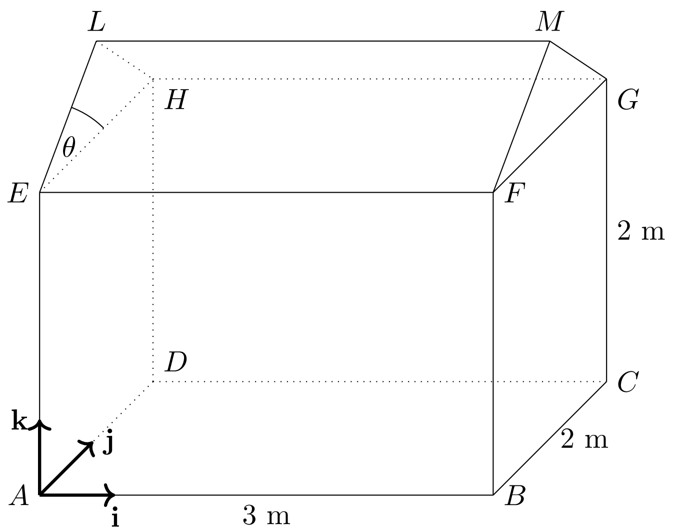
Problem 8.



The diagram shows a garden shed with horizontal base $ABCD$, where $AB = 3$ m and $BC = 2$ m. There are two vertical rectangular walls $ABFE$ and $DCGH$, where $AE = BF = CG = DH = 2$ m. The roof consists of two rectangular planes $EFML$ and $HGML$, which are inclined at an angle θ to the horizontal such that $\tan\theta = \frac{3}{4}$.

The point A is taken as the origin and the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, each of length 1 m, are taken along AB, AD and AE respectively.

(a) Verify that the plane with equation $\mathbf{r} \cdot (22\mathbf{i} + 33\mathbf{j} - 12\mathbf{k}) = 66$ passes through B, D and M.

(b) Find the perpendicular distance, in metres, from A to the plane BDM.

(c) Find a vector equation of the straight line EM.

(d) Show that the perpendicular distance from C to the straight line EM is 2.91 m, correct to 3 significant figures.

Solution.

Part (a). We have $\overrightarrow{AB} = \langle 3,\ 0,\ 0\rangle$, $\overrightarrow{BF} = \overrightarrow{AE} = \langle 0,\ 0,\ 2\rangle$ and $\overrightarrow{FG} = \overrightarrow{AD} = \langle 0,\ 2,\ 0\rangle$. Let T be the midpoint of FG. We have $\overrightarrow{FT} = \langle 0,\ 1,\ 0\rangle$ and $TM/FT = \tan\theta = 3/4$, whence $\overrightarrow{TM} = \langle 0,\ 0,\ 3/4\rangle$. Hence,

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BF} + \overrightarrow{FT} + \overrightarrow{TM} = \begin{pmatrix}3\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\2\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\3/4\end{pmatrix} = \frac{1}{4}\begin{pmatrix}12\\4\\11\end{pmatrix}.$$

Consider $\overrightarrow{AB} \cdot \langle 22,\ 33,\ -12\rangle$, $\overrightarrow{AD} \cdot \langle 22,\ 33,\ -12\rangle$ and $\overrightarrow{AM} \cdot \langle 22,\ 33,\ -12\rangle$.

$$\begin{aligned}
\overrightarrow{AB} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= \begin{pmatrix}3\\0\\0\end{pmatrix} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= 66\\
\overrightarrow{AD} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= \begin{pmatrix}0\\2\\0\end{pmatrix} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= 66\\
\overrightarrow{AM} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= \frac{1}{4}\begin{pmatrix}12\\4\\11\end{pmatrix} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= 66
\end{aligned}$$

Since $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AM}$ satisfy the equation $\mathbf{r} \cdot \langle 22,\ 33,\ -12\rangle = 66$, they all lie on the plane with said equation.

Part (b). The perpendicular distance from A to the plane BDM is given by

$$\text{Perpendicular distance} = \left|\overrightarrow{AB} \cdot \hat{\mathbf{n}}\right| = \frac{|\langle 3,\ 0,\ 0\rangle \cdot \langle 22,\ 33,\ -12\rangle|}{|\langle 22,\ 33,\ -12\rangle|} = \frac{66}{\sqrt{1717}} \text{ m.}$$

Part (c). Observe that $\overrightarrow{EM} = \overrightarrow{AM} - \overrightarrow{AE} = \frac{1}{4}\langle 12,\ 4,\ 3\rangle$. Hence, the line EM has vector equation

$$\mathbf{r} = \begin{pmatrix}0\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}12\\4\\3\end{pmatrix},\ \lambda \in \mathbb{R}.$$

Part (d). Note that $\overrightarrow{EC} = \overrightarrow{AC} - \overrightarrow{AE} = \langle 3,\ 2,\ -2\rangle$. The perpendicular distance from C to the line EM is hence given by

$$\frac{\left|\overrightarrow{EC} \times \langle 12,\ 4,\ 3\rangle\right|}{|\langle 12,\ 4,\ 3\rangle|} = \frac{1}{13}\left|\begin{pmatrix}3\\2\\-2\end{pmatrix} \times \begin{pmatrix}12\\4\\3\end{pmatrix}\right| = \frac{1}{13}\left|\begin{pmatrix}14\\-33\\-12\end{pmatrix}\right| = \frac{\sqrt{1429}}{13} = 2.91 \text{ m (3 s.f.)}.$$

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Problem 9. The planes π_1 and π_2 have equations

$$x + y - z = 0 \text{ and } 2x - 4y + z + 12 = 0$$

respectively. The point P has coordinates $(3, 8, 2)$ and O is the origin.

(a) Verify that the vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is parallel to both π_1 and π_2.

(b) Find the equation of the plane which passes through P and is perpendicular to both π_1 and π_2.

(c) Verify that $(0, 4, 4)$ is a point common to both π_1 and π_2, and hence or otherwise, find the equation of the line of intersection of π_1 and π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$.

(d) Find the coordinates of the point in which the line OP meets π_2.

(e) Find the length of projection of OP on π_1.

Solution. Note that π_1 and π_2 have vector equations $\mathbf{r} \cdot \langle 1,\ 1,\ -1\rangle = 0$ and $\mathbf{r} \cdot \langle 2,\ -4,\ 1\rangle = -12$ respectively.

Part (a). Observe that $\langle 1,\ 1,\ 2\rangle \cdot \langle 1,\ 1,\ -1\rangle = \langle 1,\ 1,\ 2\rangle \cdot \langle 2,\ -4,\ 1\rangle = 0$. Thus, the vector $\langle 1,\ 1,\ 2\rangle$ is perpendicular to the normal vectors of both π_1 and π_2 and is hence parallel to them.

Part (b). Let the required plane be π_3. Since π_3 is perpendicular to both π_1 and π_2, its normal vector is parallel to both planes. Thus, $\mathbf{n} = \langle 1,\ 1,\ 2\rangle \implies d = \langle 3,\ 8,\ 2\rangle \cdot \langle 1,\ 1,\ 2\rangle = 15$. π_3 hence has the vector equation

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 15.$$

Part (c). Since $\langle 0,\ 4,\ 4\rangle \cdot \langle 1,\ 1,\ -1\rangle = 0$ and $\langle 0,\ 4,\ 4\rangle \cdot \langle 2,\ -4,\ 1\rangle = -12$, $(0, 4, 4)$ satisfies the vector equation of both π_1 and π_2 and thus lies on both planes.

Let l be the line of intersection of π_1 and π_2. Since $(0, 4, 4)$ is a point common to both planes, l passes through it. Furthermore, since l lies on both π_1 and π_2, it is perpendicular to the normal vector of both planes and hence has direction vector $\langle 1,\ 1,\ -1\rangle \times \langle 2,\ -4,\ 1\rangle = -3\,\langle 1,\ 1,\ 2\rangle$. Thus, l can be expressed as

$$l : \mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

Part (d). Note that the line OP, denoted l_{OP} has equation

$$l_{OP} : \mathbf{r} = \mu \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix}, \ \mu \in \mathbb{R}.$$

Consider the intersection between l_{OP} and π_2.

$$\mu \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = -12 \implies -24\mu = -12 \implies \mu = \frac{1}{2}.$$

Hence, OP meets π_2 at $(3/2, 4, 1)$.

Part (e). The length of projection of OP on π_1 is given by

$$\frac{\overrightarrow{OP} \times \langle 1,\ 1,\ -1\rangle}{|\langle 1,\ 1,\ -1\rangle|} = \frac{1}{\sqrt{3}} \left| \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right| = \frac{1}{\sqrt{3}} \left| \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} \right| = \frac{5\sqrt{6}}{\sqrt{3}} = 5\sqrt{2} \text{ units.}$$

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Problem 10. The line l_1 passes through the point A, whose position vector is $3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, and is parallel to the vector $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. The line l_2 passes through the point B, whose position vector is $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, and is parallel to the vector $\mathbf{i} - \mathbf{j} - 4\mathbf{k}$. The point P on l_1 and Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. The plane Π contains PQ and l_1.

(a) Find a vector parallel to PQ.

(b) Find the equation of Π in the forms $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, $\lambda, \mu \in \mathbb{R}$ and $\mathbf{r} \cdot \mathbf{n} = D$.

(c) Find the perpendicular distance from B to Π.

(d) Find the acute angle between Π and l_2.

(e) Find the position vectors of P and Q.

Solution.

Part (a). Note that l_1 and l_2 have vector equations

$$\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}, \mu \in \mathbb{R}$$

respectively. Since PQ is perpendicular to both l_1 and l_2, it is parallel to $\langle 3, 4, 2\rangle \times \langle 1, -1, -4\rangle = \langle -14, 14, -7\rangle = -7\langle 2, -2, 1\rangle$.

Part (b). Since Π contains PQ and l_1, it is parallel to $\langle 2, -2, 1\rangle$ and $\langle 3, 4, 2\rangle$. Also note that Π contains $\langle 3, -5, -4\rangle$. Thus,

$$\Pi : \mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \lambda, \mu \in \mathbb{R}.$$

Note that $\langle 2, -2, 1\rangle \times \langle 3, 4, 2\rangle = \langle -8, -1, 14\rangle \parallel \langle 8, 1, -14\rangle$. We hence take $\mathbf{n} = \langle 8, 1, -14\rangle$, whence $d = \langle 3, -5, -4\rangle \cdot \langle 8, 1, -14\rangle = 75$. Thus, $|Pi$ is also given by

$$\Pi : \mathbf{r} \cdot \begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix} = 75.$$

Part (c). Note that $\overrightarrow{AB} = \langle -1, 8, 9\rangle$. Hence, the perpendicular distance from B to Π is given by

$$\frac{|\langle -1, 8, 9\rangle \cdot \langle 8, 1, -14\rangle|}{|\langle 8, 1, -14\rangle|} = \frac{126}{\sqrt{261}} \text{ units.}$$

Part (d). Let θ be the acute angle between Π and l_2.

$$\sin\theta = \frac{|\langle 1, -1, -4\rangle \cdot \langle 8, 1, -14\rangle|}{|\langle 1, -1, -4\rangle|\,|\langle 8, 1, -14\rangle|} = \frac{7}{\sqrt{58}} \implies \theta = 66.8^\circ \text{ (1 d.p.)}.$$

Part (e). Since P is on l_1, we have $\overrightarrow{OP} = \langle 3, -5, -4\rangle + \lambda\langle 3, 4, 2\rangle$ for some $\lambda \in \mathbb{R}$. Similarly, since Q is on l_2, we have $\overrightarrow{OQ} = \langle 2, 3, 5\rangle + \mu\langle 1, -1, 4\rangle$ for some $\mu \in \mathbb{R}$. Thus,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} - \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}.$$

Recall that PQ is parallel to $\langle 2, -2, 1\rangle$. Hence, $\overrightarrow{PQ}$ can be expressed as $\nu\langle 2, -2, 1\rangle$ for some $\nu \in \mathbb{R}$. Equating the two expressions for $\overrightarrow{PQ}$, we obtain

$$\begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} - \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} = \nu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \implies \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \nu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 3\lambda - \mu + 2\nu = -1 \\ 4\lambda + \mu - 2\nu = 8 \\ 2\lambda + 4\mu + \nu = 9 \end{cases}$$

which has the unique solution $\lambda = 1$, $\mu = 2$ and $\nu = -1$. Thus,

$$\overrightarrow{OP} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -2 \end{pmatrix}, \quad \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}.$$

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Problem 11. The equations of three planes p_1, p_2 and p_3 are

$$\begin{aligned} 2x - 5y + 3z &= 3 \\ 3x + 2y - 5z &= -5 \\ 5x + \lambda y + 17z &= \mu \end{aligned}$$

respectively, where λ and μ are constants. The planes p_1 and p_2 intersect in a line l.

(a) Find a vector equation of l.

(b) Given that all three planes meet in the line l, find λ and μ.

(c) Given instead that the three planes have no point in common, what can be said about the values of λ and μ?

(d) Find the Cartesian equation of the plane which contains l and the point $(1, -1, 3)$.

Solution.

Part (a). Consider the intersection of p_1 and p_2:

$$\begin{cases} 2x - 5y + 3z = 3 \\ 3x + 2y - 5z = -5 \end{cases}$$

The above system has solution

$$x = -1 + t, \quad y = -1 + t, \quad z = t$$

for all $t \in \mathbb{R}$. Thus, the line l has vector equation

$$l : \mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Part (b). Since all three planes meet in the line l, l must satisfy the equation of p_3. Substituting the above solution to the given equation, we have

$$5(-1 + t) + \lambda(-1 + t) + 17t = \mu \implies (22 + \lambda)t - (5 + \lambda + \mu) = 0.$$

Comparing the coefficients of t and the constant terms, we have the following system:

$$\begin{cases} \lambda + 22 = 0 \\ \lambda + \mu + 5 = 0 \end{cases}$$

which has the unique solution $\lambda = -22$ and $\mu = 17$.

Part (c). If the three planes have no point in common, we have

$$(22 + \lambda)t - (5 + \lambda + \mu) \neq 0$$

for all $t \in \mathbb{R}$. To satisfy this relation, we need $22 + \lambda = 0$ and $5 + \lambda + \mu \neq 0$, whence $\lambda = -22$ and $\mu \neq 17$.

Part (d). Note that $\langle -1, -1, 0\rangle$ lies on l and is thus contained on the required plane. Observe that $\langle -1, -1, 0\rangle - \langle 1, -1, 3\rangle = \langle -2, 0, -3\rangle$. Thus, the required plane is parallel to $\langle 1, 1, 1\rangle$ and $\langle -2, 0, -3\rangle$ and hence has vector equation

$$\mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

Observe that $\mathbf{n} = \langle 1, 1, 1\rangle \times \langle -2, 0, 3\rangle = \langle -3, 1, 2\rangle$, whence $d = \langle -1, -1, 0\rangle \cdot \langle -3, 1, 2\rangle = 2$. The required plane thus has the equation

$$\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = 2.$$

Let $\mathbf{r} = \langle x, y, z\rangle$. It follows that the plane has Cartesian equation

$$-3x + y + 2 = 2.$$

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Problem 12. The planes p_1 and p_2, which meet in line l, have equations $x - 2y + 2z = 0$ and $2x - 2y + z = 0$ respectively.

(a) Find an equation of l in Cartesian form.

The plane p_3 has equation $(x - 2y + 2z) + c(2x - 2y + z) = d$.

(b) Given that $d = 0$, show that all 3 planes meet in the line l for any constant c.

(c) Given instead that the 3 planes have no point in common, what can be said about the value of d?

Solution.

Part (a). Consider the intersection of p_1 and p_2. This gives the system

$$\begin{cases} x - 2y + 2z = 0 \\ 2x - 2y + \ \ z = 0 \end{cases}$$

which has solution $x = t$, $y = \frac{3}{2}t$ and $z = t$. Thus, l has Cartesian equation

$$x = \frac{2}{3}y = z.$$

Part (b). When $d = 0$, p_3 has equation

$$(x - 2y + 2z) + c(2x - 2y + z) = 0.$$

Observe that the line l satisfies the equations $x - 2y + 2z = 0$ and $2x - 2y + z = 0$. Hence, l also satisfies the equation that gives p_3 for all c. Thus, p_3 contains l, implying that all 3 planes meet in the line l.

Part (c). If the 3 planes have no point in common, then l does not have any point in common with p_3. That is, all points on l satisfy the relation

$$(x - 2y + 2z) + c(2x - 2y + z) \neq d.$$

Since $x - 2y + 2z = 0$ and $2x - 2y + z = 0$ for all points on l, the LHS simplifies to 0. Thus, to satisfy the above relation, we require $d \neq 0$.

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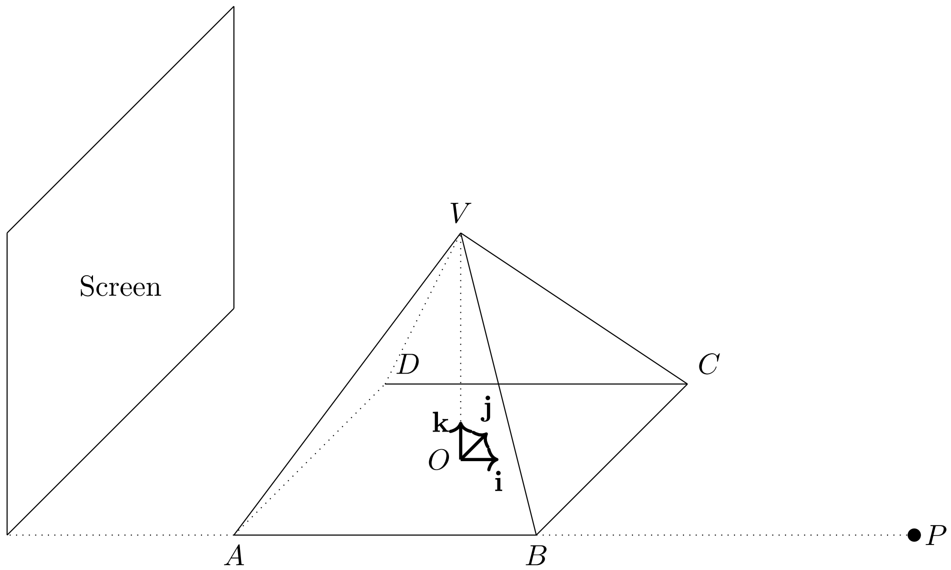
Problem 13.



A right opaque pyramid with square base $ABCD$ and vertex V is placed at ground level for a shadow display, as shown in the diagram. O is the centre of the square base $ABCD$, and the perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are in the directions of $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{OV}$ respectively. The length of AB is 8 units and the length of OV is $2h$ units.

A point light source for this shadow display is placed at the point $P(20, -4, 0)$ and a screen of height 35 units is placed with its base on the ground such that the screen lies on a plane with vector equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha$, where $\alpha < -4$.

(a) Find a vector equation of the line depicting the path of the light ray from P to V in terms of h.

(b) Find an inequality between α and h so that the shadow of the pyramid cast on the screen will not exceed the height of the screen.

The point light source is now replaced by a parallel light source whose light rays are perpendicular to the screen. It is also given that $h = 10$.

(c) Find the exact length of the shadow cast by the edge VB on the screen.

A mirror is placed on the plane VBC to create a special effect during the display.

(d) Find a vector equation of the plane VBC and hence find the angle of inclination made by the mirror with the ground.

Solution.

Part (a). Note that $\overrightarrow{OV} = \langle 0,\ 0,\ 2h\rangle$ and $\overrightarrow{OP} = \langle 20,\ -4,\ 0\rangle$, whence $\overrightarrow{PV} = \langle -20,\ 4,\ 2h\rangle = 2\langle -10,\ 2,\ h\rangle$. Thus, the line from P to V, denoted l_{PV}, has the vector equation

$$l_{PV} : \mathbf{r} = \begin{pmatrix} 20 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -10 \\ 2 \\ h \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Part (b). Let the point of intersection between l_{PV} and the screen be I.

$$\left[\begin{pmatrix} 20 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -10 \\ 2 \\ h \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha \implies 20 - 10\lambda = \alpha \implies \lambda = \frac{20-\alpha}{10}.$$

Hence, $\overrightarrow{OI} = \langle 20,\ -4,\ 0\rangle + \frac{20-\alpha}{10}\langle -10,\ 2,\ h\rangle$. To prevent the shadow from exceeding the screen, we require the $\mathbf{k}$-component of $\overrightarrow{OI}$ to be less than the height of the screen, i.e. 35 units. This gives the inequality $\frac{20-\alpha}{10} \cdot h \leq 35$, whence we obtain

$$h \leq \frac{350}{20-\alpha}.$$

Part (c). Since the light rays emitted by the light source are now perpendicular to the screen, the image of some point with coordinates (a, b, c) on the screen is given by (α, b, c). Thus, the image of $B(4, -4, 0)$ and $V(0, 0, 20)$ on the screen have coordinates $(\alpha, -4, 0)$ and $(\alpha, 0, 20)$. The length of the shadow cast by VB is thus

$$\sqrt{(\alpha-\alpha)^2 + (-4-0)^2 + (0-20)^2} = 4\sqrt{26} \text{ units.}$$

Part (d). Note that $\overrightarrow{BV} = 4\langle -1,\ 1,\ 5\rangle$ and $\overrightarrow{BC} = 8\langle 0,\ 1,\ 0\rangle$. Hence, the plane VBC is parallel to $\langle -1,\ 1,\ 5\rangle$ and $\langle 0,\ 1,\ 0\rangle$. Note that $\langle -1,\ 1,\ 5\rangle \times \langle 0,\ 1,\ 0\rangle = -\langle 5,\ 0,\ 1\rangle$. Thus, $\mathbf{n} = \langle 5,\ 0,\ 1\rangle$, whence $d = \langle 0,\ 0,\ 20\rangle \cdot \langle 5,\ 0,\ 1\rangle = 20$. Thus, the plane VBC has the vector equation

$$\mathbf{r} \cdot \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = 20.$$

Observe that the ground is given by the vector equation $\mathbf{r} \cdot \langle 0,\ 0,\ 1\rangle = 0$. Let θ be the angle of inclination made by the mirror with the ground.

$$\cos\theta = \frac{\langle 5,\ 0,\ 1\rangle \cdot \langle 0,\ 0,\ 1\rangle}{|\langle 5,\ 0,\ 1\rangle|\,|\langle 0,\ 0,\ 1\rangle|} = \frac{1}{\sqrt{26}} \implies \theta = 78.7^\circ \text{ (1 d.p.)}.$$

Self-Practice A9

Problem 1. The position vectors of the vertices of A, B and C of a triangle are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

If O is the origin, show that the area of triangle OAB is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$ and deduce an expression for the area of the triangle ABC.

Hence, or otherwise, show that the perpendicular distance from B to AC is

$$\frac{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|}{\mathbf{c}-\mathbf{a}}.$$

* * * * *

Problem 2. Points A, B, C and D have position vectors, relative to the origin O, given by $\overrightarrow{OA}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\overrightarrow{OB}=-\mathbf{i}+2\mathbf{j}+c\mathbf{k}$, $\overrightarrow{OC}=2\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and $\overrightarrow{OD}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, where c is a constant. It is given that OA and OB are perpendicular.

(a) Find the value of c.

(b) Show that OA is normal to the plane OBC.

(c) Find an equation of the plane through D and parallel to OBC.

Also, find the position vector of the point of intersection of this plane and the line AC. Find the acute angle between the plane OBC and the plane through D normal to OD.

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Problem 3. The equations of the line l_1 and the plane Π_1 are as follows:

$$l_1:\mathbf{r}=\begin{pmatrix}5\\-1\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\0\end{pmatrix},\quad \lambda\in\mathbb{R},$$
$$\Pi_1: xa+z=5a+4,\quad a\in\mathbb{R}^+.$$

(a) If the angle between l_1 and Π_1 is $\pi/6$, show that $a=1$.

Using the value of a in (a),

(b) Verify that l_1 and Π_1 intersect at the point $A(5,-1,4)$.

(c) Given that $C(7,-3,4)$, find the length of projection of $\overrightarrow{AC}$ on Π_1.

(d) Find the position vector of N, the foot of perpendicular of C to Π_1.

(e) Point C' is obtained by reflecting C about Π_1. Determine the vector equation of the line that passes through A and C'.

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Problem 4. The equation of the plane Π_1 is $x+y-2z=3$.

(a) Find the vector equation of the line l_1, which lies in both the plane Π_1 and the yz plane.

(b) Another plane Π_2 contains the line l_2 with equation $x = 1$, $\frac{y+1}{2} = z$ and is perpendicular to Π_1. Find the equation of the plane Π_2 in scalar product form. Determine whether l_1 lies on Π_2.

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Problem 5. The lines l_1 and l_2 intersect at the point P with position vector $\mathbf{i}+5\mathbf{j}+12\mathbf{k}$. The equations of l_1 and l_2 are $\mathbf{r} = (1+3\lambda)\mathbf{i} + (5+2\lambda)\mathbf{j} + (12-2\lambda)\mathbf{k}$ and $\mathbf{r} = (1+8\mu)\mathbf{i} + (5+11\mu)\mathbf{j} + (12+6\mu)\mathbf{k}$ respectively, where λ and μ are real parameters.

(a) Find an equation of the plane Π_1, which contains l_1 and l_2 in the form $\mathbf{r}\cdot\mathbf{n} = d$.

Π_2 and Π_3 are two planes with equations $2x + az = b$ and $x - 3y - z = 7$ respectively, where a and b are constants.

(b) Find the line of intersection between Π_1 and Π_3.

(c) (i) Find the condition satisfied by a if the three planes Π_1, Π_2 and Π_3 intersect at one unique point.

(ii) Given that all three planes meet in a line l, find a and b.

(iii) Given instead that the three planes have no point in common, what can be said about the values of a and b?

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Problem 6. The point A and B have position vectors $3\mathbf{i}+\mathbf{j}$ and $3\mathbf{i}+3\mathbf{j}$ respectively. The line l_1 and the planes Π_1 and Π_2 have equations as follows:

$$l_1 : \mathbf{r} = \overrightarrow{OA} + \alpha\begin{pmatrix}2\\1\\-1\end{pmatrix}, \quad \Pi_1 : x + 2z = 3, \quad \Pi_2 : \mathbf{r} = \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\1\end{pmatrix},$$

where α, λ and $\mu \in \mathbb{R}$.

It is given that the planes Π_1 and Π_2 intersect in the line l_2 and B lies on l_2.

(a) Find a vector equation of the line l_2 and show that the line l_2 is parallel to the line l_1. Hence, find the shortest distance between the lines l_1 and l_2.

(b) The plane Π_3 is parallel to the plane Π_2 and is equidistant to both point A and the plane Π_2. Show that the equation of the plane Π_3 is given by $\mathbf{r}\cdot(\mathbf{i}-\mathbf{j}+\mathbf{k}) = 1$. Find the position vector of the foot of perpendicular from the point A to the plane Π_3.

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Problem 7. The planes p_1, p_2 and p_3 have equations $x = 1$, $2x + y + az = 5$ and $x + 2y + z = b$, where a and b are real constants. Given that p_1 and p_2 intersect at the line l, show that the vector equation of l, in terms of a, is $\mathbf{r} = \mathbf{i} + (3-\lambda a)\mathbf{j} + \lambda\mathbf{k}$, where λ is a real constant.

(a) The acute angle between l and p_3 is $60°$. Without using a calculator, find the possible values of a.

(b) Given that the shortest distance from the origin to p_3 is $\sqrt{6}/3$ and without solving for the value of b, determine the possible position vectors of the foot of perpendicular from the origin to p_3.

(c) What can be said about a and b if p_1, p_2 and p_3 do not have any points in common?

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Problem 8 (🌶). The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The plane π, with vector equation $\mathbf{r} = \mathbf{b} + \lambda\mathbf{u} + \mu\mathbf{c}$, where λ and μ are real parameters, contains B but not A.

(a) Show that the perpendicular distance of A from π is p, where

$$p = \frac{|(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{b} - \mathbf{a})|}{|\mathbf{u} \times \mathbf{v}|}.$$

(b) The perpendicular from A to π meets at C, and D is the point on AB such that CD is perpendicular to AB. Show that $AD = p^2/AB$ and hence, or otherwise, show that the position vector of D is

$$\mathbf{a} + \left(\frac{p}{|\mathbf{b} - \mathbf{a}|}\right)^2 (\mathbf{b} - \mathbf{a}).$$

In the case where $\mathbf{a} = -\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$, $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, find the value of p, and show that

$$\overrightarrow{CD} = \frac{8\sqrt{2}}{9}\mathbf{x} + \frac{4}{9}\mathbf{y},$$

where $\mathbf{x}$ and $\mathbf{y}$ are the unit vectors of $\overrightarrow{CB}$ and $\overrightarrow{CA}$ respectively.

Assignment A9

Problem 1. The equation of the plane Π_1 is $y+z=0$ and the equation of the line l is $\frac{x-2}{2}=\frac{y-2}{-1}=\frac{z-2}{3}$. Find

(a) the position vector of the point of intersection of l and Π_1,

(b) the length of the perpendicular from the origin to l,

(c) the Cartesian equation for the plane Π_2 which contains l and the origin,

(d) the acute angle between the planes Π_1 and Π_2, giving your answer correct to the nearest 0.1°.

Solution. Note that Π_1 has equation $\mathbf{r}\cdot\langle 0,\ 1,\ 1\rangle=0$ and l has equation $\mathbf{r}=\langle 5,\ 2,\ 2\rangle+\lambda\langle 2,\ -1,\ 3\rangle$, $\lambda\in\mathbb{R}$.

Part (a). Let P be the point of intersection of Π_1 and l. Then $\overrightarrow{OP}=\langle 5,\ 2,\ 2\rangle+\lambda\langle 2,\ -1,\ 3\rangle$ for some $\lambda\in\mathbb{R}$. Also, $\overrightarrow{OP}\cdot\langle 0,\ 1,\ 1\rangle=0$. Hence,

$$\left[\begin{pmatrix}5\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}\right]\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}=0\implies 4+2\lambda=0\implies\lambda=-2.$$

Thus,

$$\overrightarrow{OP}=\begin{pmatrix}5\\2\\2\end{pmatrix}-2\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}1\\4\\-4\end{pmatrix}.$$

Part (b). The perpendicular distance from the origin to l is

$$\frac{|\langle 5,\ 2,\ 2\rangle\times\langle 2,\ -1,\ 3\rangle|}{|\langle 2,\ -1,\ 3\rangle|}=\frac{1}{\sqrt{14}}\left|\begin{pmatrix}8\\-11\\-9\end{pmatrix}\right|=\frac{\sqrt{266}}{\sqrt{14}}=\sqrt{19}\text{ units.}$$

Part (c). Observe that Π_2 is parallel to $\langle 5,\ 2,\ 2\rangle$ and $\langle 2,\ -1,\ 3\rangle$. Thus, $\mathbf{n}=\langle 5,\ 2,\ 2\rangle\times\langle 2,\ -1,\ 3\rangle=\langle 8,\ -11,\ -9\rangle$. Since Π_2 contains the origin, $d=0$. Hence, Π_2 has vector equation $\mathbf{r}\cdot\langle 8,\ -11,\ -9\rangle=0$, which translates to $8x-11y-9z=0$.

Part (d). Let the acute angle be θ.

$$\cos\theta=\frac{|\langle 0,\ 1,\ 1\rangle\cdot\langle 8,\ -11,\ -9\rangle|}{|\langle 0,\ 1,\ 1\rangle|\,|\langle 8,\ -11,\ -9\rangle|}=\frac{20}{\sqrt{2}\sqrt{266}}\implies\theta=29.9^\circ\text{ (1 d.p.).}$$

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Problem 2. The plane Π_1 has equation $\mathbf{r}\cdot(-\mathbf{i}+2\mathbf{k})=-4$ and the points A and P have position vectors $4\mathbf{i}$ and $\mathbf{i}+\alpha\mathbf{j}+\mathbf{k}$ respectively, where $\alpha\in\mathbb{R}$.

(a) Show that A lies on Π_1, but P does not.

(b) Find, in terms of α, the position vector of N, the foot of perpendicular of P on Π_1.

The plane Π_2 contains the points A, P and N.

(c) Show that the equation of Π_2 is $\mathbf{r}\cdot(2\alpha\mathbf{i}+5\mathbf{j}+\alpha\mathbf{k})=8\alpha$ and write down the equation of l, the line of the intersection of Π_1 and Π_2.

The plane Π_3 has equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 4$.

(d) By considering l, or otherwise, find the value of α for which the three planes intersect in a line.

Solution. Note that $\Pi_1 : \mathbf{r} \cdot \langle -1, 0, 2\rangle = -4$, $\overrightarrow{OA} = \langle 4, 0, 0\rangle$ and $\overrightarrow{OP} = \langle 1, \alpha, 1\rangle$.

Part (a). Since $\overrightarrow{OA} \cdot \langle -1, 0, 2\rangle = \langle 4, 0, 0\rangle \cdot \langle -1, 0, 2\rangle = -4$, A lies on Π_1. On the other hand, since $\overrightarrow{OP} \cdot \langle -1, 0, 2\rangle = \langle 1, \alpha, 1\rangle \cdot \langle -1, 0, 2\rangle = 1 \neq -4$, P does not lie on Π_1.

Part (b). Note that $\overrightarrow{NP} = \lambda \langle -1, 0, 2\rangle$ for some $\lambda \in \mathbb{R}$, and $\overrightarrow{ON} \cdot \langle -1, 0, 2\rangle = -4$. Hence,

$$\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON} = \begin{pmatrix}1\\ \alpha\\ 1\end{pmatrix} - \overrightarrow{ON} = \lambda \begin{pmatrix}-1\\ 0\\ 2\end{pmatrix}.$$

Thus,

$$\left[\begin{pmatrix}1\\ \alpha\\ 1\end{pmatrix} - \overrightarrow{ON}\right] \cdot \begin{pmatrix}-1\\ 0\\ 2\end{pmatrix} = \lambda \begin{pmatrix}-1\\ 0\\ 2\end{pmatrix} \cdot \begin{pmatrix}-1\\ 0\\ 2\end{pmatrix} \implies 1 - (-4) = 5\lambda \implies \lambda = 1.$$

Hence, $\overrightarrow{NP} = \langle -1, 0, 2\rangle$, whence $\overrightarrow{ON} = \overrightarrow{OP} - \overrightarrow{NP} = \langle 2, \alpha, -1\rangle$.

Part (c). Note that Π_2 is parallel to $\overrightarrow{NP} = \langle -1, 0, 2\rangle$ and $\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \langle -2, \alpha, -1\rangle$. Since $\langle -1, 0, 2\rangle \times \langle -2, \alpha, -1\rangle = -\langle 2\alpha, 5, \alpha\rangle$, we take $\mathbf{n} = \langle 2\alpha, 5, \alpha\rangle$, whence $d = \langle 4, 0, 0\rangle \cdot \langle 2\alpha, 5, \alpha\rangle = 8\alpha$. Thus, Π_2 has vector equation $\mathbf{r} \cdot \langle 2\alpha, 5, \alpha\rangle = 8\alpha$ which translates to $\mathbf{r} \cdot (2\alpha\mathbf{i} + 5\mathbf{j} + \alpha\mathbf{k}) = 8\alpha$.

Meanwhile, the line of intersection between Π_1 and Π_2 has equation

$$l : \begin{pmatrix}4\\ 0\\ 0\end{pmatrix} + \mu \begin{pmatrix}-2\\ \alpha\\ -1\end{pmatrix}, \quad \mu \in \mathbb{R}.$$

Part (d). If the three planes intersect in a line, they must intersect at l. Hence, l lies on Π_3.

$$\left[\begin{pmatrix}4\\ 0\\ 0\end{pmatrix} + \mu \begin{pmatrix}-2\\ \alpha\\ -1\end{pmatrix}\right] \cdot \begin{pmatrix}1\\ 1\\ 2\end{pmatrix} = 4 \implies 4 + (\alpha - 4)\mu = 4 \implies (\alpha - 4)\mu = 0.$$

Since $(\alpha - 4)\mu = 0$ must hold for all $\mu \in \mathbb{R}$, we must have $\alpha = 4$.

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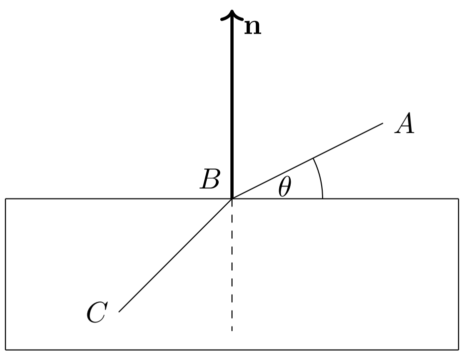
Problem 3. When a light ray passes from air to glass, it is deflected through an angle. The light ray ABC starts at point $A(1, 2, 2)$ and enters a glass object at point $B(0, 0, 2)$. The surface of the glass object is a plane with normal vector $\mathbf{n}$. The diagram shows a cross-section of the glass object in the plane of the light ray and $\mathbf{n}$.



(a) Find a vector equation of the line AB.

The surface of the glass object is a plane with equation $x+z=2$. AB makes an acute angle θ with the plane.

(b) Calculate the value of θ, giving your answer in degrees.

The line BC makes an angle of 45° with the normal to the plane, and BC is parallel to the unit vector $\langle -2/3,\, p,\, q\rangle$.

(c) By considering a vector perpendicular to the plane containing the light ray and $\mathbf{n}$, or otherwise, find the values of p and q.

The light ray leaves the glass object through a plane with equation $3x+3z=-4$.

(d) Find the exact thickness of the glass object, taking one unit as one cm.

(e) Find the exact coordinates of the point at which the light ray leaves the glass object.

Solution. Let Π_G be the plane representing the surface of the glass object.

Part (a). Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 0,\, 0,\, 2\rangle - \langle 1,\, 2,\, 2\rangle = -\langle 1,\, 2,\, 0\rangle$. Hence,

$$l_{AB} : \mathbf{r} = \begin{pmatrix}0\\0\\2\end{pmatrix} + \lambda \begin{pmatrix}1\\2\\0\end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Part (b). Observe that Π_G has equation $\mathbf{r}\cdot\langle 1,\, 0,\, 1\rangle = 2$. Hence,

$$\sin\theta = \frac{|\langle 1,\, 0,\, 1\rangle \cdot \langle 1,\, 2,\, 0\rangle|}{|\langle 1,\, 0,\, 1\rangle|\,|\langle 1,\, 2,\, 0\rangle|} = \frac{1}{\sqrt{2}\sqrt{5}} \implies \theta = 71.6^\circ \text{ (1 d.p.)}.$$

Part (c). Since line BC makes an angle of 45° with $\mathbf{n}_G$,

$$\sin 45^\circ = \frac{|\langle 1,\, 0,\, 1\rangle \cdot \langle -2/3,\, p,\, q\rangle|}{|\langle 1,\, 0,\, 1\rangle|\,|\langle -2/3,\, p,\, q\rangle|} \implies \frac{1}{\sqrt{2}} = \frac{|q-2/3|}{\sqrt{2}\cdot 1} \implies \left|q - \frac{2}{3}\right| = 1.$$

Hence, $q=-1/3$. Note that we reject $q=5/3$ since $\langle -2/3,\, p,\, q\rangle$ is a unit vector, which implies that $|q|\leq 1$.

Let Π_L be the plane containing the light ray. Note that Π_L is parallel to $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Hence, $\mathbf{n}_L = \langle 1,\, 2,\, 0\rangle \times \langle -2/3,\, p,\, q\rangle = \frac{1}{3}\langle 6q,\, -3q,\, 3p+4\rangle$. Since Π_L contains $\mathbf{n}_G$, we have that $\mathbf{n}_L \perp \mathbf{n}_G$, whence $\mathbf{n}_L \cdot \mathbf{n}_G = 0$. This gives us

$$\begin{pmatrix}6q\\-3q\\3p+4\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} = 0 \implies 6q+3p+4=0 \implies 6\left(-\frac{1}{3}\right)+3p+4=0 \implies p=-\frac{2}{3}.$$

Part (d). Let Π_G' be the plane with equation $3x+3z=-4$. Observe that Π_G is parallel to Π_G'. Also note that $(-4/3, 0, 0)$ is a point on Π_G'. Hence, the distance between Π_G and Π_G' is given by

$$\frac{|2-\langle -4/3,\, 0,\, 0\rangle \cdot \langle 1,\, 0,\, 1\rangle|}{|\langle 1,\, 0,\, 1\rangle|} = \frac{10}{3\sqrt{2}} \text{ cm}.$$

Part (e). Observe that $\langle -2/3,\, p,\, q\rangle = \langle -2/3,\, -2/3,\, -1/3\rangle = -\frac{1}{3}\langle 2,\, 2,\, 1\rangle$, whence the line BC has equation $\mathbf{r} = \langle 0,\, 0,\, 2\rangle + \mu\langle 2,\, 2,\, 1\rangle$, $\mu\in\mathbb{R}$. Let P be the intersection between line BC and Π_G'. Also note that $\overrightarrow{OP} = \langle 0,\, 0,\, 2\rangle + \mu\langle 2,\, 2,\, 1\rangle$ for some $\mu\in\mathbb{R}$, and $\overrightarrow{OP}\cdot\langle 3,\, 0,\, 3\rangle = -4$. Hence,

$$\left[\begin{pmatrix}0\\0\\2\end{pmatrix} + \mu\begin{pmatrix}2\\2\\1\end{pmatrix}\right] \cdot \begin{pmatrix}3\\0\\3\end{pmatrix} = -4 \implies 6-9\mu=-4 \implies \mu = -\frac{10}{9}.$$

Hence, $\overrightarrow{OP} = \langle 0,\, 0,\, 2\rangle - \frac{10}{9}\langle 2,\, 2,\, 1\rangle = \langle -20/9,\, -20/9,\, 8/9\rangle$. The coordinates of the point are hence $(-20/9, -20/9.8/9)$.

A10.1 Complex Numbers - Complex Numbers in Cartesian Form

Tutorial A10.1

Problem 1. Given that $z = 3 - 2\mathrm{i}$ and $w = 1 + 4\mathrm{i}$, express in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$:

(a) $z + 2w$

(b) zw

(c) z/w

(d) $(w - w^*)^3$

(e) z^4

Solution.

Part (a).

$$z + 2w = (3 - 2\mathrm{i}) + 2(1 + 4\mathrm{i}) = 3 - 2\mathrm{i} + 2 + 8\mathrm{i} = 5 + 6\mathrm{i}.$$

Part (b).

$$zw = (3 - 2\mathrm{i})(1 + 4\mathrm{i}) = 3 + 12\mathrm{i} - 2\mathrm{i} + 8 = 11 + 10\mathrm{i}.$$

Part (c).

$$\frac{z}{w} = \frac{3 - 2\mathrm{i}}{1 + 4\mathrm{i}} = \frac{(3 - 2\mathrm{i})(1 - 4\mathrm{i})}{(1 + 4\mathrm{i})(1 - 4\mathrm{i})} = \frac{3 - 12\mathrm{i} - 2\mathrm{i} - 8}{1^2 + 4^2} = \frac{-5 - 14\mathrm{i}}{17} = -\frac{5}{17} - \frac{14}{17}\mathrm{i}.$$

Part (d).

$$(w - w^*)^3 = [2\mathrm{Im}(w)\mathrm{i}]^3 = (8\mathrm{i})^3 = -512\mathrm{i}.$$

Part (e).

$$\begin{aligned} z^4 = (3 - 2\mathrm{i})^4 &= 3^4 + 4 \cdot 3^3(-2\mathrm{i}) + 6 \cdot 3^2(-2\mathrm{i})^2 + 4 \cdot 3(-2\mathrm{i})^3 + (-2\mathrm{i})^4 \\ &= 81 - 216\mathrm{i} - 216 + 96\mathrm{i} + 16 = -119 - 120\mathrm{i}. \end{aligned}$$

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Problem 2. Is the following true or false in general?

(a) $\mathrm{Im}(zw) = \mathrm{Im}(z)\mathrm{Im}(w)$

(b) $\mathrm{Re}(zw) = \mathrm{Re}(z)\mathrm{Re}(w)$

Solution. Let $z = a+bi$ and $w = c+di$. Then $zw = (a+bi)(c+di) = (ac-bd)+(ad+bc)i$.

Part (a). Observe that

$$\mathrm{Im}(zw) = ad + bc \neq bd = \mathrm{Im}(z)\mathrm{Im}(w).$$

Hence, the statement is false in general.

Part (b). Observe that

$$\mathrm{Re}(zw) = ac - bd \neq ac = \mathrm{Re}(z)\mathrm{Re}(w).$$

Hence, the statement is false in general.

Problem 3.

(a) Find the complex number z such that $\frac{z-2}{z} = 1 + \mathrm{i}$.

(b) Given that $u = 2 + \mathrm{i}$ and $v = -2 + 4\mathrm{i}$, find in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$, the complex number z such that $\frac{1}{z} = \frac{1}{u} + \frac{1}{v}$.

Solution.

Part (a).

$$\frac{z-2}{z} = 1 + \mathrm{i} \implies z - 2 = z + \mathrm{i}z \implies \mathrm{i}z = -2 \implies z = -\frac{2}{\mathrm{i}} = 2\mathrm{i}.$$

Part (b).

$$\frac{1}{z} = \frac{1}{u} + \frac{1}{v} \implies z = \frac{1}{1/u + 1/v} = \frac{uv}{u+v} = \frac{(2+\mathrm{i})(-2+4\mathrm{i})}{(2+\mathrm{i}) + (-2+4\mathrm{i})} = \frac{-8+6\mathrm{i}}{5\mathrm{i}} = \frac{6}{5} + \frac{8}{5}\mathrm{i}.$$

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Problem 4. The complex numbers z and w are $1 + a\mathrm{i}$ and $b - 2\mathrm{i}$ respectively, where a and b are real and a is negative. Given that $zw^* = 8i$, find the exact values of a and b.

Solution. Note that

$$zw^* = (1 + a\mathrm{i})(b + 2\mathrm{i}) = (b - 2a) + (2 + ab)\mathrm{i}.$$

Comparing real and imaginary parts, we have $b - 2a = 0 \implies b = 2a$ and $2 + ab = 8$. Hence, $2a^2 = 6$, giving $a = -\sqrt{3}$ and $b = -2\sqrt{3}$.

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Problem 5. Find, in the form $x + \mathrm{i}y$, the two complex numbers z satisfying both of the equations

$$\frac{z}{z^*} = \frac{3}{5} + \frac{4}{5}\mathrm{i} \quad \text{and} \quad zz^* = 5.$$

Solution. Multiplying both equations together, we have $z^2 = 3 + 4\mathrm{i}$. Let $z = x + \mathrm{i}y$, with $x, y \in \mathbb{R}$. We thus have $z^2 = x^2 - y^2 + 2xy\mathrm{i} = 3 + 4\mathrm{i}$. Comparing real and imaginary parts, we obtain the following system:

$$x^2 - y^2 = 3, \quad 2xy = 4.$$

Squaring the second equation yields $x^2y^2 = 4$. From the first equation, we have $x^2 = 3 + y^2$. Thus, $y^2(3 + y^2) = 4 \implies y^2 = 1 \implies y = \pm 1 \implies x = \pm 2$. Hence, $z = 2 + \mathrm{i}$ or $z = -2 - \mathrm{i}$.

Problem 6.

(a) Given that $\mathrm{i}w + 3z = 2 + 4\mathrm{i}$ and $w + (1 - \mathrm{i})z = 2 - \mathrm{i}$, find z and w in the form of $x + \mathrm{i}y$, where x and y are real numbers.

(b) Determine the value of k such that $z = \frac{1-k\mathrm{i}}{\sqrt{3}+\mathrm{i}}$ is purely imaginary, where $k \in \mathbb{R}$.

Solution.

Part (a). Let $w = a + b\mathrm{i}$ and $z = c + d\mathrm{i}$. Then

$$\mathrm{i}w + 3z = \mathrm{i}(a + b\mathrm{i}) + 3(c + d\mathrm{i}) = (-b + 3c) + (a + 3d)\mathrm{i} = 2 + 4\mathrm{i}$$

and

$$w + (1 - \mathrm{i})z = (a + b\mathrm{i}) + (1 - \mathrm{i})(c + d\mathrm{i}) = (a + c + d) + (b - c + d)\mathrm{i} = 2 - \mathrm{i}.$$

Comparing the real and imaginary parts of both equations yields the following system:

$$\begin{cases} \quad -b + 3c \quad = 2 \\ a \qquad\qquad + 3d = 4 \\ a \quad + c + d = 2 \\ \quad b - c + d = -1 \end{cases}$$

which has the unique solution $a = 1$, $b = -2$, $c = 0$ and $d = 1$. Hence, $w = 1 - 2\mathrm{i}$ and $z = \mathrm{i}$.

Part (b).

$$z = \frac{1 - k\mathrm{i}}{\sqrt{3} + \mathrm{i}} = \frac{(1 - k\mathrm{i})(\sqrt{3} - \mathrm{i})}{\sqrt{3}^2 + 1^2} = \frac{1}{4}(\sqrt{3} - \mathrm{i} - k\sqrt{3}\mathrm{i} - k) = \frac{1}{4}\left[(\sqrt{3} - k) - (1 + k\sqrt{3})\mathrm{i}\right].$$

Since z is purely imaginary, $\operatorname{Re}(z) = 0$. Hence, $\frac{1}{4}(\sqrt{3} - k) = 0 \implies k = \sqrt{3}$.

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Problem 7.

(a) The complex number $x + \mathrm{i}y$ is such that $(x + iy)^2 = \mathrm{i}$. Find the possible values of the real numbers x and y, giving your answers in exact form.

(b) Hence, find the possible values of the complex number w such that $w^2 = -\mathrm{i}$.

Solution.

Part (a). Note that $(x + \mathrm{i}y)^2 = x^2 - y^2 + 2xy\mathrm{i} = \mathrm{i}$. Comparing real and imaginary parts, we have

$$x^2 - y^2 = 0, \quad 2xy = 1.$$

Note that the second equation implies that both x and y have the same sign. Hence, from the first equation, we have $x = y$. Thus, $x^2 = y^2 = 1/2 \implies x = y = \pm 1/\sqrt{2}$.

Part (b).

$$w^2 = -\mathrm{i} \implies (w^*)^2 = \mathrm{i} \implies w^* = \pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}\mathrm{i} \implies w = \pm\frac{1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}}\mathrm{i}.$$

Problem 8.

(a) The roots of the equation $z^2 = -8\mathrm{i}$ are z_1 and z_2. Find z_1 and z_2 in Cartesian form $x + \mathrm{i}y$, showing your working.

(b) Hence, or otherwise, find in Cartesian form the roots w_1 and w_2 of the equation $w^2 + 4w + (4 + 2\mathrm{i}) = 0$.

Solution.

Part (a). Let $z = x + \mathrm{i}y$ where $x, y \in \mathbb{R}$. Then $(x + \mathrm{i}y)^2 = x^2 - y^2 + 2xy\mathrm{i} = -8\mathrm{i}$. Comparing real and imaginary parts, we have the following system:

$$x^2 - y^2 = 0, \quad 2xy = 8.$$

From the second equation, we know that x and y have opposite signs. Hence, from the first equation, we have that $x = -y$. Thus, $x^2 = 4 \implies x = \pm 2 \implies y = \mp 2$. Thus, $z = \pm 2(1 - \mathrm{i})$, whence $z_1 = 2 - 2\mathrm{i}$ and $z_2 = -2 + 2\mathrm{i}$.

Part (b).

$$\begin{aligned} w^2 + 4w + (4 + 2\mathrm{i}) = 0 \implies (w + 2)^2 = -2\mathrm{i} \implies (2w + 4)^2 = -8\mathrm{i} \\ \implies 2w + 4 = \pm 2(1 - \mathrm{i}) \implies w = 2 \pm (1 - \mathrm{i}). \end{aligned}$$

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Problem 9. One of the roots of the equations $2x^3 - 9x^2 + 2x + 30 = 0$ is $3 + \mathrm{i}$. Find the other roots of the equation.

Solution. Let $P(x) = 2x^3 - 9x^2 + 2x + 30$. Since $P(x)$ is a polynomial with real coefficients, by the conjugate root theorem, we have that $(3 + \mathrm{i})^* = 3 - \mathrm{i}$ is also a root of $P(x)$. Let α be the third root of $P(x)$. Then

$$P(x) = 2x^3 - 9x^2 + 2x + 30 = 2(x - \alpha)\left[x - (3 + i)\right]\left[x - (3 - i)\right].$$

Comparing constants,

$$2(-\alpha)(-3 - \mathrm{i})(-3 + \mathrm{i}) = 30 \implies \alpha = -\frac{15}{(-3 - \mathrm{i})(-3 + \mathrm{i})} = -\frac{3}{2}.$$

Hence, the other roots of the equation are $3 - \mathrm{i}$ and $-3/2$.

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Problem 10. Obtain a cubic equation having 2 and $\frac{5}{4} - \frac{\sqrt{7}}{4}\mathrm{i}$ as two of its roots, in the form $az^3 + bz^2 + cz + d = 0$, where a, b, c and d are real integral coefficients to be determined.

Solution. Let $P(z) = az^3 + bz^2 + cz + d$. Since $P(z)$ is a polynomial with real coefficients, by the conjugate root theorem, we have that $\left(\frac{5}{4} - \frac{\sqrt{7}}{4}\mathrm{i}\right)^* = \frac{5}{4} + \frac{\sqrt{7}}{4}\mathrm{i}$ is also a root of $P(z)$. We can thus write $P(z)$ as

$$\begin{aligned} P(z) &= k(z - 2)\left[z - \left(\frac{5}{4} - \frac{\sqrt{7}}{4}\mathrm{i}\right)\right]\left[z - \left(\frac{5}{4} + \frac{\sqrt{7}}{4}\mathrm{i}\right)\right] \\ &= k(z - 2)\left[\left(z - \frac{5}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2\right] = k(z - 2)\left(z^2 - \frac{5}{2}z + 2\right) \\ &= \frac{1}{2}k(2z^3 - 9z^2 + 14z - 8), \end{aligned}$$

where k is an arbitrary real number. Taking $k = 2$, we have $P(z) = 2z^3 - 9z^2 + 14z - 8$, whence $a = 2$, $b = -9$, $c = 14$ and $d = -8$.

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Problem 11.

(a) Verify that $-1 + 5\mathrm{i}$ is a root of the equation $w^2 + (-1 - 8\mathrm{i})w + (-17 + 7\mathrm{i}) = 0$. Hence, or otherwise, find the second root of the equation in Cartesian form, $p + \mathrm{i}q$, showing your working.

(b) The equation $z^3 - 5z^2 + 16z + k = 0$, where k is a real constant, has a root $z = 1 + a\mathrm{i}$, where a is a positive real constant. Find the values of a and k, showing your working.

Solution.

Part (a). Let $P(w) = w^2 + (-1 - 8\mathrm{i})w + (-17 + 7\mathrm{i})$. Consider $P(-1 + 5\mathrm{i})$.

$$\begin{aligned} P(-1 + 5\mathrm{i}) &= (-1 + 5\mathrm{i})^2 + (-1 - 8\mathrm{i})(-1 + 5\mathrm{i}) + (-17 + 7\mathrm{i}) \\ &= (1 - 10\mathrm{i} - 25) + (1 - 5\mathrm{i} + 8\mathrm{i} + 40) + (-17 + 7\mathrm{i}) = 0. \end{aligned}$$

Hence, $-1 + 5\mathrm{i}$ is a root of $w^2 + (-1 - 8\mathrm{i})w + (-17 + 7\mathrm{i}) = 0$.

Let α be the other root of the equation. By Vieta's formula, we have

$$\alpha + (-1 + 5\mathrm{i}) = -\left(\frac{-1 - 8\mathrm{i}}{1}\right) = 1 + 8\mathrm{i} \implies \alpha = 2 + 3\mathrm{i}.$$

Part (b). Let $P(z) = z^3 - 5z^2 + 16z + k$. Then $P(1 + a\mathrm{i}) = 0$. Note that

$$\begin{aligned} P(1 + a\mathrm{i}) &= (1 + a\mathrm{i})^3 - 5(1 + a\mathrm{i})^2 + 16(1 + a\mathrm{i}) + k \\ &= \left[1 + 3a\mathrm{i} - 3a^2 - a^3\mathrm{i}\right] - 5(1 + 2a\mathrm{i} - a^2) + (16 + 16a\mathrm{i}) + k \\ &= (12 + k + 2a^2) + (9 - a^2)a\mathrm{i}. \end{aligned}$$

Comparing real and imaginary parts, we have $a(9 - a^2) = 0 \implies a = 3$ (since $a > 0$) and $12 + k + 2a^2 = 0 \implies k = -30$.

Self-Practice A10.1

Problem 1. By writing $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, solve the simultaneous equations

$$z^2 + zw - 2 = 0 \quad \text{and} \quad z^* = \frac{w}{1+\mathrm{i}},$$

where z^* is the conjugate of z.

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Problem 2. Given that the complex numbers w and z satisfy the equations

$$w^* + 2z = \mathrm{i} \quad \text{and} \quad w + (1 - 2\mathrm{i})z = 3 + 3\mathrm{i},$$

find w and z in the form $a + b\mathrm{i}$, where a and b are real.

* * * * *

Problem 3.

(a) Determine the complex numbers u and v for which

$$z^2 + (6 - 2\mathrm{i})z = (z - u)^2 - v, \quad \forall z \in \mathbb{C}.$$

(b) Write down the square roots of $7 - 24\mathrm{i}$. Hence, solve the quadratic equation $z^2 + (6 - 2\mathrm{i})z = -1 - 18\mathrm{i}$.

* * * * *

Problem 4. If $z = i$ is a root of the equation $z^3 + (1-3\mathrm{i})z^2 - (2+3\mathrm{i})z - 2 = 0$, determine the other roots. Hence, find the roots of the equation $w^3 + (1+3\mathrm{i})w^2 + (3\mathrm{i} - 2)w - 2 = 0$.

* * * * *

Problem 5. Show that the equation $z^4 - 2z^3 + 6z^2 - 8z + 8 = 0$ has a root of the form $k\mathrm{i}$, where k is real. Hence, solve the equation $z^4 - 2z^3 + 6z^2 - 8z + 8 = 0$.

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Problem 6. Verify that $-2 + \mathrm{i}$ is a root of the equation $z^4 + 24z + 55 = 0$. Hence, determine the other roots.

Assignment A10.1

Problem 1. The complex number w is such that $ww^* + 2w = 3 + 4\mathrm{i}$, where w^* is the complex conjugate of w. Find w in the form $a + \mathrm{i}b$, where a and b are real.

Solution. Note $ww^* = (\operatorname{Re} w)^2 + (\operatorname{Im} w)^2 \in \mathbb{R}$.

Taking the imaginary part of the given equation,

$$\operatorname{Im}(ww^* + 2w) = \operatorname{Im}(3 + 4i) \implies 2 \operatorname{Im} w = 4 \implies \operatorname{Im} w = 2.$$

Taking the real part of the given equation,

$$\operatorname{Re}(ww^* + 2w) = \operatorname{Re}(3 + 4i) \implies \left[(\operatorname{Re} w)^2 + (\operatorname{Im} w)^2\right] + 2 \operatorname{Re} w = 3$$
$$\implies (\operatorname{Re} w)^2 + 2\mathrm{Re}(w) + 1 = 0 \implies (\operatorname{Re} w + 1)^2 = 0 \implies \mathrm{Re}(w) = -1.$$

Hence, $w = -1 + 2\mathrm{i}$.

* * * * *

Problem 2. Express $(3 - \mathrm{i})^2$ in the form $a + \mathrm{i}b$.

Hence, or otherwise, find the roots of the equation $(z + \mathrm{i})^2 = -8 + 6\mathrm{i}$.

Solution. We have

$$(3 - \mathrm{i})^2 = 3^2 - 6\mathrm{i} + \mathrm{i}^2 = 8 - 6\mathrm{i}.$$

Consider $(z + \mathrm{i})^2 = -8 + 6\mathrm{i}$. Note that $-(z + \mathrm{i})^2 = (\mathrm{i}z - 1)^2$.

$$(z + \mathrm{i})^2 = -8 + 6\mathrm{i} \implies (\mathrm{i}z - 1)^2 = 8 - 6\mathrm{i} \implies \mathrm{i}z - 1 = \pm(3 - \mathrm{i})$$
$$\implies z = \frac{1}{\mathrm{i}}(1 \pm (3 - \mathrm{i})) = -\mathrm{i}(1 \pm (3 - \mathrm{i})) = -1 - 4\mathrm{i} \text{ or } 1 + 2\mathrm{i}.$$

* * * * *

Problem 3.

(a) It is given that $z_1 = 1 + \sqrt{3}\mathrm{i}$. Find the value of z_1^3, showing clearly how you obtain your answer.

(b) Given that $1 + \sqrt{3}\mathrm{i}$ is a root of the equation

$$2z^3 + az^2 + bz + 4 = 0$$

find the values of the real numbers a and b. Hence, solve the above equation.

Solution.

Part (a). We have

$$z_1^3 = \left(1 + \sqrt{3}\mathrm{i}\right)^3 = 1 + 3\left(\sqrt{3}\mathrm{i}\right) + 3\left(\sqrt{3}\mathrm{i}\right)^2 + \left(\sqrt{3}\mathrm{i}\right)^3 = 1 + 3\sqrt{3}\mathrm{i} - 9 - 3\sqrt{3}\mathrm{i} = -8.$$

Part (b). Since $1 + \sqrt{3}\mathrm{i}$ is a root of the given equation, we have

$$2\left(1 + \sqrt{3}\mathrm{i}\right)^3 + a\left(1 + \sqrt{3}\mathrm{i}\right)^2 + b\left(1 + \sqrt{3}\mathrm{i}\right) + 4 = 0$$
$$\implies -16 + a\left(-2 + 2\sqrt{3}\mathrm{i}\right) + b\left(1 + \sqrt{3}\mathrm{i}\right) + 4 = 0 \implies (-2a + b) + \sqrt{3}(2a + b)\mathrm{i} = 12.$$

Comparing real and imaginary parts, we obtain $-2a + b = 12$ and $2a + b = 0$, whence $a = -3$ and $b = 6$.

Since the coefficients of $2z^3 + az^2 + bz + 4$ are all real, the second root is $(1+\sqrt{3}\mathrm{i})^* = 1-\sqrt{3}\mathrm{i}$. Let the third root be α. By Vieta's formula,

$$(1+\sqrt{3}\mathrm{i})(1-\sqrt{3}\mathrm{i})\alpha = -\frac{4}{2} \implies 4\alpha = -2 \implies \alpha = -\frac{1}{2}.$$

The roots of the equation are hence $1+\sqrt{3}\mathrm{i}$, $1-\sqrt{3}\mathrm{i}$ and $-\frac{1}{2}$.

* * * * *

Problem 4. The complex number z is such that $az^2 + bz + a = 0$ where a and b are real constants. It is given that $z = z_0$ is a solution to this equation where $\mathrm{Im}(z_0) \neq 0$.

(a) Verify that $z = \frac{1}{z_0}$ is the other solution. Hence, show that $|z_0| = 1$.

Take $\mathrm{Im}(z_0) = 1/2$ for the rest of the question.

(b) Find the possible complex numbers for z_0.

(c) If $\mathrm{Re}(z_0) > 0$, find b in terms of a.

Solution.

Part (a).

$$a\left(\frac{1}{z_0}\right)^2 + b\left(\frac{1}{z_0}\right) + a = \left(\frac{1}{z_0}\right)^2\left(a + bz_0 + az_0^2\right) = 0$$

Hence, $z = 1/z_0$ is a root of the given equation.

Since $a, b \in \mathbb{R}$, by the conjugate root theorem, $z_0^* = 1/z_0$. Hence,

$$z_0 z_0^* = 1 \implies \mathrm{Re}(z_0)^2 + \mathrm{Im}(z_0)^2 = |z_0|^2 = 1 \implies |z_0| = 1.$$

Part (b). Let $z_0 = x + \frac{1}{2}\mathrm{i}$. Then

$$\left|x + \frac{1}{2}i\right| = 1 \implies x^2 + \left(\frac{1}{2}\right)^2 = 1^2 \implies x^2 = \frac{3}{4} \implies x = \pm\frac{\sqrt{3}}{2}.$$

Hence, $z_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$ or $z_0 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$.

Part (c). Since $\mathrm{Re}(z_0) > 0$, we have $z_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$. By Vieta's formula,

$$-\frac{b}{a} = z_0 + \frac{1}{z_0} = z_0 + z_0^* = 2\mathrm{Re}(z_0) = \sqrt{3} \implies b = -\sqrt{3}a.$$

A10.2 Complex Numbers - Complex Numbers in Polar Form

Tutorial A10.2

Problem 1. Is the following true or false in general?

(a) $\left|w^2\right| = |w|^2$

(b) $|z + 2w| = |z| + |2w|$

Solution.
Part (a). Let $w = r\mathrm{e}^{\mathrm{i}\theta}$, where $r, \theta \in \mathbb{R}$. Note that $\left|\mathrm{e}^{\mathrm{i}\theta}\right| = \left|\mathrm{e}^{2\mathrm{i}\theta}\right| = 1$.

$$\left|w^2\right| = \left|r^2\mathrm{e}^{2\mathrm{i}\theta}\right| = r^2\left|\mathrm{e}^{2\mathrm{i}\theta}\right| = r^2 = r^2\left|\mathrm{e}^{\mathrm{i}\theta}\right|^2 = \left|r\mathrm{e}^{\mathrm{i}\theta}\right|^2 = |w|^2 .$$

The statement is hence true in general.
Part (b). Take $z = 1$ and $w = -1$.

$$|z + 2w| = |1 - 2| = 1 \neq 3 = |1| + |2(-1)| = |z| + |2w| .$$

The statement is hence false in general.

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Problem 2. Express the following complex numbers z in polar form $r(\cos\theta + \mathrm{i}\sin\theta)$ with exact values.

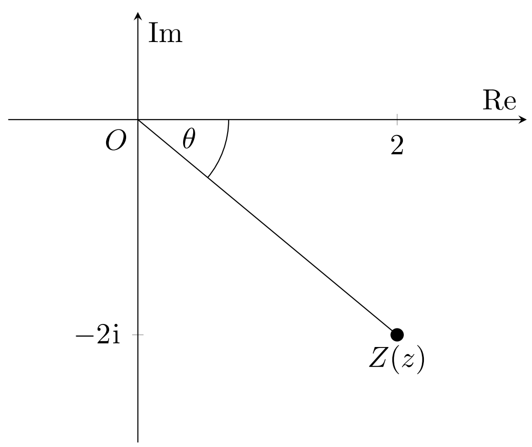
(a) $z = 2 - 2\mathrm{i}$

(b) $z = -1 + \mathrm{i}\sqrt{3}$

(c) $z = -5\mathrm{i}$

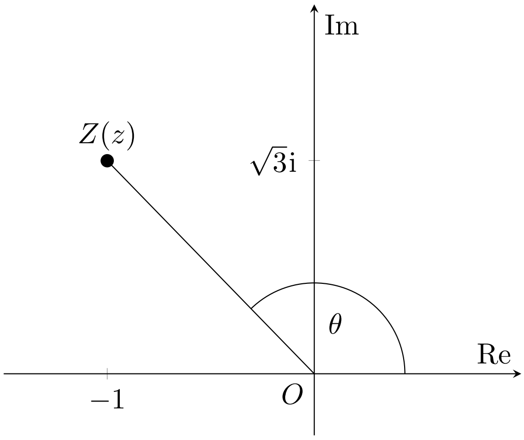
(d) $z = -2\sqrt{3} - 2\mathrm{i}$

Solution.
Part (a).



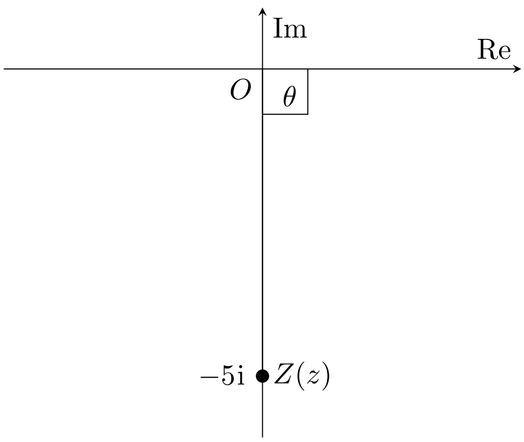
We have $r^2 = 2^2 + (-2)^2 \implies r = 2\sqrt{2}$ and $\tan\theta = -2/2 \implies \theta = -\pi/4$. Hence, $2 - 2\mathrm{i} = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]$.

Part (b).



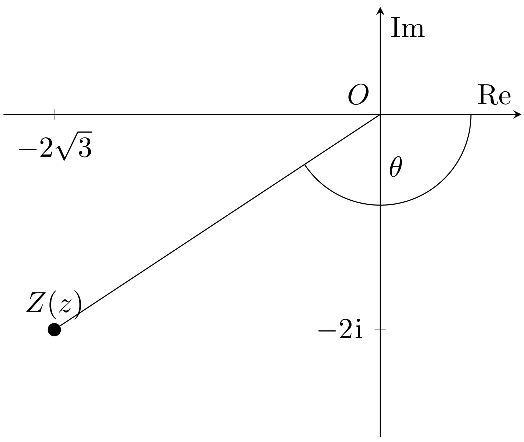
We have $r^2 = (-1)^2 + (\sqrt{3})^2 \implies r = 2$ and $\tan t = \sqrt{3}/(-1) \implies \theta = 2\pi/3$. Hence, $-1 + \sqrt{3}\mathrm{i} = 2\left[\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right]$.

Part (c).



We have $r = 5$ and $\theta = -\pi/2$. Hence, $-5\mathrm{i} = 5\left[\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right]$.

Part (d).



We have $r^2 = (-2\sqrt{3})^2 + (-2)^2 \implies r = 4$ and $\tan t = -2/(-2\sqrt{3}) \implies \theta = -5\pi/6$. Hence, $-2\sqrt{3} - 2\mathrm{i} = 4\left[\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right]$.

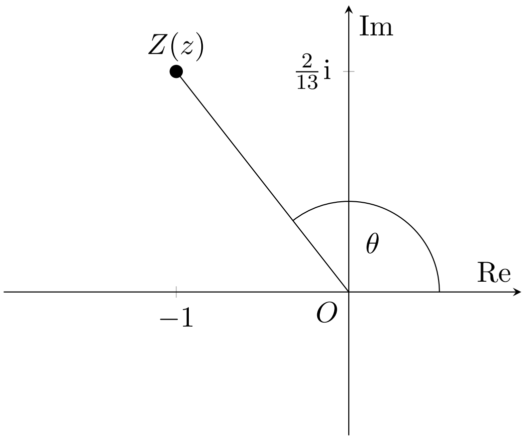
Problem 3. Express the following complex numbers z in exponential form $re^{i\theta}$.

(a) $z = -1 + \frac{2}{13}i$

(b) $z = \cos 50° - i \sin 50°$

Solution.

Part (a).



We have $r^2 = (-1)^2 + \left(\frac{2}{13}\right)^2 \implies r = 1.01$ (3 s.f.) and $\tan t = \frac{2/13}{-1} \implies \theta = 2.99$ (3 s.f.). Hence, $-1 + \frac{2}{13}i = 1.01e^{2.99i}$.

Part (b). We have $r = 1$ and $\theta = -50° = -\frac{5}{18}\pi$. Hence, $\cos 50° + i \sin 50° = e^{-i\frac{5}{18}\pi}$.

* * * * *

Problem 4. Express the following complex numbers z in Cartesian form.

(a) $z = 7e^{1-5i}$

(b) $z = 6\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

Solution.

Part (a). We have

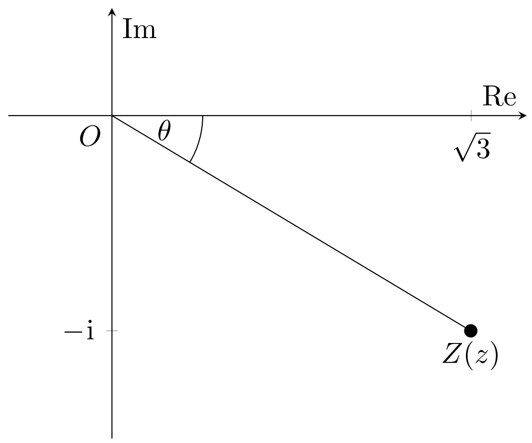
$$z = 7e^{1-5i} = 7e \cdot e^{-5i} = 7e\left[\cos(-5) + i\sin(-5)\right] = 5.40 + 18.2i \text{ (3 s.f.)}.$$

Part (b). We have

$$z = 6\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right) = 5.54 - 2.30i \text{ (3 s.f.)}.$$

Problem 5. Given that $z = \sqrt{3} - \mathrm{i}$, find the exact modulus and argument of z. Hence, find the exact modulus and argument of $1/z^2$ and z^{10}.

Solution.



We have $r^2 = (\sqrt{3})^2 + (-1)^2 \implies r = 2$ and $\tan\theta = -1/\sqrt{3} \implies \theta = -\pi/6$. Hence, $|z| = 2$ and $\arg z = -\pi/6$.

Note that $\left|1/z^2\right| = |z|^{-2} = 1/4$. Also, $\arg\left(1/z^2\right) = -2\arg z = \pi/3$.

Note that$\left|z^{10}\right| = |z|^1 0 = 1024$. Also, $\arg z^{10} = 10\arg z = -5\pi/3 \equiv \pi/3$.

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Problem 6. If $\arg(z - 1/2) = \pi/5$, determine $\arg(2z - 1)$.

Solution.

$$\arg(2z-1) = \arg\left(\frac{1}{2}\left[z - \frac{1}{2}\right]\right) = \arg\left(z - \frac{1}{2}\right) = \frac{\pi}{5}.$$

* * * * *

Problem 7. In an Argand diagram, points P and Q represent the complex numbers $z = 1 + \mathrm{i}$ and $w = 1 + 2\mathrm{i}$ respectively, and O is the origin.

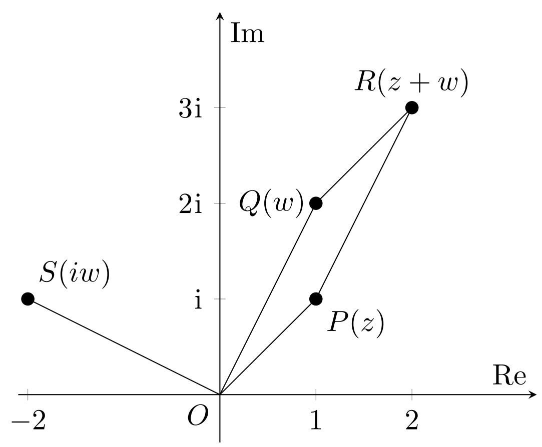
(a) Mark on the Argand diagram the points P and Q, and the points R and S which represent $z + w$ and iw respectively.

(b) What is the geometrical shape of $OPRQ$?

(c) State the angle SOP.

Solution.

Part (a).



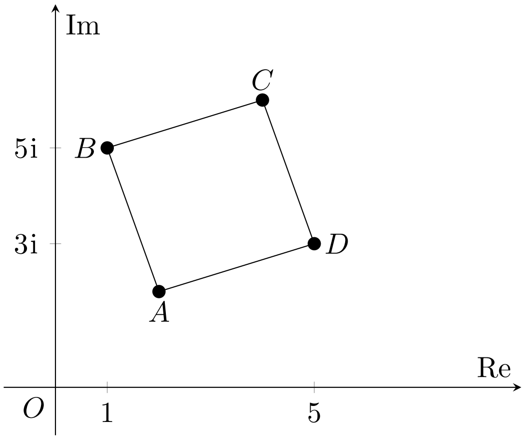
Part (b). $OPRQ$ is a parallelogram.

Part (c). $\angle SOP = \pi/2$.

* * * * *

Problem 8. B and D are points in the Argand diagram representing the complex numbers $1+5\mathrm{i}$ and $5+3\mathrm{i}$ respectively. Given that BD is a diagonal of the square $ABCD$, calculate the complex numbers represented by A and C.

Solution.



Let $A(x+\mathrm{i}y)$. Since $AB \perp AD$, we have $b-a = \mathrm{i}(d-a)$.

$$b-a=\mathrm{i}(d-a) \implies (1+5\mathrm{i})-(x+\mathrm{i}y) = i\left[(5+3\mathrm{i})-(x+\mathrm{i}y)\right]$$
$$\implies (1-x)+(5-y)\mathrm{i} = (-3+y)+(5-x)\mathrm{i} \implies (x+y)+(y-x)\mathrm{i} = 4.$$

Comparing real and imaginary parts, we obtain $x=y=2$. Hence, $A(2+2\mathrm{i})$.

Let $C(u+\mathrm{i}v)$. Since $CB \perp CD$, we have $d-c=\mathrm{i}(b-c)$.

$$d-c=\mathrm{i}(b-c) \implies (5+3\mathrm{i})-(u+\mathrm{i}v) = \mathrm{i}\left[(1+5\mathrm{i})-(u+\mathrm{i}v)\right]$$
$$\implies (5-u)+(3-v)\mathrm{i} = (-5+v)+(1-u)\mathrm{i} \implies (u+v)+(v-u)\mathrm{i} = 10+2\mathrm{i}.$$

Comparing real and imaginary parts, we obtain $u=4$ and $v=6$. Hence, $C(4+6\mathrm{i})$.

* * * * *

Problem 9.

(a) Given that $u = 2\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$ and $w = 4\left(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}\right)$, find the modulus and argument of u^*/w^3 in exact form.

(b) Let z be the complex number $-1+\mathrm{i}\sqrt{3}$. Find the value of the real number a such that $\arg\left(z^2+az\right) = -\pi/2$.

Solution.

Part (a). Note that $|u|=2$, $\arg u = \pi/6$, $|w|=4$ and $\arg w = -\pi/3$. Hence,

$$\left|\frac{u^*}{w^3}\right| = \frac{|u^*|}{|w^3|} = \frac{|u|}{|w|^3} = \frac{2}{4^3} = \frac{1}{32}$$

and

$$\arg\frac{u^*}{w^3} = \arg u^* - \arg w^3 = -\arg u - 3\arg w = -\frac{\pi}{6} - 3\left(-\frac{\pi}{3}\right) = \frac{5}{6}\pi.$$

Part (b). Since $\arg(z^2+az) = -\pi/2$, we have that z^2+az is purely imaginary, with a negative imaginary part. Since

$$z^2 + az = \left(-1+\mathrm{i}\sqrt{3}\right)^2 + a\left(-1+\mathrm{i}\sqrt{3}\right) = \left(-2-2\sqrt{3}\mathrm{i}\right) + a\left(-1+\mathrm{i}\sqrt{3}\right).$$

Hence,

$$\operatorname{Re}(z^2+az) = 0 \implies -2-a=0 \implies a=-2.$$

* * * * *

Problem 10. The complex number w has modulus r and argument θ, where $0<\theta<\pi/2$, and w^* denotes the conjugate of w. State the modulus and argument of p, where $p = w/w^*$. Given that p^5 is real and positive, find the possible values of θ.

Solution. Clearly, $|p| = 1$ and $\arg p = 2\theta$.

Since p^5 is real and positive, we have $\arg p^5 = 2\pi n$, where $n \in \mathbb{Z}$. Thus, $\arg p = 2\pi n/5 = 2\theta \implies \theta = \pi n/5$. Since $0<\theta<\pi/2$, the possible values of θ are $\pi/5$ and $2\pi/5$.

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Problem 11. The complex number w has modulus $\sqrt{2}$ and argument $-3\pi/4$, and the complex number z has modulus 2 and argument $-\pi/3$. Find the modulus and argument of wz, giving each answer exactly.

By first expressing w and z in the form $x+\mathrm{i}y$, find the exact real and imaginary parts of wz.

Hence, show that $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Solution. Note that

$$|wz| = |w|\,|z| = 2\sqrt{2}$$

and

$$\arg(wz) = \arg w + \arg z = -\frac{3}{4}\pi - \frac{1}{3}\pi = -\frac{13}{12}\pi \equiv \frac{11}{12}\pi.$$

Also,

$$w = \sqrt{2}\left[\cos\left(-\frac{3}{4}\pi\right) + \mathrm{i}\sin\left(-\frac{3}{4}\pi\right)\right] = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right) = -1-\mathrm{i}$$

and

$$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right] = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) = 1-\sqrt{3}\mathrm{i}.$$

Hence,

$$wz = (-1-\mathrm{i})(1-\sqrt{3}\mathrm{i}) = (-1+\sqrt{3}-\mathrm{i}-\sqrt{3}) = (-1-\sqrt{3}) + (\sqrt{3}-1)\mathrm{i},$$

whence $\operatorname{Re}(wz) = -1-\sqrt{3}$ and $\operatorname{Im}(wz) = \sqrt{3}-1$.

From the first part, we have that $wz = 2\sqrt{2}\left[\cos\left(\frac{11}{12}\pi\right) + \mathrm{i}\sin\left(\frac{11}{12}\pi\right)\right]$. Thus, $\operatorname{Im}(wz) = 2\sqrt{2}\sin\left(\frac{11}{12}\pi\right) = 2\sqrt{2}\sin\frac{\pi}{12}$. Equating the result for $\operatorname{Im}(wz)$ found in the second part, we have

$$2\sqrt{2}\sin\frac{\pi}{12} = \sqrt{3}-1 \implies \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Problem 12. Given that $\frac{5+z}{5-z} = e^{i\theta}$, show that z can be written as $5i\tan\frac{\theta}{2}$.

Solution. Note that

$$\frac{5+z}{5-z} = e^{i\theta} \implies 5+z = e^{i\theta}(5-z) \implies z + e^{i\theta}z = 5e^{i\theta} - 5 \implies z = 5\left(\frac{e^{i\theta}-1}{e^{i\theta}+1}\right).$$

Hence,

$$z = 5\left(\frac{e^{i\theta}-1}{e^{i\theta}+1}\right) = 5\left(\frac{e^{i\theta/2}-e^{-i\theta/2}}{e^{i\theta/2}+e^{-i\theta/2}}\right) = 5\left(\frac{2i\sin(\theta/2)}{2\cos(\theta/2)}\right) = 5i\tan\frac{\theta}{2}.$$

* * * * *

Problem 13. The polynomial $P(z)$ has real coefficients. The equation $P(z) = 0$ has a root $re^{i\theta}$, where $r > 0$ and $0 < \theta < \pi$.

(a) Write down a second root in terms of r and θ, and hence show that a quadratic factor of $P(z)$ is $z^2 - 2rz\cos\theta + r^2$.

(b) Given that 3 roots of the equation $z^6 = -64$ are $2e^{i\frac{\pi}{6}}$, $2e^{i\frac{\pi}{2}}$ and $2e^{-i\frac{5\pi}{6}}$, express $z^6 + 64$ as a product of three quadratic factors with real coefficients, giving each factor in non-trigonometric form.

(c) Represent all roots of $z^6 = -64$ on an Argand diagram and interpret the geometrical shape formed by joining the roots.

Solution.

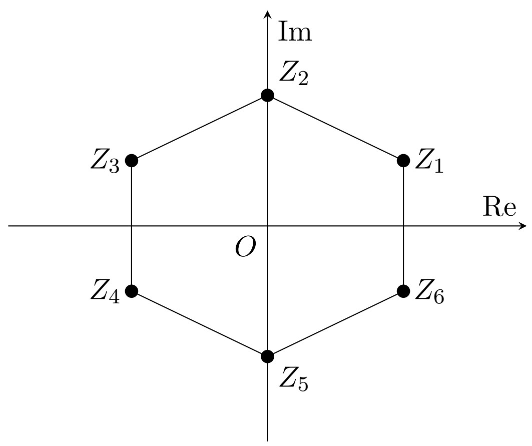
Part (a). Since $P(z)$ has real coefficients, by the conjugate root theorem, $\left(re^{i\theta}\right)^* = re^{-i\theta}$ is also a root of $P(z)$. By the factor theorem, a quadratic factor of $P(z)$ is

$$(z - re^{i\theta})(z - re^{-i\theta}) = z^2 - rz(e^{i\theta} + e^{-i\theta}) + r^2e^{i\theta}e^{-i\theta} = z^2 - 2rz\cos\theta + r^2.$$

Part (b). Let $r_1 = r_2 = r_3 = 2$ and $\theta_1 = \pi/6$, $\theta_2 = \pi/2$ and $\theta_3 = -5\pi/6$.

$$\begin{aligned} z^6 + 64 &= \left(z^2 - 2r_1z\cos\theta_1 + r_1^2\right)\left(z^2 - 2r_2z\cos\theta_2 + r_2^2\right)\left(z^2 - 2r_3z\cos\theta_3 + r_3^2\right) \\ &= \left(z^2 - 4z\cos\left(\frac{\pi}{6}\right) + 4\right)\left(z^2 - 4z\cos\left(\frac{\pi}{2}\right) + 4\right)\left(z^2 - 4z\cos\left(-\frac{5}{6}\pi\right) + 4\right) \\ &= \left(z^2 - 2\sqrt{3}z + 4\right)\left(z^2 + 4\right)\left(z^2 + 2\sqrt{3}z + 4\right) \end{aligned}$$

Part (c).



The geometrical shape formed is a regular hexagon.

Self-Practice A10.2

Problem 1. The complex numbers $2e^{i\pi/12}$ and $2e^{i(5\pi/12)}$ are represented by the points A and B respectively in an Argand diagram with origin O. Show that the triangle OAB is equilateral.

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Problem 2. The complex numbers z and w are such that

$$|z| = 2, \quad \arg(z) = -\frac{2\pi}{3}, \qquad \text{and} \qquad |w| = 5, \quad \arg(w) = \frac{3\pi}{4}.$$

(a) Find the exact values of the modulus and argument of w/z^2. Hence, represent z, w and w/z^2 clearly in an Argand diagram.

(b) Express w/z^2 in the exponential form. Hence, or otherwise, find the smallest positive integer n such that $(w/z^2)^n$ is a real number.

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Problem 3. Express $\frac{\cot\theta + i}{\cot\theta - i}$ in the exponential form.

* * * * *

Problem 4. Do not use a calculator in answering this question.
Two complex numbers are $z_1 = 2\left(\cos\frac{\pi}{18} - i\sin\frac{\pi}{18}\right)$ and $z_2 = 2i$.

(a) Show that

$$\frac{z_1^2}{z_1^*} + z_2 = \sqrt{3} = i.$$

(b) A third complex number, z_3, is such that

$$\left(\frac{z_1^2}{z_1^*} + z_2\right) z_3 \in \mathbb{R} \qquad \text{and} \qquad \left|\left(\frac{z_1^2}{z_1^*} + z_2\right) z_3\right| = \frac{2}{3}.$$

Find the possible values of z_3 in the form of $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.

* * * * *

Problem 5. Do not use a calculator in answering this question.
The complex numbers z and w satisfy the following equations:

$$w - z = 1 - \sqrt{3}, \qquad iz + w = \left(\sqrt{3} + 1\right) i.$$

Find w in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give r and θ in exact form.

Hence, find the three smallest positive whole number values of n for which $(iw)^n$ is an imaginary number.

* * * * *

Problem 6 (🌶). It is given that $z = \cos\theta + i\sin\theta$, where $0 < \theta < \pi/2$.

(a) Show that $e^{i(\theta - \pi/2)} = \sin\theta - i\cos\theta$.

(b) Hence, or otherwise, show that $\arg\left(1 - z^2\right) = \theta - \pi/2$ and find the modulus of $1 - z^2$.

(c) Hence, represent the complex number $1 - z^2$ on an Argand diagram.

(d) Given that $\frac{z^*}{z^3(1 - z^2)}$ is real, find the possible values of θ.

Assignment A10.2

Problem 1. On an Argand diagram, mark and label clearly the points P and Q representing the complex numbers p and q respectively, where

$$p = \cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}, \qquad q = 2\cos\frac{\pi}{4} + 2\mathrm{i}\sin\frac{\pi}{4}.$$

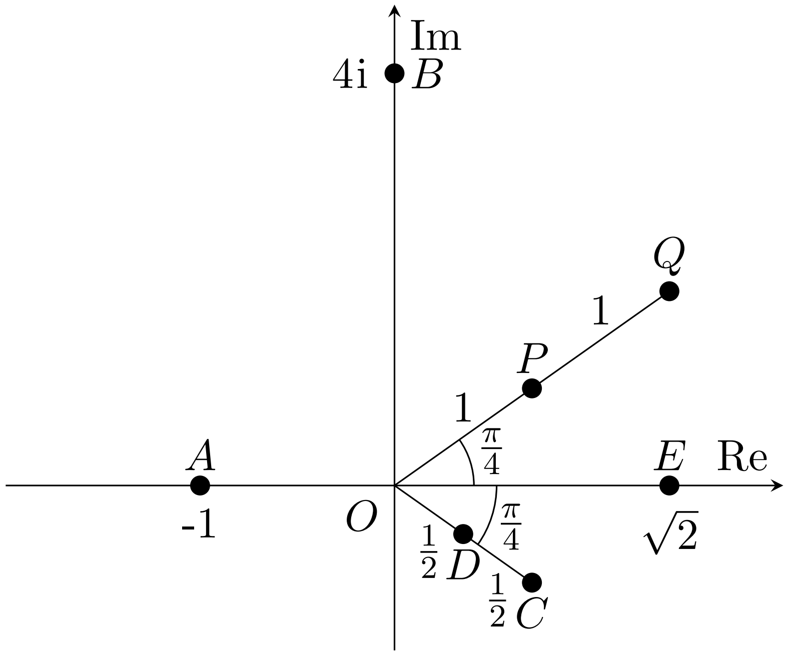
Find the moduli and arguments of the complex numbers a, b, c, d and e, where $a = p^4$, $b = q^2$, $c = -ip$, $d = 1/q$, $e = p + p^*$.

On your Argand diagram, mark and label the points A, B, C, D and E representing these complex numbers.

Find the area of triangle COQ.

Find the modulus and argument of $p^{13/3}q^{45/2}$.

Solution.



Note that $p = \mathrm{e}^{\mathrm{i}\pi/4}$ and $q = 2\mathrm{e}^{\mathrm{i}\pi/4}$.

$$a = p^4 = \left(\mathrm{e}^{\mathrm{i}\pi/4}\right)^4 = \mathrm{e}^{\mathrm{i}\pi}, \quad b = q^2 = \left(2\mathrm{e}^{\mathrm{i}\pi/4}\right)^2 = 4\mathrm{e}^{\mathrm{i}\pi/2}$$

$$c = -\mathrm{i}p = \mathrm{e}^{-\mathrm{i}\pi/2}\mathrm{e}^{\mathrm{i}\pi/4} = \mathrm{e}^{-\mathrm{i}\pi/4}, \quad d = \frac{1}{q} = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi/4}$$

$$e = p + p^* = 2\,\mathrm{Re}\,p = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}$$

z	$\lvert z\rvert$	$\arg z$
a	1	π
b	4	$\pi/2$
c	1	$-\pi/4$
d	$1/2$	$-\pi/4$
e	$\sqrt{2}$	0

Since $\angle COQ = \pi/2$, we have $[\triangle COQ] = \frac{1}{2}(2)(1) = 1$ units^2.

We have

$$p^{\frac{13}{3}}q^{\frac{45}{2}} = \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^{\frac{13}{3}}\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^{\frac{45}{2}} = 2^{\frac{45}{2}}\mathrm{e}^{\mathrm{i}\frac{161\pi}{24}} = 2^{\frac{45}{2}}\mathrm{e}^{\mathrm{i}\frac{17\pi}{24}}.$$

Hence, $\left|p^{13/3}q^{45/2}\right| = \mathrm{e}^{45/2}$ and $\arg\left(p^{13/3}q^{45/2}\right) = \frac{17}{24}\pi$.

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Problem 2. The complex number q is given by $q = \frac{\mathrm{e}^{\mathrm{i}2\theta}}{1-\mathrm{e}^{\mathrm{i}2\theta}}$, where $0 < \theta < 2\pi$. In either order,

(a) find the real part of q,

(b) show that the imaginary part of q is $\frac{1}{2}\cot\theta$.

Solution. We have

$$q = \frac{\mathrm{e}^{\mathrm{i}2\theta}}{1-\mathrm{e}^{\mathrm{i}2\theta}} = \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}} = \frac{\cos\theta + \mathrm{i}\sin\theta}{-2\mathrm{i}\sin\theta} = -\frac{1}{2} - \frac{1}{2\mathrm{i}}\cot\theta = -\frac{1}{2} + \frac{\mathrm{i}}{2}\cot\theta.$$

Hence, $\operatorname{Re} q = -\frac{1}{2}$ and $\operatorname{Im} q = \frac{1}{2}\cot\theta$.

* * * * *

Problem 3. The complex numbers z and w are such that $z = 4\left(\cos\frac{3}{4}\pi + \mathrm{i}\sin\frac{3}{4}\pi\right)$ and $w = 1 - \mathrm{i}\sqrt{3}$. z^* denotes the conjugate of z.

(a) Find the modulus r and the argument θ of w^2/z^*, where $r > 0$ and $-\pi < \theta < \pi$.

(b) Given that $\left(w^2/z^*\right)^n$ is purely imaginary, find the set of values that n can take.

Solution.

Part (a). Note that $z = 4\mathrm{e}^{\mathrm{i}3\pi/4}$ and $w = 2\left(\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right) = 2\mathrm{e}^{-\mathrm{i}\pi/3}$. Hence,

$$\frac{w^2}{z^*} = \frac{\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)^2}{4\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}} = \frac{4\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}}{4\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{12}}.$$

Thus, $r = 1$ and $\theta = \pi/12$.

Part (b). Note that $\left(w^2/z^*\right)^n = \left(\mathrm{e}^{\mathrm{i}\pi/12}\right)^n = \mathrm{e}^{\mathrm{i}n\pi/12}$. Since $\left(w^2/z^*\right)^n$ is purely imaginary, we have $\arg\left(w^2/z^*\right)^n = \pi/2 + \pi k$, where $k \in \mathbb{Z}$. Thus, $n\pi/12 = \pi/2 + \pi k$, whence $n = 6 + 12k$. Hence, $\{n \in \mathbb{Z} : n = 6 + 12k,\ k \in Z\}$.

* * * * *

Problem 4. The complex number w has modulus $\sqrt{2}$ and argument $\pi/4$ and the complex number z has modulus $\sqrt{2}$ and argument $5\pi/6$.

(a) By first expressing w and z in the form $x + \mathrm{i}y$, find the exact real and imaginary parts of $w + z$.

(b) On the same Argand diagram, sketch the points P, Q, R representing the complex numbers z, w, and $z+w$ respectively. State the geometrical shape of the quadrilateral $OPRQ$.

(c) Referring the Argand diagram in part (b), find $\arg(w + z)$ and show that $\tan\frac{11}{24}\pi = \frac{a+\sqrt{2}}{\sqrt{6}+b}$, where a and b are constants to be determined.

Solution.

Part (a). Note that

$$w = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i$$

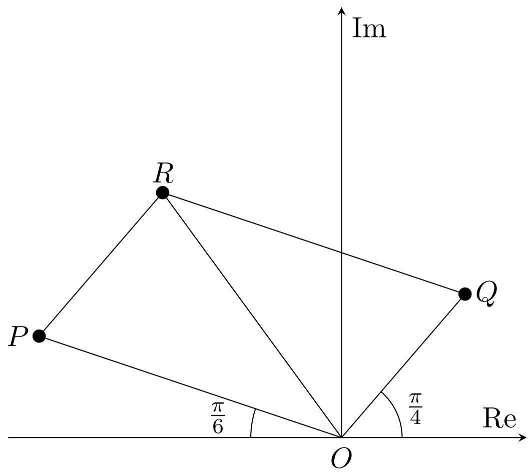
and

$$z = \sqrt{2}e^{i5\pi/6} = \sqrt{2}\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\frac{\sqrt{3}}{\sqrt{2}} + i\frac{1}{\sqrt{2}}.$$

Hence,

$$w + z = (1 + i) + \left(-\frac{\sqrt{3}}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \left(1 - \frac{\sqrt{3}}{\sqrt{2}}\right) + i\left(1 + \frac{1}{\sqrt{2}}\right).$$

Part (b).



$OPRQ$ is a rhombus.

Part (c). Note that $\angle POQ = \pi - \frac{\pi}{6} - \frac{\pi}{4} = \frac{7}{12}\pi$. Since $|z| = |w|$, we have $OP = OQ$, whence $\angle ROQ = \frac{1}{2} \cdot \frac{7}{12}\pi = \frac{7}{24}\pi$. Hence, $\arg(w + z) = \frac{\pi}{4} + \frac{7}{24}\pi = \frac{13}{24}\pi$. Thus,

$$\tan\left(\frac{13}{24}\pi\right) = \frac{1 + 1/\sqrt{2}}{1 - \sqrt{3}/\sqrt{2}} = \frac{\sqrt{2} + 1}{\sqrt{2} - \sqrt{3}} = \frac{2 + \sqrt{2}}{2 - \sqrt{6}}$$

However, $\tan\left(\frac{13}{24}\pi\right) = -\tan\left(\pi - \frac{13}{24}\right) = -\tan\left(\frac{11}{24}\pi\right)$. Hence,

$$\tan\left(\frac{11}{24}\pi\right) = -\frac{2 + \sqrt{2}}{2 - \sqrt{6}} = \frac{2 + \sqrt{2}}{\sqrt{6} - 2},$$

whence $a = 2$ and $b = -2$.

* * * * *

Problem 5. The complex number z is given by $z = 2(\cos\beta + i\sin\beta)$ where $0 < \beta < \frac{\pi}{2}$.

(a) Show that $\frac{z}{4-z^2} = (k\csc\beta)i$, where k is positive real constant to be determined.

(b) State the argument of $\frac{z}{4-z^2}$, giving your reasons clearly.

(c) Given the complex number $w = -\sqrt{3} + i$, find the three smallest positive integer values of n such that $\left(\frac{z}{4-z^2}\right)(w^*)^n$ is a real number.

Solution.

Part (a). Observe that $z = 2(\cos\beta + \mathrm{i}\sin\beta) = 2\mathrm{e}^{\mathrm{i}\beta}$. Hence,

$$\frac{z}{4-z^2} = \frac{2\mathrm{e}^{\mathrm{i}\beta}}{4-4\mathrm{e}^{\mathrm{i}2\beta}} = \frac{1}{2}\left(\frac{1}{\mathrm{e}^{-\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\beta}}\right) = \frac{1}{2}\left(\frac{1}{-2\mathrm{i}\sin\beta}\right) = \left(\frac{1}{4}\csc\beta\right)\mathrm{i},$$

thus $k = 1/4$.

Part (b). Since $0 < \beta < \pi/2$, we know that $\csc\beta > 0$. Hence, $\operatorname{Im}\left(\frac{z}{4-z^2}\right) > 0$. Furthermore, $\operatorname{Re}\left(\frac{z}{4-z^2}\right) = 0$. Thus, $\arg\left(\frac{z}{4-z^2}\right) = \pi/2$.

Part (c). Note that $w = -\sqrt{3} + \mathrm{i} = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) = 2\mathrm{e}^{-\mathrm{i}5\pi/6}$. Hence,

$$\arg\left(\left(\frac{z}{4-z^2}\right)(w^*)^n\right) = \frac{\pi}{2} + n\left(-\frac{5\pi}{6}\right) = \pi\left(\frac{1}{2} - \frac{5n}{6}\right).$$

For $\left(\frac{z}{4-z^2}\right)(w^*)^n$ to be a real number, we require $\arg\left(\left(\frac{z}{4-z^2}\right)(w^*)^n\right) = \pi k$, where $k \in \mathbb{Z}$. Hence,

$$\pi\left(\frac{1}{2} - \frac{5}{6}n\right) = \pi k \implies \frac{1}{2} - \frac{5}{6}n = k \implies 3 - 5n = 6k \implies n \equiv 3 \pmod 6.$$

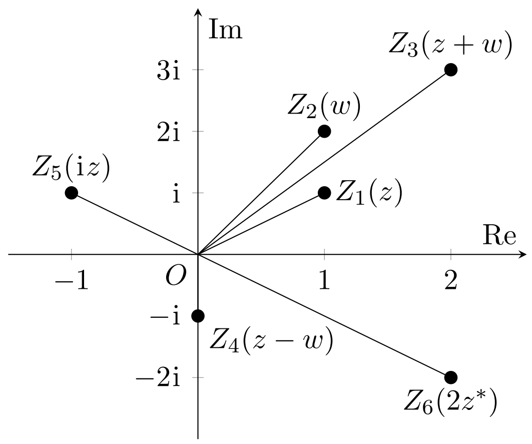
Hence, the three smallest possible values of n are 3, 9 and 15.

A10.3 Complex Numbers - Geometrical Effects and De Moivre's Theorem

Tutorial A10.3

Problem 1. Given that $z = 1 + \mathrm{i}$ and $w = 1 + 2\mathrm{i}$, mark on an Argand diagram, the positions representing: z, w, $z + w$, $z - w$, $\mathrm{i}z$ and $2z^*$.

Solution.



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Problem 2.

(a) Write down the exact values of the modulus and the argument of the complex number $\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$.

(b) The complex numbers z and w satisfy the equation

$$z^2 - zw + w^2 = 0.$$

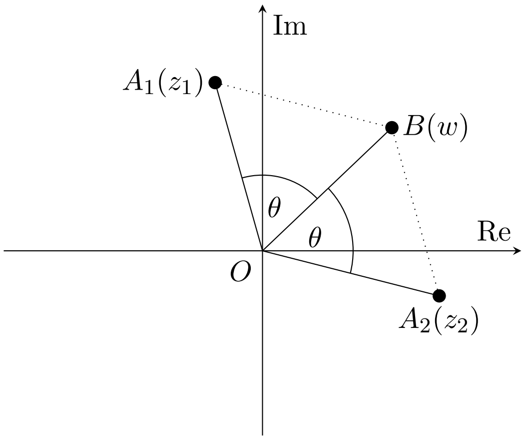
Find z in terms of w. In an Argand diagram, the points O, A and B represent the complex numbers 0, z and w respectively. Show that $\triangle OAB$ is an equilateral triangle.

Solution.

Part (a). We have $r^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \implies r = 1$ and $\tan\theta = \frac{\sqrt{3}/2}{1/2} \implies \theta = \frac{\pi}{3}$. Hence, $\left|\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right| = 1$ and $\arg\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = \frac{\pi}{3}$.

Part (b). From the quadratic formula, we have

$$z = \frac{w \pm \sqrt{w^2 - 4w^2}}{2} = w\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}\right).$$



Since $\left|\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}\right| = 1$, we have that $OB = OA_1 = OA_2$. Further, since $\arg\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}\right) = \pm\pi/3$, we know $\angle A_1OB = \angle A_2OB = \pi/3$, whence $\triangle A_1OB$ and $\triangle A_2OB$ are both equilateral.

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Problem 3. Find the exact roots of the equations

(a) $z^3 = 1$

(b) $(z-1)^4 = -16$

in the form $x + \mathrm{i}y$.

Solution.

Part (a). Note that

$$z^3 = 1 = \mathrm{e}^{\mathrm{i}2\pi n} \implies z = \mathrm{e}^{\mathrm{i}2\pi n/3} = \cos\frac{2\pi n}{3} + \mathrm{i}\sin\frac{2\pi n}{3},$$

for $n \in \mathbb{Z}$. Evaluating z in the $n = 0, 1, 2$ cases, we see that the roots of $z^3 = 1$ are

$$z = 1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}.$$

Part (b). Note that $(z-1)^4 = -16 = 16\mathrm{e}^{\mathrm{i}\pi(2n+1)}$. Hence,

$$z = 1 + 2\mathrm{e}^{\mathrm{i}\pi(2n+1)/4} = 1 + 2\left[\cos\left(\frac{2n+1}{4}\pi\right) + \mathrm{i}\sin\left(\frac{2n+1}{4}\pi\right)\right],$$

where $n \in \mathbb{Z}$. Evaluating z in the $n = 0, 1, 2, 3$ cases, we see that the roots of $(z-1)^4 = -16$ are

$$z = (1+\sqrt{2}) + \mathrm{i}\sqrt{2},\ (1-\sqrt{2}) + \mathrm{i}\sqrt{2},\ (1-\sqrt{2}) - \mathrm{i}\sqrt{2},\ (1+\sqrt{2}) - \mathrm{i}\sqrt{2}.$$

* * * * *

Problem 4.

(a) Write down the 5 roots of the equation $z^5 - 1 = 0$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Show that the roots of the equation $(5+z)^5 - (5-z)^5 = 0$ can be written in the form $5\mathrm{i}\tan\frac{k\pi}{5}$, where $k = 0, \pm 1, \pm 2$.

Solution.

Part (a). Note that

$$z^5 = 1 = \mathrm{e}^{\mathrm{i}2\pi n} \implies z = \mathrm{e}^{\mathrm{i}2\pi n/5}.$$

Since $-\pi < \theta \leq \pi$, we have

$$z = \mathrm{e}^{-\mathrm{i}4\pi/5},\ \mathrm{e}^{-\mathrm{i}2\pi/5},\ 1,\ \mathrm{e}^{\mathrm{i}2\pi/5},\ \mathrm{e}^{\mathrm{i}4\pi/5}.$$

Part (b). Note that

$$(5+z)^5 - (5-z)^5 = 0 \implies \left(\frac{5+z}{5-z}\right)^5 - 1 = 0 \implies \frac{5+z}{5-z} = \mathrm{e}^{\mathrm{i}2k\pi/5}.$$

Solving for z, we get

$$z = 5\left(\frac{\mathrm{e}^{\mathrm{i}2k\pi/5} - 1}{\mathrm{e}^{\mathrm{i}2k\pi/5} + 1}\right) = 5\left(\frac{\mathrm{e}^{\mathrm{i}k\pi/5} - \mathrm{e}^{-\mathrm{i}k\pi/5}}{\mathrm{e}^{\mathrm{i}k\pi/5} + \mathrm{e}^{-\mathrm{i}k\pi/5}}\right) = 5\left[\frac{2\mathrm{i}\sin(k\pi/5)}{2\cos(k\pi/5)}\right] = 5\mathrm{i}\tan\frac{k\pi}{5}.$$

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Problem 5. De Moivre's theorem for a positive integral exponent states that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

Use de Moivre's theorem to show that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta.$$

Hence, obtain the roots of the equation

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

in the form $\cos q\pi$, where q is a rational number.

Solution. Taking $n = 7$, we have $\cos 7\theta + \mathrm{i}\sin 7\theta = (\cos\theta + \mathrm{i}\sin\theta)^7$, whence $\cos 7\theta = \operatorname{Re}(\cos\theta + \mathrm{i}\sin\theta)^7$. Let $c = \cos\theta$ and $s = \sin\theta$. By the binomial theorem,

$$\cos 7\theta = \operatorname{Re}(c + \mathrm{i}s)^7 = \operatorname{Re}\sum_{k=0}^{7}\binom{7}{k}\mathrm{i}^k s^k c^{7-k}.$$

Note that $\operatorname{Re} i^k$ is given by

$$\operatorname{Re} i^k = \begin{cases} 0, & k = 1, 3 \pmod 4 \\ 1, & k = 0 \pmod 4 \\ -1, & k = 2 \pmod 4 \end{cases}$$

We hence have

$$\begin{aligned}\cos 7\theta &= c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6 = c^7 - 21c^5\left(1-c^2\right) + 35c^3\left(1-c^2\right)^2 - 7c\left(1-c^2\right)^3 \\ &= 64c^7 - 112c^5 + 56c^3 - 7c = 64\cos^7\theta - 112\cos^5 + 56\cos^3\theta - 7\cos\theta.\end{aligned}$$

Observe that we can manipulate the given equation into

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0 \implies 64x^7 - 112x^5 + 56x^3 - 7x = -\frac{1}{2}.$$

Under the substitution $x = \cos\theta$, we see that

$$\cos 7\theta = -\frac{1}{2} \implies 7\theta = \frac{2}{3}\pi + 2\pi n \implies \theta = \frac{2\pi}{21}(3n+1),$$

where $n \in \mathbb{Z}$. Taking $0 \le n < 7$,

$$\begin{aligned} x &= \cos\frac{2\pi}{21}, \cos\frac{8\pi}{21}, \cos\frac{14\pi}{21}, \cos\frac{20\pi}{21}, \cos\frac{26\pi}{21}, \cos\frac{32\pi}{21}, \cos\frac{38\pi}{21} \\ &= \cos\frac{2\pi}{21}, \cos\frac{4\pi}{21}, \cos\frac{8\pi}{21}, \cos\frac{10\pi}{21}, \cos\frac{14\pi}{21}, \cos\frac{16\pi}{21}, \cos\frac{20\pi}{21}. \end{aligned}$$

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Problem 6. By considering $\sum_{n=1}^{N} z^{2n-1}$, where $z = \mathrm{e}^{\mathrm{i}\theta}$, or by any method, show that

$$\sum_{n=1}^{N} \sin(2n-1)\theta = \frac{\sin^2 N\theta}{\sin\theta},$$

provided $\sin\theta \neq 0$.

Solution. Observe that

$$\sum_{n=1}^{N} \sin(2n-1)\theta = \operatorname{Im}\sum_{n=1}^{N}\left[\cos(2n-1)\theta + \mathrm{i}\sin(2n-1)\theta\right] = \operatorname{Im}\sum_{n=1}^{N} z^{2n-1}.$$

Since

$$\begin{aligned} \sum_{n=1}^{N} z^{2n-1} &= \frac{1}{z}\sum_{n=1}^{N}\left(z^2\right)^n = \frac{1}{z}\left(\frac{z^2\left[\left(z^2\right)^N - 1\right]}{z^2 - 1}\right) = \frac{z^{2N}-1}{z - z^{-1}} \\ &= z^N\left(\frac{z^N - z^{-N}}{z - z^{-1}}\right) = z^N\left(\frac{2\mathrm{i}\sin N\theta}{2\mathrm{i}\sin\theta}\right) = z^N\left(\frac{\sin N\theta}{\sin\theta}\right), \end{aligned}$$

we have

$$\sum_{n=1}^{N}\sin(2n-1)\theta = \left(\frac{\sin N\theta}{\sin\theta}\right)\operatorname{Im}\left(z^N\right) = \left(\frac{\sin N\theta}{\sin\theta}\right)\sin N\theta = \frac{\sin^2 N\theta}{\sin\theta}.$$

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Problem 7. By considering the series $\sum_{n=0}^{N}\left(\mathrm{e}^{2\mathrm{i}\theta}\right)^n$, show that, provided $\sin\theta \neq 0$,

$$\sum_{n=0}^{N}\cos 2n\theta = \frac{\sin(N+1)\theta\cos N\theta}{\sin\theta}$$

and deduce that

$$\sum_{n=0}^{N}\sin^2 n\theta = \frac{N}{2} + \frac{1}{2} - \frac{\sin(N+1)\theta\cos N\theta}{2\sin\theta}.$$

Solution. Let $z = \mathrm{e}^{\mathrm{i}\theta}$. Then

$$\sum_{n=0}^{N}\cos 2n\theta = \operatorname{Re}\sum_{n=0}^{N}\left(\cos 2n\theta + \mathrm{i}\sin 2n\theta\right) = \operatorname{Re}\sum_{n=0}^{N}\mathrm{e}^{\mathrm{i}2n\theta} = \operatorname{Re}\sum_{n=0}^{N}\left(z^2\right)^n.$$

Observe that

$$\sum_{n=0}^{N}\left(z^2\right)^n=\frac{\left(z^2\right)^{N+1}-1}{z^2-1}=\frac{z^{N+1}}{z}\left(\frac{z^{N+1}-z^{-(N+1)}}{z-z^{-1}}\right)=z^N\left(\frac{\sin(N+1)\theta}{\sin\theta}\right).$$

Hence,

$$\sum_{n=0}^{N}\cos 2n\theta=\left(\frac{\sin(N+1)\theta}{\sin\theta}\right)\operatorname{Re}\left(z^N\right)=\frac{\sin(N+1)\theta\cos N\theta}{\sin\theta}.$$

Recall that $\cos 2n\theta=1-2\sin^2 n\theta \implies \sin^2 n\theta=\frac{1}{2}(1-2\cos 2n\theta)$. Thus,

$$\sum_{n=0}^{N}\sin^2 n\theta=\frac{1}{2}\sum_{n=0}^{N}(1-\cos 2n\theta)=\frac{N+1}{2}-\frac{\sin(N+1)\theta\cos N\theta}{2\sin\theta}.$$

* * * * *

Problem 8. Given that $z=\mathrm{e}^{\mathrm{i}\theta}$, show that $z^k+1/z^k=2\cos k\theta$, $k\in\mathbb{Z}$.

Hence, show that $\cos^8\theta=\frac{1}{128}\left(\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35\right)$.

Find, correct to three decimal places, the values of θ such that $0<\theta<\frac{1}{2}\pi$ and $\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+1=0$.

Solution. Note that

$$\begin{aligned}z^k+\frac{1}{z^k}&=z^k+z^{-k}=\left(\mathrm{e}^{\mathrm{i}\theta}\right)^k+\left(\mathrm{e}^{\mathrm{i}\theta}\right)^{-k}=\mathrm{e}^{\mathrm{i}k\theta}+\mathrm{e}^{-\mathrm{i}k\theta}\\&=[\cos(k\theta)+\mathrm{i}\sin(k\theta)]+[\cos(-k\theta)+\mathrm{i}\sin(-k\theta)]=2\cos(k\theta).\end{aligned}$$

Observe that

$$\begin{aligned}\cos^8\theta&=\frac{1}{256}(2\cos\theta)^8=\frac{1}{256}(z+z^{-1})^8=\frac{1}{256}z^{-8}\left(z^2+1\right)^8\\&=\frac{1}{256}\left(z^{-8}+8z^{-6}+28z^{-4}+56z^{-2}+70+56z^2+28z^4+8z^6+z^8\right)\\&=\frac{1}{128}\left[\left(\frac{z^8+z^{-8}}{2}\right)+8\left(\frac{z^6+z^{-6}}{2}\right)+28\left(\frac{z^4+z^{-4}}{2}\right)+56\left(\frac{z^2+z^{-2}}{2}\right)+\frac{70}{2}\right]\\&=\frac{1}{128}\left(\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35\right).\end{aligned}$$

Note that we rewrite the equation as

$$\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35=128\cos^8\theta=34.$$

Thus,

$$\cos\theta=\sqrt[8]{\frac{34}{128}}\implies\theta=0.560\text{ (3 s.f.)}.$$

Self-Practice A10.3

Problem 1. Express $\frac{\cot\theta+\mathrm{i}}{\cot\theta-\mathrm{i}}$ in the exponential form. Hence, show that one of the roots of the equation

$$z^4 = \frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$$

is $\mathrm{e}^{\mathrm{i}\pi/12}$, and find three more roots in the exponential form.

* * * * *

Problem 2. Find the cube roots of the complex number $1+\mathrm{i}\sqrt{3}$. Give your answers exactly, in the form $r\mathrm{e}^{\mathrm{i}\theta}$. Hence, solve the equation $z^6-2z^3+4=0$. Give your answers exactly, in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

* * * * *

Problem 3. Express $8\left(\sqrt{3}-\mathrm{i}\right)$ in the form $r\left(\cos\theta+\mathrm{i}\sin\theta\right)$, where $r>0$ and $-\pi\le\theta\le\pi$, giving θ in terms of π. Hence, obtain the roots of the equation $z^4=8\left(\sqrt{3}-\mathrm{i}\right)$ in the same form.

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Problem 4. Write down, in any form, the five complex numbers which satisfy the equation $z^5-1=0$. Hence, show that the five complex numbers which satisfy the equation

$$\left(\frac{2w+1}{w}\right)^5 = 1$$

are

$$\frac{-2+\cos\left(\frac{2}{5}\pi k\right)-\mathrm{i}\sin\left(\frac{2}{5}\pi k\right)}{5-4\cos\left(\frac{2}{5}\pi k\right)},$$

where $k=0,1,2,3,4$.

* * * * *

Problem 5.

(a) Show that, for all complex numbers z and all real numbers α,

$$\left(z-\mathrm{e}^{\mathrm{i}\alpha}\right)\left(z-\mathrm{e}^{-\mathrm{i}\alpha}\right) = z^2-2z\cos\alpha+1.$$

(b) Write down, in any form, the seven complex numbers which satisfy the equation $z^7-1=0$.

(c) Hence, show that, for all complex numbers z,

$$z^7-1=(z-1)\left[z^2-2z\cos\frac{2\pi}{7}+1\right]\left[z^2-2z\cos\frac{4\pi}{7}+1\right]\left[z^2-2z\cos\frac{6\pi}{7}+1\right].$$

* * * * *

Problem 6. Use De Moivre's theorem to show that

$$\cos 6\theta = 32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1.$$

Deduce that, for all θ,

$$0\le\cos^6\theta-\frac{3}{2}\cos^4\theta+\frac{9}{16}\cos^2\theta\le\frac{1}{16}.$$

Problem 7. Show that for $z \neq -1$,

$$z - z^2 + z^3 - \cdots + z^7 = \frac{z + z^8}{1 + z}.$$

Hence, by substituting $z = \mathrm{e}^{\mathrm{i}\theta}$, show that

$$\sum_{k=1}^{7} (-1)^{k-1} \sin k\theta = \frac{\sin 4\theta \cos \frac{7}{2}\theta}{\cos \frac{1}{2}\theta},$$

where θ is not an odd multiple of π.

Assignment A10.3

Problem 1.

(a) Solve $z^4 = -4 - 4\sqrt{3}\mathrm{i}$, expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Sketch the roots on an Argand diagram.

(c) Hence, solve $w^4 = -1 + \sqrt{3}\mathrm{i}$, expressing your answers in a similar form.

Solution.

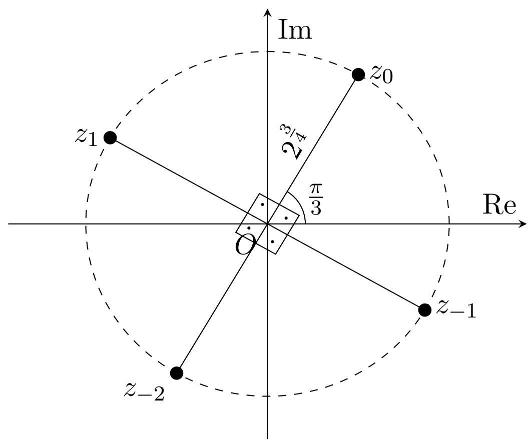
Part (a). Observe that $-4 - 4\sqrt{3}\mathrm{i} = 8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) = 8\mathrm{e}^{\mathrm{i}\frac{4}{3}\pi + 2k\pi\mathrm{i}}$ for all $k \in \mathbb{Z}$. Hence,

$$z^4 = 8\mathrm{e}^{\mathrm{i}\frac{4}{3}\pi + 2k\pi\mathrm{i}} \implies z = 8^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\frac{1}{3}\pi + \frac{1}{2}k\pi\mathrm{i}} = 2^{\frac{3}{4}}\mathrm{e}^{\mathrm{i}\frac{2+3k}{6}\pi}.$$

Taking $k = -2, -1, 0, 1$, we see that the roots are

$$z_{-2} = 2^{\frac{3}{4}}\mathrm{e}^{-\mathrm{i}\frac{2}{3}\pi}, \quad z_{-1} = 2^{\frac{3}{4}}\mathrm{e}^{-\mathrm{i}\frac{1}{6}\pi}, \quad z_0 = 2^{\frac{3}{4}}\mathrm{e}^{\mathrm{i}\frac{1}{3}\pi}, \quad z_1 = 2^{\frac{3}{4}}\mathrm{e}^{\mathrm{i}\frac{5}{6}\pi}.$$

Part (b).



Part (c). Observe that $w^4 = -1 + \sqrt{3}\mathrm{i} = \frac{1}{4}(-4 + 4\sqrt{3}\mathrm{i}) = 2^{-2}(z^*)^4$. Hence, $w = 2^{-1/2}z^*$. Thus, the roots are

$$w_{-2} = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\frac{2}{3}\pi}, \quad w_{-1} = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\frac{1}{6}\pi}, \quad w_0 = 2^{\frac{1}{4}}\mathrm{e}^{-\mathrm{i}\frac{1}{3}\pi}, \quad w_1 = 2^{\frac{1}{4}}\mathrm{e}^{-\mathrm{i}\frac{5}{6}\pi}.$$

* * * * *

Problem 2. Let

$$C = 1 - \binom{2n}{1}\cos\theta + \binom{2n}{2}\cos 2\theta - \binom{2n}{3}\cos 3\theta + \ldots + \cos 2n\theta$$
$$S = -\binom{2n}{1}\sin\theta + \binom{2n}{2}\sin 2\theta - \binom{2n}{3}\sin 3\theta + \ldots + \sin 2n\theta$$

where n is a positive integer.

Show that $C = (-4)^n \cos(n\theta)\sin^{2n}(\theta/2)$, and find the corresponding expression for S.

Solution. Clearly,

$$C = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \cos k\theta, \quad S = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \sin k\theta.$$

Hence,

$$\begin{aligned}
C + \mathrm{i}S &= \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (\cos k\theta + \mathrm{i}\sin k\theta) = \sum_{k=0}^{2n} \binom{2n}{k} (-\mathrm{e}^{\mathrm{i}\theta})^k = \left(1 - \mathrm{e}^{\mathrm{i}\theta}\right)^{2n} \\
&= \left(\mathrm{e}^{\mathrm{i}\theta/2}\right)^{2n} \left(\mathrm{e}^{-\mathrm{i}\theta/2} - \mathrm{e}^{\mathrm{i}\theta/2}\right)^{2n} = \mathrm{e}^{\mathrm{i}n\theta} \left(2\mathrm{i}\sin\frac{\theta}{2}\right)^{2n} = \mathrm{e}^{\mathrm{i}n\theta}(-4)^n \sin^{2n}\frac{\theta}{2} \\
&= (\cos n\theta + \mathrm{i}\sin n\theta)(-4)^n \sin^{2n}\frac{\theta}{2}.
\end{aligned}$$

Comparing real and imaginary parts, we have

$$C = (-4)^n \cos(n\theta)\sin^{2n}\frac{\theta}{2}, \quad S = (-4)^n \sin(n\theta)\sin^{2n}\frac{\theta}{2}.$$

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Problem 3. Given that $z = \cos\theta + \mathrm{i}\sin\theta$, show that

(a) $z - 1/z = 2\mathrm{i}\sin\theta$,

(b) $z^n + z^{-n} = 2\cos n\theta$.

Hence, show that

$$\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$$

Find a similar expression for $\cos^6\theta$, and hence express $\cos^6\theta - \sin^6\theta$ in the form $a\cos 2\theta + b\cos 6\theta$.

Solution.

Part (a). Note that

$$z - \frac{1}{z} = z - z^{-1} = \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} = [\cos\theta + \mathrm{i}\sin\theta] - [\cos(-\theta) + \mathrm{i}\sin(-\theta)] = 2\mathrm{i}\sin\theta.$$

Part (b). Note that

$$z^n + z^{-n} = \mathrm{e}^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta} = [\cos n\theta + \mathrm{i}\sin n\theta] + [\cos(-n\theta) + \mathrm{i}\sin(n\theta)] = 2\cos n\theta.$$

Observe that

$$\begin{aligned}
\sin^6\theta &= \frac{1}{(2\mathrm{i})^6}(2\mathrm{i}\sin\theta)^6 = -\frac{1}{64}(z - z^{-1})^6 \\
&= -\frac{1}{64}\left(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}\right) \\
&= -\frac{1}{32}\left[-\frac{20}{2} + 15\left(\frac{z^2 + z^{-2}}{2}\right) - 6\left(\frac{z^4 + z^{-4}}{2}\right) + \left(\frac{z^6 + z^{-6}}{2}\right)\right] \\
&= \frac{1}{32}\left(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cos^6\theta &= \frac{1}{2^6}(2\cos\theta)^6 = \frac{1}{64}(z+z^{-1})^6 \\
&= \frac{1}{64}\left[z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}\right] \\
&= \frac{1}{32}\left[\frac{20}{2} + 15\left(\frac{z^2+z^{-2}}{2}\right) + 6\left(\frac{z^4+z^{-4}}{2}\right) + \left(\frac{z^6+z^{-6}}{2}\right)\right] \\
&= \frac{1}{32}\left(10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta\right).
\end{aligned}$$

Hence,

$$\cos^6\theta - \sin^6\theta = \frac{1}{32}(30\cos 2\theta + 2\cos 6\theta) = \frac{15}{16}\cos 2\theta + \frac{1}{16}\cos 6\theta,$$

whence $a = 15/16$ and $b = 1/16$.

A10.4 Complex Numbers - Loci in Argand Diagram

Tutorial A10.4

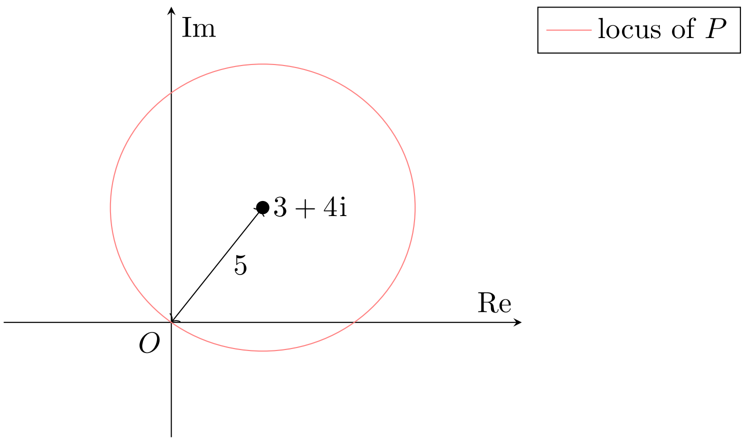
Problem 1. A complex number z is represented in an Argand diagram by the point P. Sketch, on separate Argand diagrams, the locus of P. Describe geometrically the locus of P and determine its Cartesian equation.

(a) $|2z - 6 - 8\mathrm{i}| = 10$

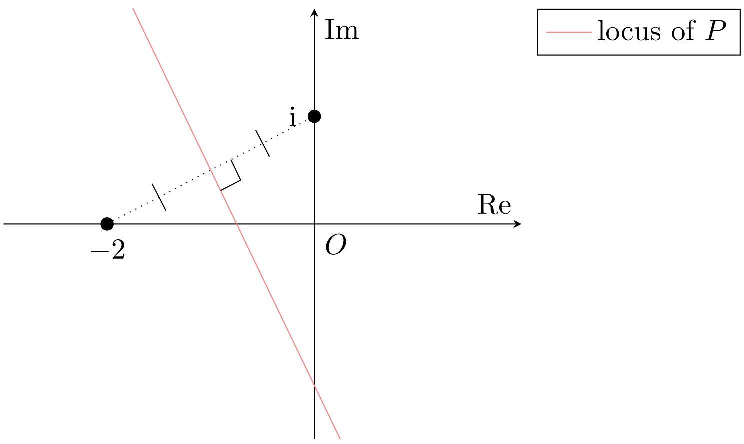
(b) $|z + 2| = |z - \mathrm{i}|$

(c) $\arg(z + 2 - \mathrm{i}) = -\pi/4$

Solution.
Part (a). Note that $|2z - 6 - 8\mathrm{i}| = 10 \implies |z - (3 + 4\mathrm{i})| = 5$.

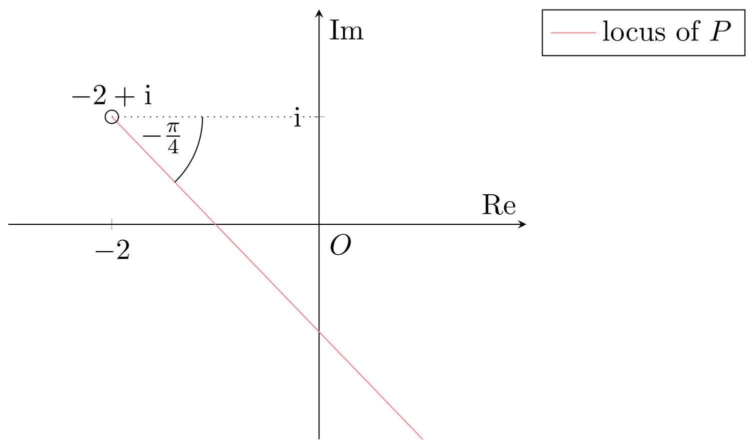


The locus of P is a circle with centre $(3, 4)$ and radius 5. Its Cartesian equation is $(x - 3)^2 + (y - 4)^2 = 5^2$.
Part (b). Note that $|z + 2| = |z - \mathrm{i}| \implies |z - (-2)| = |z - \mathrm{i}|$.



The locus of P is the perpendicular bisector of the line segment joining $(-2, 0)$ and $(0, 1)$. Its Cartesian equation is $y = -2x - 1.5$.

Part (c). Note that $\arg(z + 2 - \mathrm{i}) = -\pi/4 \implies \arg(z - (-2 + \mathrm{i})) = -\pi/4$.



The locus of P is the half-line starting from $(-2, 1)$ and inclined at an angle $-\pi/4$ to the positive real axis. Its Cartesian equation is $y = -x - 1$

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Problem 2. Sketch the following loci on separate Argand diagrams.

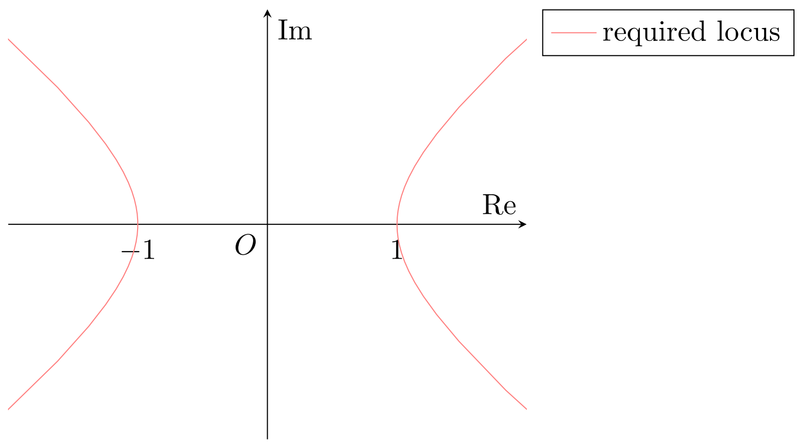
(a) $\mathrm{Re}(z^2) = 1$

(b) $|6 - \mathrm{i}z| = 2$,

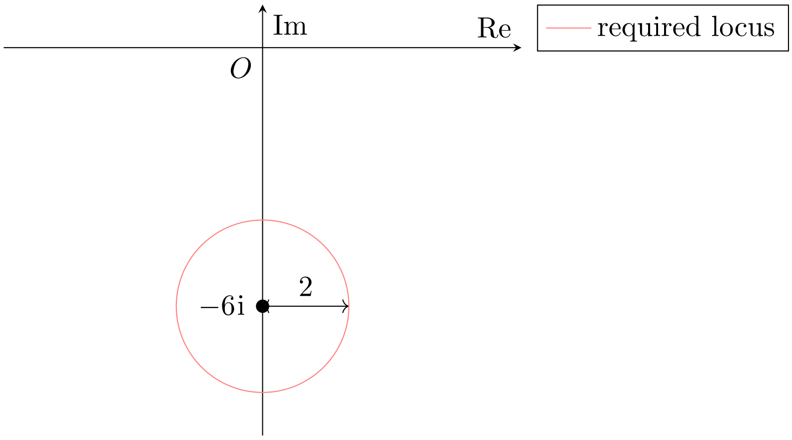
(c) $\arg\left(\frac{\mathrm{i}z}{1-\sqrt{3}\mathrm{i}}\right) = \pi$

Solution.

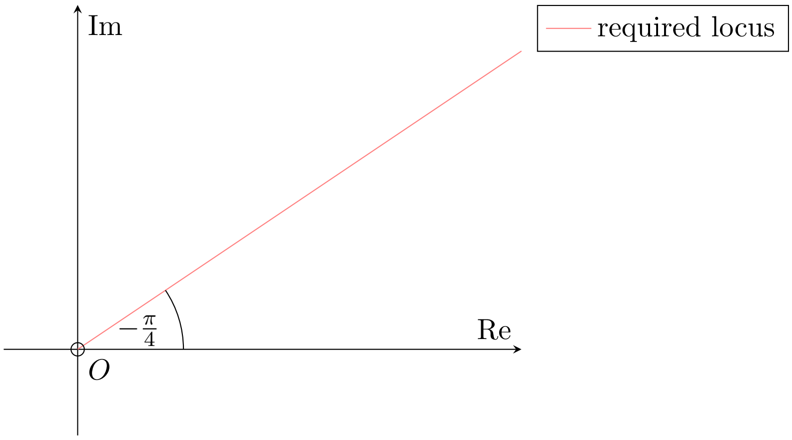
Part (a). Let $z = r(\cos\theta + \mathrm{i}\sin\theta)$. Then $\mathrm{Re}(z^2) = 1 \implies r^2 \cos 2\theta = 1 \implies r^2 = \sec 2\theta$.



Part (b). Note $|6 - \mathrm{i}z| = 2 \implies |-\mathrm{i}(z + 6i)| = 2 \implies |z + 6\mathrm{i}| = 2 \implies |z - (6\mathrm{i})| = 2$.



Part (c). Note $\arg\left(\frac{\mathrm{i}z}{1-\sqrt{3}\mathrm{i}}\right) = \pi \implies \frac{\pi}{2} + \arg(z) - \left(-\frac{\pi}{3}\right) \implies \arg(z) = \frac{\pi}{6}$.



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Problem 3. Sketch, on separate Argand diagrams, the set of points satisfying the following inequalities.

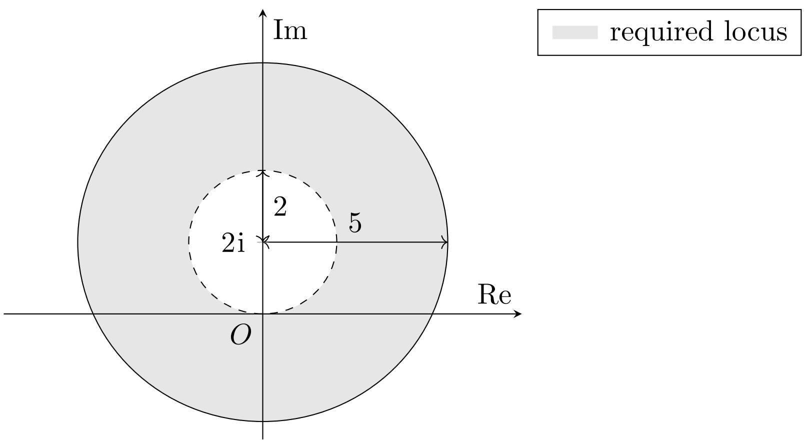
(a) $2 < |z - 2\mathrm{i}| \leq |3 - 4\mathrm{i}|$

(b) $|z + \mathrm{i}| > |z + 1 - \mathrm{i}|$

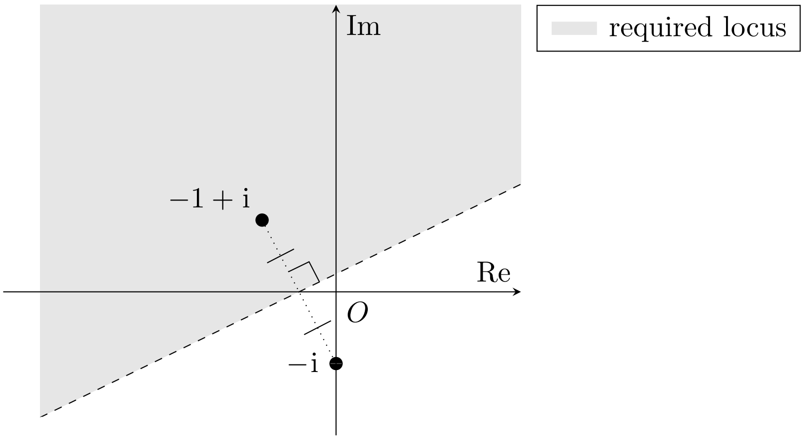
(c) $\frac{\pi}{4} < \arg\left(\frac{1}{z}\right) \leq \frac{\pi}{2}$

Solution.

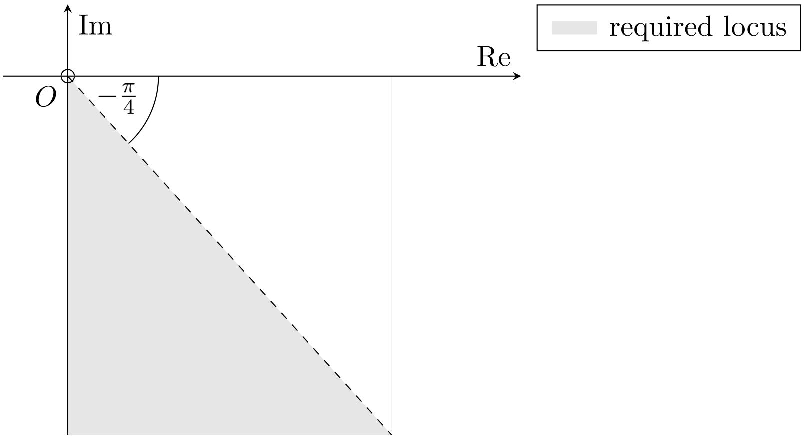
Part (a). Note $2 < |z - 2\mathrm{i}| \leq |3 - 4\mathrm{i}| \implies 2 < |z - 2\mathrm{i}| \leq 5$.



Part (b). Note $|z+\mathrm{i}| > |z+1-\mathrm{i}| \implies |z-(-\mathrm{i})| > |z-(-1+\mathrm{i})|$.



Part (c). Note $\frac{\pi}{4} < \arg\left(\frac{1}{z}\right) \leq \frac{\pi}{2} \implies \frac{\pi}{4} < -\arg(z) \leq \frac{\pi}{2} \implies -\frac{\pi}{2} \geq \arg(z) > -\frac{\pi}{4}$.



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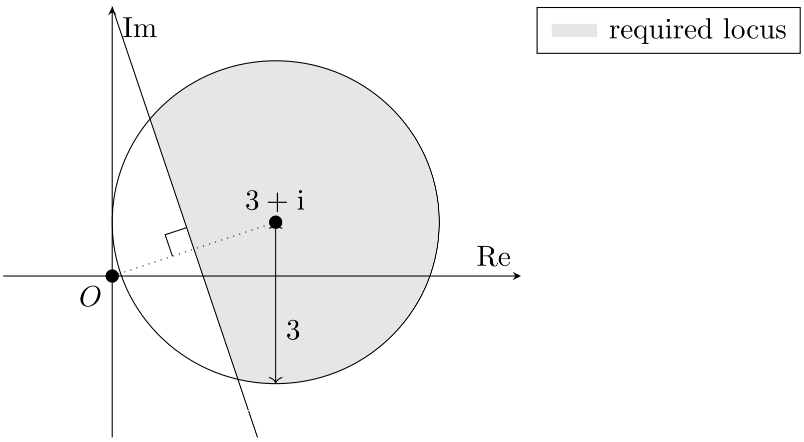
Problem 4. Sketch on separate Argand diagrams for (a) and (b) the set of points representing all complex numbers z satisfying both of the following inequalities.

(a) $|z-3-\mathrm{i}| \leq 3$ and $|z| \geq |z-3-\mathrm{i}|$

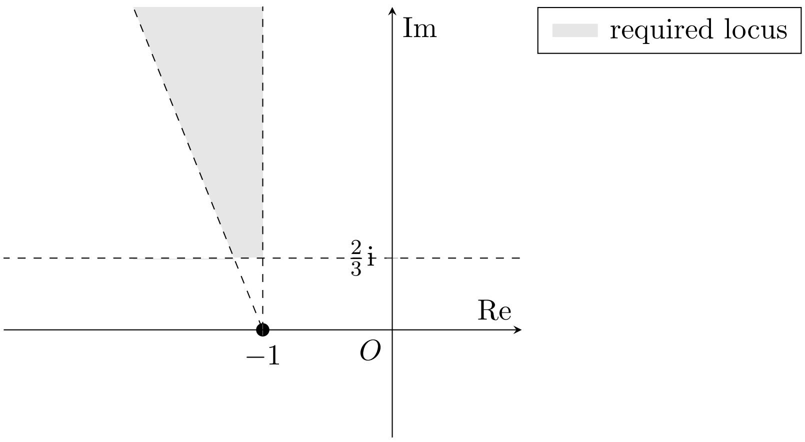
(b) $\frac{\pi}{2} < \arg(z+1) \leq \frac{2}{3}\pi$ and $3\mathrm{Im}(z) > 2$

Solution.

Part (a). Note $|z-3-\mathrm{i}| \leq 3 \implies |z-(3+\mathrm{i})| \leq 3$ and $|z| \geq |z-3-\mathrm{i}| \implies |z| \geq |z-(3+\mathrm{i})|$.



Part (b). Note $\frac{\pi}{2} < \arg(z+1) < \frac{2}{3}\pi \implies \frac{\pi}{2} < \arg(z-(-1)) < \frac{2}{3}\pi$ and $3\mathrm{Im}(z) > 2 \implies \mathrm{Im}(z) > \frac{2}{3}$.



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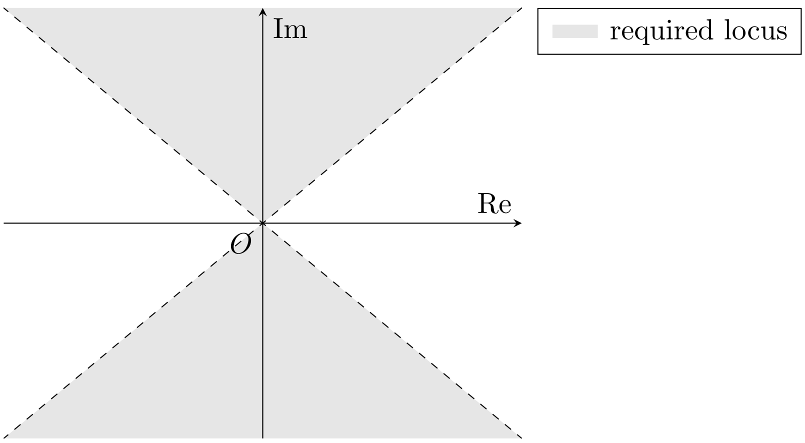
Problem 5. Illustrate, in separate Argand diagrams, the set of points z for which

(a) $\mathrm{Re}\left(z^2\right) < 0$

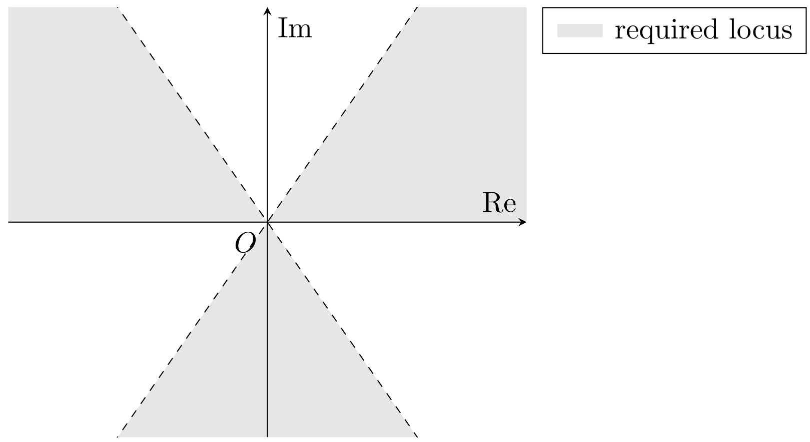
(b) $\mathrm{Im}\left(z^3\right) > 0$

Solution.

Part (a). Let $z = r(\cos\theta + \mathrm{i}\sin\theta)$, $0 \leq \theta < 2\pi$. Then $\mathrm{Re}(z^2) < 0 \implies r^2\cos 2\theta < 0 \implies \cos 2\theta < 0 \implies 2\theta \in \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right) \cup \left(\frac{5}{2}\pi, \frac{7}{2}\pi\right) \implies \theta \in \left(\frac{1}{4}\pi, \frac{3}{4}\pi\right) \cup \left(\frac{5}{4}\pi, \frac{7}{4}\pi\right)$.



Part (b). Let $z = r(\cos\theta + \mathrm{i}\sin\theta)$, $0 \leq \theta < 2\pi$. Then $\operatorname{Im}(z^3) > 0 \implies r^3 \sin 3\theta > 0 \implies \sin 3\theta > 0 \implies 3\theta \in (0, \pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi) \implies \theta \in \left(0, \frac{1}{3}\pi\right) \cup \left(\frac{2}{3}\pi, \pi\right) \cup \left(\frac{4}{3}\pi, \frac{5}{3}\pi\right)$.



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Problem 6. The complex number z satisfies $|z + 4 - 4\mathrm{i}| = 3$.

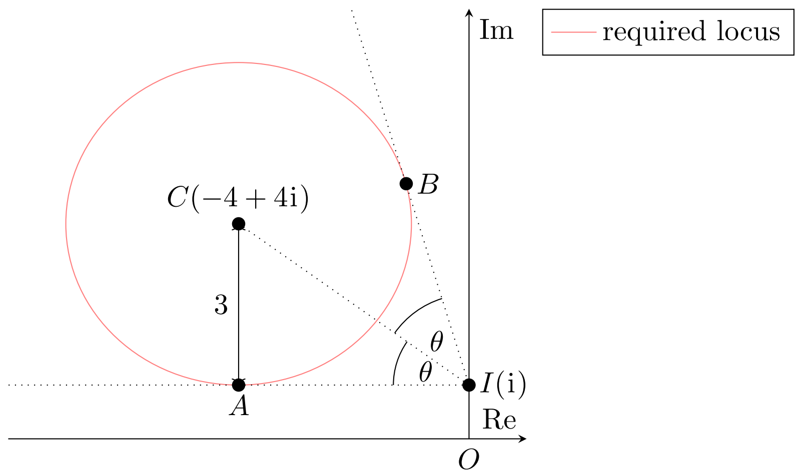
(a) Describe, with the aid of a sketch, the locus of the point which represents z in an Argand diagram.

(b) Find the least possible value of $|z - \mathrm{i}|$.

(c) Find the range of values of $\arg(z - \mathrm{i})$.

Solution.

Part (a). Note $|z + 4 - 4\mathrm{i}| = 3 \implies |z - (-4 + 4\mathrm{i})| = 3$.



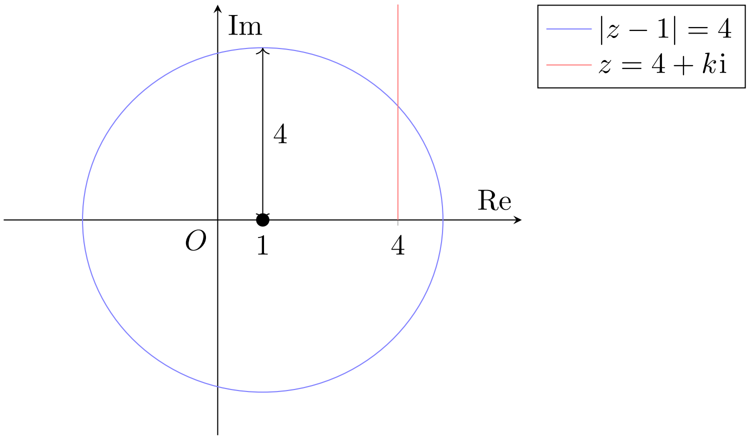
Part (b). Observe that the distance CI is equal to the sum of the radius of the circle and $\min|z - \mathrm{i}|$. Hence,

$$\min|z - \mathrm{i}| = \sqrt{(-4 - 0)^2 + (4 - 1)^2} - 3 = 2.$$

Part (c). Let A and B be points on the circle such that AI and BI are tangent to the circle. Let $\angle CIA = \theta$. Then $\tan\theta = \frac{3}{4} \implies \theta = \arctan\frac{3}{4}$. By symmetry, we also have $\angle CIB = \theta$, whence $\angle AIB = 2\theta = 2\arctan\frac{3}{4}$. Hence, $\min\arg(z - \mathrm{i}) = \pi - 2\arctan\frac{3}{4}$ (at B) and $\max\arg(z - \mathrm{i}) = \pi$ (at A). Thus, $\pi - 2\arctan\frac{3}{4} \leq \arg(z - \mathrm{i}) \leq \pi$.

Problem 7. Sketch, on the same Argand diagram, the two loci representing the complex number z for which $z = 4 + k\mathrm{i}$, where k is a positive real variable, and $|z-1| = 4$. Write down, in the form $x + \mathrm{i}y$, the complex number satisfying both conditions.

Solution.



Note that z is of the form $4 + k\mathrm{i}$, $k \in \mathbb{R}^+$. Since $|z-1| = 4$, we have $|3 + k\mathrm{i}| = 4 \implies 3^2 + k^2 = 4 \implies k = \sqrt{7}$. Note that we reject $k = -\sqrt{7}$ since $k > 0$. Thus, $z = 4 + \sqrt{7}\mathrm{i}$.

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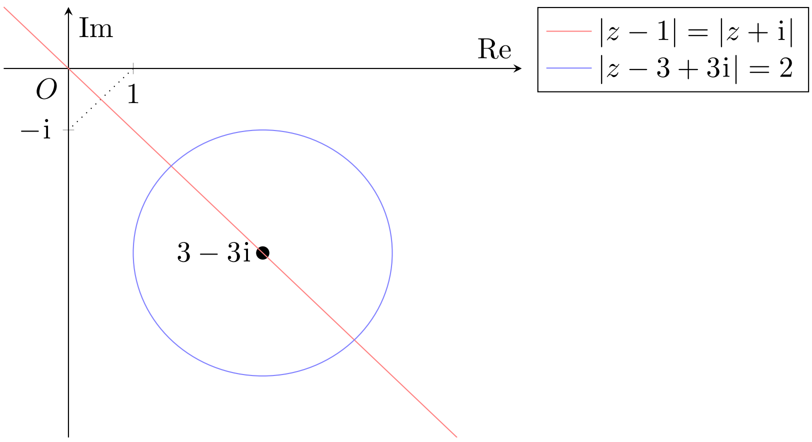
Problem 8. Describe, in geometrical terms, the loci given by $|z-1| = |z+\mathrm{i}|$ and $|z-3+3\mathrm{i}| = 2$ and sketch both loci on the same diagram.

Obtain, in the form $a + \mathrm{i}b$, the complex numbers representing the points of intersection of the loci, giving the exact values of a and b.

Solution. Note that $|z-1| = |z+\mathrm{i}| \implies |z-1| = |z-(-\mathrm{i})|$ and $|z-3+3\mathrm{i}| = 2 \implies |z-(3-3\mathrm{i})| = 2$.

The locus given by $|z-1| = |z+\mathrm{i}|$ is the perpendicular bisector of the line segment joining 1 and $-i$. The locus given by $|z-3+3\mathrm{i}| = 2$ is a circle with centre $3-3\mathrm{i}$ and radius 2.



Observe that the locus of $|z-1| = |z+\mathrm{i}|$ has Cartesian equation $y = -x$ and the locus of $|z-3+3\mathrm{i}| = 2$ has Cartesian equation $(x-3)^2 + (y+3)^2 = 2^2$. Solving both equations simultaneously, we have

$$(x-3)^2 + (y+3)^2 = (x-3)^2 + (3-x)^2 = 2^2 \implies x^2 - 6x + 7 = 0$$
$$\implies x = 3 \pm \sqrt{2} \implies y = -3 \mp \sqrt{2}.$$

Hence, the complex numbers representing the points of intersections of the loci are $(3+\sqrt{2})+(-3-\sqrt{2})\mathrm{i}$ and $(3-\sqrt{2})+(-3+\sqrt{2})\mathrm{i}$.

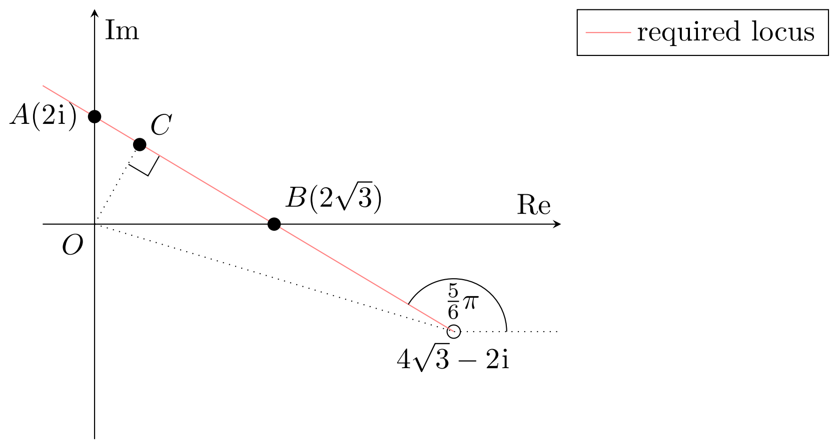
* * * * *

Problem 9. Sketch the locus for $\arg\big(z-(4\sqrt{3}-2\mathrm{i})\big)=\frac{5}{6}\pi$ in an Argand diagram.

(a) Verify that the points $2\mathrm{i}$ and $2\sqrt{3}$ lie on it.

(b) Find the minimum value of $|z|$ and the range of values of $\arg(z)$.

Solution.



Part (a). Note that

$$\arg\Big(2\mathrm{i}-(4\sqrt{3}-2\mathrm{i})\Big)=\arg\Big(-\sqrt{3}+\mathrm{i}\Big)=\arctan\frac{1}{-\sqrt{3}}=\frac{5}{6}\pi$$

and

$$\arg\Big(2\sqrt{3}-(4\sqrt{3}-2\mathrm{i})\Big)=\arg\Big(-\sqrt{3}+\mathrm{i}\Big)=\arctan\frac{1}{-\sqrt{3}}=\frac{5}{6}\pi.$$

Hence, the points $2\mathrm{i}$ and $2\sqrt{3}$ satisfy the equation $\arg\big(z-(4\sqrt{3}-2\mathrm{i})\big)=\frac{5}{6}\pi$ and thus lie on its locus.

Part (b). Let $A(2\mathrm{i})$ and $B(2\sqrt{3})$. Let C be the point on the required locus such that $OC\perp AB$. Observe that $\triangle OAB$, $\triangle COB$ and $\triangle CAO$ are all similar to one another. Hence,

$$\frac{OC}{CB}=\frac{AO}{BO}=\frac{1}{\sqrt{3}}\implies AC=\frac{1}{\sqrt{3}}OC,\quad \frac{OC}{CA}=\frac{BO}{OA}=\frac{\sqrt{3}}{1}\implies BC=\sqrt{3}OC.$$

Hence, $AB=AC+CB=\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)OC$, whence

$$\min|z|=OC=\frac{AB}{\sqrt{3}+1/\sqrt{3}}=\frac{\sqrt{2^2+(2\sqrt{3})^2}}{\sqrt{3}+1\sqrt{3}}=\frac{4\sqrt{3}}{4}=\sqrt{3}.$$

Observe that $\max\arg(z)=\frac{5}{6}\pi$ and $\min\arg(z)=\min\arg\big(4\sqrt{3}-2\mathrm{i}\big)=\arctan\frac{-2}{4\sqrt{3}}=-\arctan\frac{1}{2\sqrt{3}}$. Thus, $-\arctan\frac{1}{2\sqrt{3}}<\arg(z)\leq\frac{5}{6}\pi$.

Problem 10. The complex number z satisfies $|z-3-3\mathrm{i}| \geq |z-1-\mathrm{i}|$ and $\frac{\pi}{6} < \arg(z) \leq \frac{\pi}{3}$.

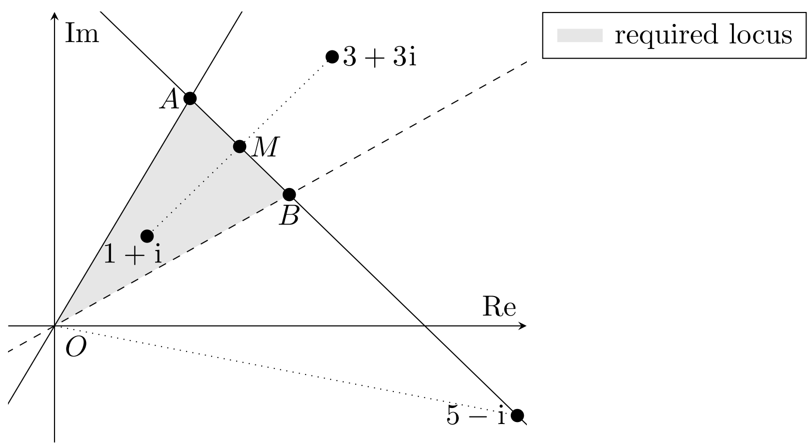
(a) On an Argand diagram, sketch the region in which the point representing z can lie.

(b) Find the area of the region in part (a).

(c) Find the range of values of $\arg(z-5+\mathrm{i})$.

Solution.

Part (a). Note that $|z-3-3\mathrm{i}| \leq |z-1-\mathrm{i}| \implies |z-(3+3\mathrm{i})| \leq |z-(1+\mathrm{i})|$.



Part (b). Note that the locus of $|z-3-3\mathrm{i}| = |z-1-\mathrm{i}|$ has Cartesian equation $y = -x+4$, while the loci of $\frac{\pi}{6} = \arg(z)$ and $\arg(z) = \frac{\pi}{3}$ have Cartesian equations $y = \frac{1}{\sqrt{3}}x$ and $y = \sqrt{3}x$ respectively. Let A and B be the intersections between $y = -x+4$ with $y = \sqrt{3}x$ and $y = \frac{1}{\sqrt{3}}x$ respectively.

At A, we have $y = \sqrt{3}x = -x+4$, whence $A\left(\frac{4}{1+\sqrt{3}}, \frac{4\sqrt{3}}{1+\sqrt{3}}\right)$. Thus,

$$OA = \sqrt{\left(\frac{4}{1+\sqrt{3}}\right)^2 + \left(\frac{4\sqrt{3}}{1+\sqrt{3}}\right)^2} = \frac{8}{1+\sqrt{3}}.$$

By symmetry, we also have $OA = OB$. Finally, since $\angle AOB = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$,

$$[\triangle AOB] = \frac{1}{2}(OA)(OB)\sin\angle AOB = \frac{1}{2}\left(\frac{8}{1+\sqrt{3}}\right)^2 \frac{1}{2} = \frac{16}{\left(1+\sqrt{3}\right)^2} = 4\left(1-\sqrt{3}\right)^2.$$

Part (c). Observe that $\min \arg(z-(5-\mathrm{i})) = \frac{3}{4}\pi$ and $\max \arg(z-(5-\mathrm{i})) = \arctan\frac{-1}{5} + \pi = \pi - \arctan\frac{1}{5}$. Hence, $\frac{3}{4}\pi \leq \arg(z-5+\mathrm{i}) < \pi - \arctan\frac{1}{5}$.

Problem 11. Sketch on an Argand diagram the set of points representing all complex numbers z satisfying both inequalities

$$|\mathrm{i}z - 2\mathrm{i} - 2| \leq 2 \qquad \text{and} \qquad \mathrm{Re}(z) > \left|1 + \sqrt{3}\mathrm{i}\right|$$

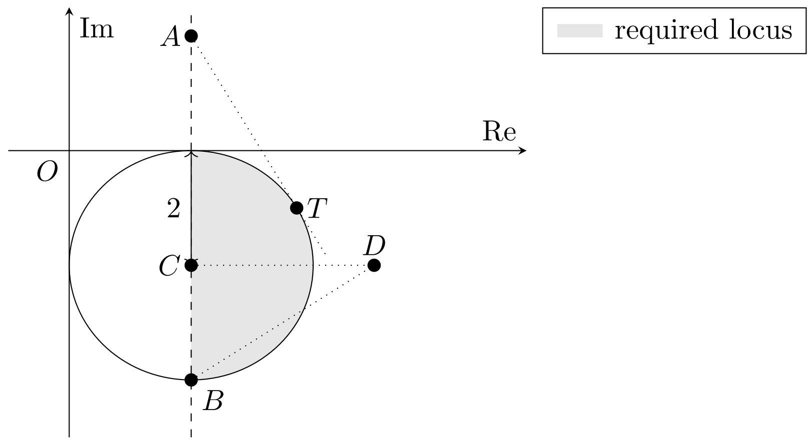
Find

(a) the range of $\arg(z - 2 - 2\mathrm{i})$,

(b) the complex number z where $\arg(z - 2 - 2\mathrm{i})$ is a maximum.

The locus of the complex number w is defined by $|w - 5 + 2\mathrm{i}| = k$, where k is a real and positive constant. Find the range of values of k such that the loci of w and z will intersect.

Solution. Note $|\mathrm{i}z - 2i - 2| \leq 2 \implies |\mathrm{i}(z - 2 + 2i)| \leq 2 \implies |z - (2 - 2\mathrm{i})| \leq 2$ and $\mathrm{Re}(z) > \left|1 + \sqrt{3}\mathrm{i}\right| = 2$.



Part (a). Note $|z - 2 - 2\mathrm{i}| = \arg(z - (2 + 2\mathrm{i}))$. Let $A(2+2\mathrm{i})$ and $C(2-2\mathrm{i})$. Let T be the point at which AT is tangent to the circle. Then $\angle ATC = \frac{\pi}{2}$, $AC = 4$ and $TC = 2$. Hence, $\angle CAT = \arcsin \frac{2}{4} = \frac{\pi}{6}$. Thus, $\min \arg(z - 2 - 2\mathrm{i}) = -\frac{\pi}{2}$ and $\max \arg(z - 2 - 2\mathrm{i}) = \min \arg(z - 2 - 2\mathrm{i}) + \angle CAT = -\frac{\pi}{2} + \frac{\pi}{6} = -\frac{\pi}{3}$. Hence, $-\frac{\pi}{2} < \arg(z - 2 - 2i) \leq -\frac{\pi}{3}$.

Part (b). Relative to C, T is given by $2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right) = \sqrt{3} + \mathrm{i}$. Thus, $T = (\sqrt{3} + \mathrm{i}) + (2 - 2\mathrm{i}) = 2 + \sqrt{3} - \mathrm{i}$.

Note $|w - 5 + 2\mathrm{i}| = k \implies |w - (5 - 2\mathrm{i})| = k$. Let $D(5 - 2\mathrm{i})$. Observe that CD is given by the sum of the radius of the circle and $\min k$. Hence, $\min k = 3 - 2 = 1$. Let $B(2 - 4\mathrm{i})$. Then $\max k$ is given by the distance between B and D. By the Pythagorean Theorem, we have $\max k = \sqrt{(5-2)^2 + (-2 - (-4))^2} = \sqrt{13}$. Thus, $1 \leq k \leq \sqrt{13}$.

Self-Practice A10.4

Problem 1. If $\arg(z-2) = 2\pi/3$ and $|z| = 2$, determine $\arg(z)$.

* * * * *

Problem 2. z is a complex number such that $\arg(z-1) = \pi/3$ and $\arg(z-\mathrm{i}) = \pi/6$. By finding the Cartesian equations of the two half-lines, or otherwise, find the value of $\arg(z)$.

* * * * *

Problem 3. The complex number z is given by $z = re^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta \le \pi/2$.

(a) Given that $w = \left(1 - \mathrm{i}\sqrt{3}\right) z$, find $|w|$ in terms of r and $\arg w$ in terms of θ.

(b) Given that r has a fixed value, draw an Argand diagram to show the locus of z as θ varies. On the same Argand diagram, show the corresponding locus of w. You should identify the modulus and argument of the end-point of each locus.

* * * * *

Problem 4. The complex number z satisfies the equation $|z| = |z+2|$. Show that the real part of z is -1. The complex number z also satisfies the equation $|z| = 3$. The two possible values of z are represented by the points P and Q in an Argand diagram. Draw a sketch showing the positions of P and Q, and calculate the two possible values of $\arg z$, giving your answers in radians correct to 3 significant figures.

It is given that P and Q lie on the locus $|z-a| = b$, where a and b are real, and $b > 0$. Give a geometrical description of this locus, and hence find the least possible value of b and the corresponding value of a.

* * * * *

Problem 5. The complex number z is given by $z = x + \mathrm{i}y$, where $x > 0$ and $y > 0$. Sketch an Argand diagram, with origin O, showing points P, Q and R representing z, $2\mathrm{i}z$ and $(z + 2\mathrm{i}z)$ respectively. State the size of angle POQ, and describe briefly the geometrical relationship between O, P, Q and R.

(a) Given that $x = 2y$, show that R lies on the imaginary axis.

(b) Given that $y = 2x$, show that the point representing z^2 is collinear with the origin and the point R.

(c) Given that $|z| \le 2$ and $\arctan \frac{1}{2} \le \arg z \le \arctan 2$, calculate the area of the region in which the point P can lie.

* * * * *

Problem 6. A complex number z satisfies $|z-a| = a$, $a \in \mathbb{R}^+$.

(a) The point P represents the complex number w, where $w = 1/z$, in an Argand diagram. Show that the locus of P is a straight line.

(b) Sketch both loci on the same diagram and show that the two loci do not intersect if $0 < a, 1/2$.

(c) For $a = 1/2$, find the range of values of $\arg(z - 1/a)$, giving your answer correct to $0.1°$. State the limit of $\arg(z - 1/a)$ when a approaches 0.

* * * * *

Problem 7. Sketch, on an Argand diagram, the locus representing the complex number z for which

$$|z - 4 - 4\mathrm{i}| = 2.$$

(a) Given that a is the least possible value of $|z|$, find a.

(b) The complex number p is such that

$$|p - 4 - 3\mathrm{i}| = 2 \quad \text{and} \quad |p| = a.$$

State the exact value of $\arg p$.

(c) Deduce the greatest value of $\arg(z/p)$, giving your answer correct to 2 decimal places.

* * * * *

Problem 8 (🌶). On an Argand diagram, the point U represents the complex number z, and the points V and W represent the complex numbers z^2 and $z^2 + 1$ respectively.

(a) (i) Given that $\arg(z) = \alpha$, where $\pi/4 < \alpha < \pi/2$, so that U lies on the half-line L_1 with equation $y = x \tan\alpha$ for $x > 0$, show that V lies on the half-line L_2 with equation $y = x \tan 2\alpha$ for $x < 0$. Find the equation of the locus L_3 of W.

(ii) The points E and F represent the values of z for which W coincides with U. Find the value of α for which the common point of L_1 and L_3 is either E or F.

(b) Given instead that $|z| = k$, where $k > 0$, so that U lies on a circle C, show that W lies on a circle C', and find its centre and radius. Find the value of k for which the common points of C and C' are E and F.

Assignment A10.4

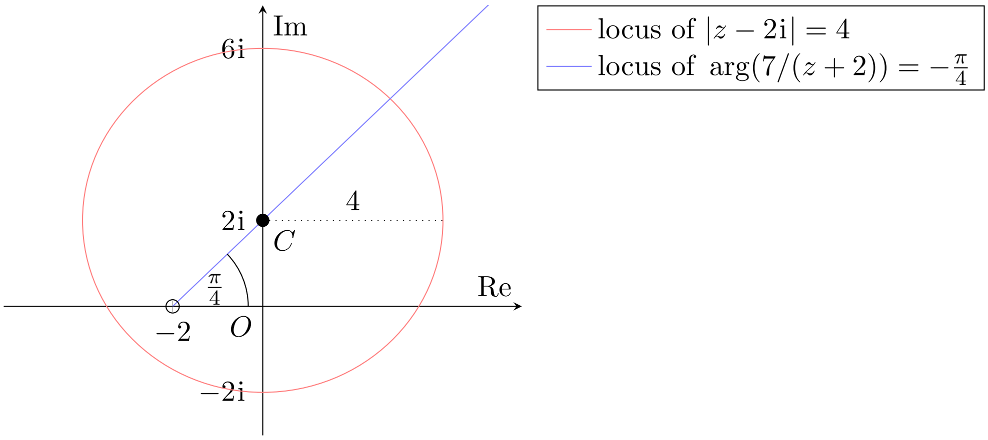
Problem 1. On a single Argand diagram, sketch the following loci.

(a) $|z - 2\mathrm{i}| = 4$.

(b) $\arg\left(\frac{7}{z+2}\right) = -\frac{\pi}{4}$.

Hence, or otherwise, find the exact value of z satisfying both equations in part (a) and (b).

Solution. Note that $\arg\left(\frac{7}{z+2}\right) = -\frac{\pi}{4} \implies \arg(z - (-2)) = \frac{\pi}{4}$.



Solving both equations simultaneously,

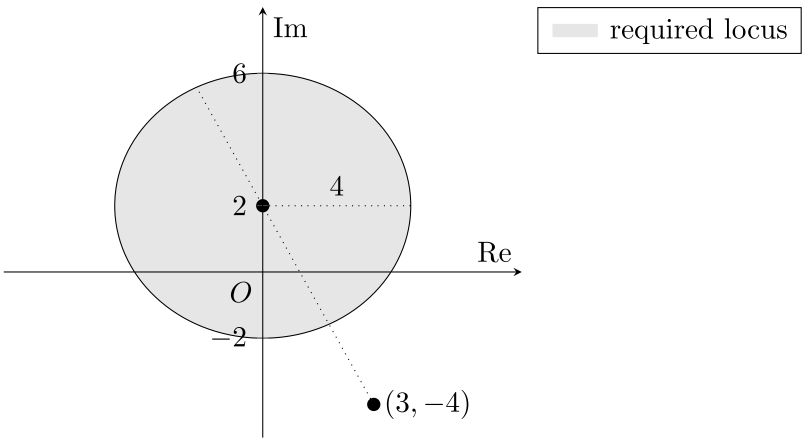
$$z = 2\mathrm{i} + \left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) = 2\mathrm{i} + \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + \left(2 + \frac{\sqrt{2}}{2}\right)\mathrm{i}.$$

* * * * *

Problem 2. Given that $|z - 2\mathrm{i}| \leq 4$, illustrate the locus of the point representing the complex number z in an Argand diagram.

Hence, find the greatest and least possible value of $|z - 3 + 4\mathrm{i}|$, given that $|z - 2\mathrm{i}| \leq 4$.

Solution.



Note that $|z - 3 + 4\mathrm{i}| = |z - (3 - 4\mathrm{i})|$ represents the distance between z and the point $(3, -4)$. By Pythagoras' Theorem, the distance between the centre of the circle $(0, 2)$

and $(3,-4)$ is $\sqrt{(0-3)^2+(2+4)^2} = 3\sqrt{5}$. Hence, $\max|z-3+4\mathrm{i}| = 3\sqrt{5}+4$, while $\min|z-3+4\mathrm{i}| = 3\sqrt{5}-4$. Thus, $\max|z-3+4\mathrm{i}| = 3\sqrt{5}+4$, $\min|z-3+4\mathrm{i}| = 3\sqrt{5}-4$.

* * * * *

Problem 3. The point A on an Argand diagram represents the fixed complex number a, where $0 < \arg a < \frac{\pi}{2}$. The complex numbers z and w are such that $|z-2\mathrm{i}a| = |a|$ and $|w| = |w+\mathrm{i}a|$.

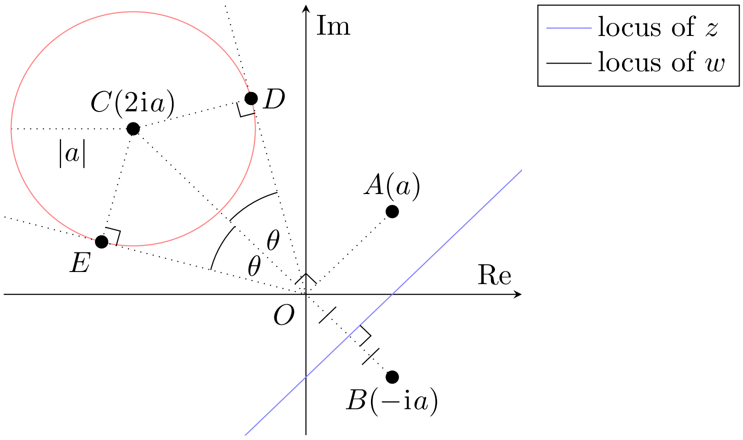
Sketch, on a single diagram, the loci of the point representing z and w.

Find

(a) the minimum value of $|z-w|$ in terms of $|a|$,

(b) the range of values of $\arg\frac{1}{z}$ in terms of $\arg a$.

Solution. Note that $|w| = |w+ia| \implies |w-0| = |w-(-ia)|$.



Part (a). Let $B(-\mathrm{i}a)$ and $C(2\mathrm{i}a)$. Note that $W\left(-\frac{1}{2}\mathrm{i}a\right)$ lies on the locus of w as well as the line passing through OC. Since CW is perpendicular to the locus of w, it follows that the minimum value of $|z-w|$ is given by

$$CW - |a| = \left|2\mathrm{i}a + \frac{1}{2}\mathrm{i}a\right| - |a| = \frac{5}{2}|a|\,|\mathrm{i}| - |a| = \frac{3}{2}|a|.$$

Part (b). Let D and E be such that OD and OE are tangent to the circle given by the locus of z. Let $\angle COD = \theta$. Observe that $\sin\theta = \frac{CD}{CO} = \frac{|a|}{|2\mathrm{i}a|} = \frac{1}{2}$, whence $\theta = \frac{\pi}{6}$. Since $\angle COA = \arg\mathrm{i} = \frac{\pi}{2}$, it follows that $\angle DOA = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \arcsin\frac{1}{2} = \frac{\pi}{3}$. Thus, $\min\arg z = \arg a + \angle DOA = \arg a + \frac{\pi}{3}$. Meanwhile, $\angle COE = \angle COD = \theta$, whence $\max\arg z = \arg a + \frac{\pi}{2} + \theta = \arg a + \frac{2}{3}\pi$. Since $\arg\frac{1}{z} = -\arg z$, we thus have $\arg\frac{1}{z} \in \left[-\left(\arg a + \frac{2}{3}\pi\right), -\left(\arg a + \frac{\pi}{3}\right)\right]$.

* * * * *

Problem 4.

(a) Solve the equation

$$z^7 - (1+\mathrm{i}) = 0,$$

giving the roots in the form $r\mathrm{e}^{\mathrm{i}\alpha}$, where $r > 0$ and $-\pi < \alpha \le \pi$.

(b) Show the roots on an Argand diagram.

(c) The roots represented by z_1 and z_2 are such that $0 < \arg z_1 < \arg z_2 < \frac{\pi}{2}$. Explain why the locus of all points z such that $|z-z_1| = |z-z_2|$ passes through the origin. Draw this locus on your Argand diagram and find its Cartesian equation.

(d) Describe the transformation that will map the points representing the roots of the equation $z^7 - (1 + \mathrm{i}) = 0$ to the points representing the roots of the equation $(z - 2)^7 - (1 + \mathrm{i}) = 0$ on the Argand diagram.

Solution.

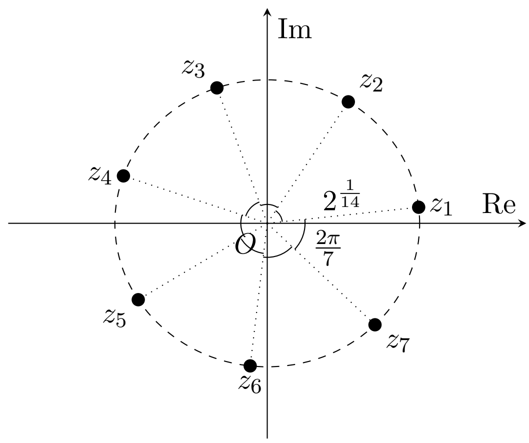
Part (a). Note that $1 + \mathrm{i} = 2^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+2k)}$, where $k \in \mathbb{Z}$. Hence,

$$z^7 = 1 + \mathrm{i} = 2^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+2k)} \implies z = 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+2k)/7} = 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi(1+8k)/28}.$$

Taking $k \in \{-3, -2, \dots, 2, 3\}$, we have

$$z = 2^{\frac{1}{14}} \mathrm{e}^{-\mathrm{i}\pi\frac{23}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{-\mathrm{i}\pi\frac{15}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{-\mathrm{i}\pi\frac{7}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi\frac{1}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi\frac{9}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi\frac{17}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi\frac{25}{28}}.$$

Part (b).



Part (c). Since $|z_1| = |z_2| = 2^{\frac{1}{14}}$, the distance between z_1 and the origin and the distance between z_2 and the origin are equal. Since the locus of $|z - z_1| = |z - z_2|$ represents all points equidistant from z_1 and z_2, it passes through the origin.

Observe that the midpoint of z_1 and z_2 will have argument $\frac{1}{2}\left(\frac{1}{28}\pi + \frac{9}{28}\pi\right) = \frac{5}{28}\pi$. Thus, the Cartesian equation of the locus of z is given by $y = \tan(5\pi/28)x$.

Part (d). Translate the points 2 units in the positive real direction.

A11 Permutations and Combinations

Tutorial A11

Problem 1. In a particular country, the alphabet contains 25 letters. A car registration number consists of two different letters of the alphabet followed by an integer n such that $100 \leq n \leq 999$. Find the number of possible car registration numbers.

Solution. Note that the number of possible n is $999 - 100 + 1 = 900$. Hence, the number of possible car registration numbers is given by ${}^{25}C_2 \cdot 900 = 540000$.

* * * * *

Problem 2. A girl wishes to phone a friend but cannot remember the exact number. She knows that it is a five-digit number that is even, and that it consists of the digits 2, 3, 4, 5, and 6 in some order. Using this information, find the greatest number of wrong telephone numbers she could try.

Solution. Since the number is odd, there are only 3 possibilities for the last digit. Hence, the maximum wrong numbers she could try is $3 \cdot 4! - 1 = 71$.

* * * * *

Problem 3. How many ways are there to select a committee of

(a) 3 students

(b) 5 students

out of a group of 8 students?

Solution.
Part (a). There are ${}^8C_3 = 56$ ways.
Part (b). There are ${}^8C_5 = 56$ ways.

* * * * *

Problem 4. How many ways are there for 2 men, 2 women and 2 children to sit a round table?

Solution. Since the men, women and children are all distinct, there are $(2+2+2-1)! = 120$ ways.

* * * * *

Problem 5. Find the number of different arrangements of the eight letters of the word NONSENSE if

(a) there is no restriction on the arrangement,

(b) the two letters E are together,

(c) the two letters E are not together,

(d) the letters N are all separated,

(e) only two of the letters N are together.

Solution.

Part (a). Note that N, S and E are repeated 3, 2, and 2 times respectively. Thus, the total number of arrangements is given by $\frac{8!}{3!\,2!\,2!} = 1680$.

Part (b). Consider the two E's as one unit. Altogether, there are 7 units. Hence, the required number of arrangements is given by $\frac{7!}{3!\,2!} = 420$.

Part (c). From part (a) and part (b), the required number of arrangements is given by $1680 - 420 = 1260$.

Part (d). There are $\frac{5!}{2!\,2!}$ ways to arrange the non-N letters, and ${}^6C_3$ ways to slot in the 3 N's into the 6 gaps in between the non-N letters. Thus, the required number of arrangements is given by $\frac{5!}{2!\,2!} \cdot {}^6C_3 = 600$.

Part (e). Consider the three N's as one unit. Altogether there are 6 units. Hence, the number of arrangements where all 3 N's are together is given by $\frac{6!}{2!\,2!} = 180$. Thus, from parts (a) and (d), the required number of arrangements is given by $1680-600-180 = 900$.

* * * * *

Problem 6. Find the number of teams of 11 that can be select from a group of 15 players

(a) if there is no restriction on choice,

(b) if the youngest two players and at most one of the oldest two players are to be included.

Solution.

Part (a). The number of teams is given by ${}^{15}C_{11} = 1365$.

Part (b). Given that the youngest two players are always included, we are effectively finding the number of teams of 9 from a group of 13 players with the restriction that at most one of the oldest two players are to be included.

Disregarding the restriction, the total number of teams is given by ${}^{13}C_9 = 715$.

Consider now that number of teams where both of the 2 oldest players are included. This is given by ${}^{11}C_7 = 330$.

Thus, the required number of teams is $715 - 330 = 385$.

* * * * *

Problem 7. A ten-digit number is formed by writing down the digits 0, 1, . . . , 9 in some order. No number is allowed to start with 0. Find how many such numbers are

(a) odd,

(b) less than 2 500 000 000.

Solution.

Part (a). Since the number is odd, there are 5 possibilities for the last digit. Furthermore, since no number is allowed to start with 0, there are $10 - 2 = 8$ possibilities for the first digit. The remaining 8 digits are free. Hence, the required number of numbers is $5 \cdot 8 \cdot 8! = 1612800$.

Part (b). *Case 1*: *Number starts with 1.* Since there are no further restrictions, the number of valid numbers in this case is 9!.

Case 2: *Number starts with 2.* Given the restriction that the number be less than 2 500 000 000, the second digit must be strictly less than 5, thus giving 4 possibilities for the second digit. The remaining 8 digits are free, for a total number of valid numbers of $4 \cdot 8!$.

Thus, the required number of numbers is $9! + 4 \cdot 8! = 524160$.

* * * * *

Problem 8. Eleven cards each bear a single letter, and together, they can be made to spell the word "EXAMINATION".

(a) Three cards are selected from the eleven cards, and the order of selection is not relevant. Find how many possible selections can be made

(i) if the three cards all bear different letters,

(ii) if two of the three cards bear the same letter.

(b) Two cards bearing the letter N have been taken away. Find the number of different arrangements for the remaining cards that can be made with no two adjacent letters the same.

Solution.

Part (a).

Part (a)(i). Observe that there are 8 distinct letters in "EXAMINATION". Hence, the number of possible selections is ${}^8C_3 = 56$.

Part (a)(ii). Note that there are 3 letters that appear twice in "EXAMINATION". Hence, the number of possible selections is given by ${}^3C_1 \cdot {}^7C_1 = 21$.

Part (b). Note that there are now 2 letters that appear twice, namely A and I. Hence, the total number of possible arrangements is $\frac{9!}{2!\,2!}$.

Consider "AA" and "II" as one unit each. Altogether, there are 7 units. The number of arrangements with two pairs of adjacent letters that are the same is hence given by 7!.

Consider "AA" as one unit, and suppose the two I's are not adjacent to each other. Observe that the non-I letters comprise 6 units, hence giving 6! ways of arranging them. Also observe that there are ${}^7C_2$ ways to slot in the two I's (which guarantee that they are not adjacent to each other). There are hence $6! \cdot {}^7C_2$ possible arrangements in this case. A similar argument follows for the case where the two I's are adjacent but the A's are not.

From the above discussion, it follows that the required number of arrangements is given by $\frac{9!}{2!\,2!} - 7! - 2 \cdot 6! \cdot {}^7C_2 = 55440$.

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Problem 9. Find how many three-letter code words can be formed from the letters of the word:

(a) PEAR.

(b) APPLE.

(c) BANANA.

Solution.

Part (a). Since all 4 letters are distinct, the number of code-words is given by ${}^4P_3 = 24$.

Part (b). Tally of letters: 2 'P', 1 'A', 1 'L', 1 'E' (5 letters, 4 distinct).

Case 1: *All letters distinct.* Since there are 4 distinct letters, the number of code-words in this case is ${}^4P_3 = 24$.

Case 2: *2 letters the same, 1 different.* Note that 'P' is the only letter repeated more than once. Reserving two spaces for 'P' leaves one space left for three remaining letters. Hence, there are ${}^1C_1 \cdot {}^3C_1 = 3$ different combinations that can be formed, with $\frac{3!}{2!} = 3$ ways to arrange each combination. Hence, the number of code-words in this case is $3 \cdot 3 = 9$.

Thus, the total number of code-words is $24 + 9 = 33$.

Part (c). Tally of letters: 3 ‘A’, 2 ‘N’, 1 ‘B’ (6 letters, 3 distinct).

Case 1: *All letters distinct.* Since there are only 3 distinct letters, the number of code-words in this case is ${}^3P_3 = 6$.

Case 2: *2 letters the same, 1 different.* Observe that both ‘A’ and ‘N’ are repeated more than once. Reserving 2 spaces for either letter leaves one space left for the two remaining letters. Hence, there are ${}^2C_1 \cdot {}^2C_1 = 4$ different combinations that can be formed, with $\frac{3!}{2!} = 3$ ways to arrange each combination. Hence, the number of code-words in this case is $4 \cdot 3 = 12$.

Case 3: *All letters the same.* Observe that ‘A’ is the only letter repeated thrice. Hence, the number of code-words in this case is 1.

Altogether, the total number of code-words is $6 + 12 + 1 = 19$.

* * * * *

Problem 10. A group of diplomats is to be chosen to represent three islands, K, L and M. The group is to consist of 8 diplomats and is chosen from a set of 12 diplomats consisting of 3 from K, 4 from L and 5 from M. Find the number of ways in which the group can be chosen if it includes

(a) 2 diplomats from K, 3 from L and 3 from M,

(b) diplomats from L and M only,

(c) at least 4 diplomats from M,

(d) at least 1 diplomat from each island.

Solution.

Part (a). Note that there are ${}^3C_2$ ways to select 2 diplomats from K, ${}^4C_3$ ways to select 3 diplomats from L, and ${}^5C_3$ ways to select 3 diplomats from M. Thus, the number of possible groups is given by ${}^3C_2 \cdot {}^4C_3 \cdot {}^5C_3 = 120$.

Part (b). There are a total of 9 diplomats from L and M. Hence, the number of possible groups is ${}^9C_8 = 9$.

Part (c). *Case 1*: *4 diplomats from M.* Note that there are ${}^5C_4$ combinations for the 4 diplomats from M. Furthermore, since M contributes 4 diplomats, K and L must contribute the other 4 diplomats. Since K and L have a total of 7 diplomats, this gives a total of ${}^5C_4 \cdot {}^7C_4$ possibilities.

Case 2: *5 diplomats from M.* Since M has 5 diplomats, there is only one way for M to send 5 diplomats (all of them have to be chosen). Meanwhile, K and L must contribute the other 3 diplomats from a pool of 7. This gives a total of ${}^7C_3$ possibilities.

Altogether, there are ${}^5C_4 \cdot {}^7C_4 + {}^7C_3 = 210$ total possibilities.

Part (d). Observe that K and M have a total of 8 diplomats. Hence, there is only one possibility where the group only consists of diplomats from K and M.

Since K and L have a total of 7 diplomats, it is impossible for the group to only come from K and L.

From part (b), we know that there are 9 ways where the group consists only of diplomats from L and M.

Note that there are a total of ${}^{12}C_8$ possible ways to choose the group.

Altogether, the required number of possibilities is given by ${}^{12}C_8 - 9 - 1 = 485$.

Problem 11. Alisa and Bruce won a hamper at a competition. The hamper comprises 9 different items.

(a) How many ways can the 9 items be divided among Alisa and Bruce if each of them gets at least one item each?

(b) How many ways can a set of 3 or more items be selected from the 9 items?

Solution.

Part (a). Note that the total number of ways to distribute the items is given by $2^9 = 512$. Also note that the only way either of them does not receive an item is when the other party gets all the items. This can only occur twice (once when Alisa receives nothing, and once when Bruce receives nothing). Thus, the number of ways where both of them gets at least one item each is $512 - 2 = 510$.

Part (b). Observe that the number of ways to choose a set of n items from the original 9 is given by ${}^9C_n$. Hence, the required number of ways is given by $512-({}^9C_0+{}^9C_1+{}^9C_2) = 466$.

* * * * *

Problem 12. In how many ways can 12 different books be distributed among students A, B, C and D

(a) if A gets 5, B gets 4, C gets 2 and D gets 1?

(b) if each student gets 3 books each?

Solution.

Part (a). At the start, A gets to pick 5 books from the 12 available books. There are ${}^{12}C_5$ ways to do so. Next, B gets to pick 4 books from the $12 - 5 = 7$ remaining books. There are ${}^7C_4$ ways to do so. Similarly, there are ${}^3C_2$ ways for C to pick his book, and ${}^1C_1$ ways for D to pick his. Hence, there are a total of ${}^{12}C_5 \cdot {}^7C_4 \cdot {}^3C_2 \cdot {}^1C_1 = 83160$ ways for the 12 books to be distributed.

Part (b). Following a similar argument as in part (a), the number of ways the 12 books can be distributed is given by ${}^{12}C_3 \cdot {}^9C_3 \cdot {}^6C_3 \cdot {}^3C_3 = 369600$.

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Problem 13. 3 men, 2 women and 2 children are arranged to sit around a round table with 7 non-distinguishable seats. Find the number of ways if

(a) (i) the 3 men are to be together,

(ii) the 3 men are to be together, and the seats are numbered,

(b) no 2 men are to be adjacent to each other,

(c) only 2 men are adjacent to each other.

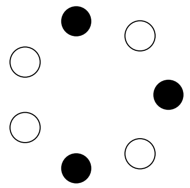
Solution.

Part (a).

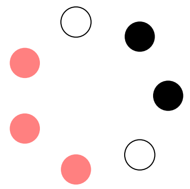
Part (a)(i). Consider the 3 men as one unit. Altogether, there are a total of 5 units, which gives a total of $(5 - 1)! = 4!$ ways for the 5 units to be arranged around the table. Since there are $3!$ ways to arrange the men, there are a total of $4! \cdot 3! = 144$ arrangements.

Part (a)(ii). Since there are a total of 7 distinguishable seats, the total number of arrangements is 7 times that of the number of arrangements with non-distinguishable seats. From part (a), this gives $144 \cdot 7 = 1008$ total arrangements.

Part (b). Observe that there is only one possible layout for no 2 men to be adjacent to each other (as shown in the diagram below). Since there are 4! ways to arrange the non-men, and 3! ways to arrange the men, there are a total of $4! \cdot 3! = 144$ arrangements.



Part (c). Observe that there are 3 possible layouts for only 2 men to be adjacent to each other (as shown in the diagram below). Since there are 4! ways to arrange the non-men, and 3! ways to arrange the men, there are a total of $3 \cdot 4! \cdot 3! = 432$ arrangements.



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Problem 14. Find the number of ways for 4 men and 4 boys to be seated alternately if they sit

(a) in a row,

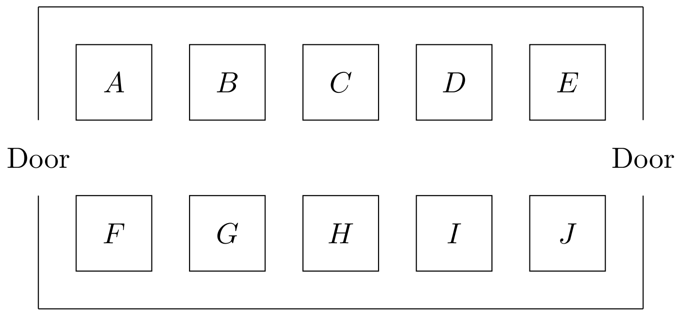
(b) at a round table.

Solution.

Part (a). Note that there are 2 possible layouts: one where a man sits at the start of the row, and one where a boy sits at the start of the row. Since there are 4! ways to arrange both the men and boys, there are a total of $2 \cdot 4! \cdot 4! = 1152$ arrangements.

Part (b). Given the rotational symmetry of the circle, there is now only one possible layout. Fixing one man, there are 3! ways to arrange the other men and 4! ways to arrange the boys, giving a total of $3! \cdot 4! = 144$ arrangements.

Problem 15. A rectangular shed, with a door at each end, contains ten fixed concrete bases marked A, B, C, $\ldots$, J, five on each side (see diagram). Ten canisters, each containing a different chemical, are placed with one canister on each base. In how many ways can the canisters be placed on the bases?



Find the number of ways in which the canisters can be placed

(a) if 2 particular canisters must not be placed on any of the 4 bases A, E, F and J next to a door,

(b) if 2 particular canisters must not be placed next to each other on the same side.

Solution. There are $10! = 3628800$ ways to place the canisters on the bases.

Part (a). Observe that there are ${}^6P_2$ possible placements for the two particular canisters. Since the other 8 canisters have no restrictions, the total number of ways to place the canisters is given by ${}^6P_2 \cdot 8! = 1209600$.

Part (b). Consider the number of ways the two particular canisters can be placed adjacently. There are $2 \cdot (5 - 1) = 8$ possible arrangements per side, giving a total of $2 \cdot 8 = 16$ possible arrangements. Since the other 8 canisters have no restrictions, the total number of ways to place the canisters is given by $16 \cdot 8! = 645120$. The required number of ways is thus given by $3628800 - 645120 = 2983680$.

Self-Practice A11

Problem 1. Find the number of three-letter code words that can be made using the letters of the word "THREE" if at least one of the letters is E.

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Problem 2. Eight people go to the theatre and sit in a particular group of eight adjacent reserved seats in the front row. Three of the eight belong to one family and sit together.

(a) If the other five people do not mind where they sit, find the number of possible seating arrangements for all eight people.

(b) If the other five people do not mind where they sit, except that two of them refuse to sit together, find the number of possible seating arrangements for all eight people.

* * * * *

Problem 3. A panel of judges in an essay competition has to select, and place in order of merit, 4 winning entries from a total entry of 20. Find the number of ways in which this can be done.

As a first step in the selection, 5 finalists are selected, without being placed in order. Find the number of ways in which this can be done.

All 20 essays are subsequently assessed by three panels of judges for content, accuracy and style, respectively, and three special prizes are awarded, one by each panel. Find the number of ways in which this can be done, assuming that an essay may win more than one prize.

* * * * *

Problem 4.

(a) A bookcase has four shelves with ten books on each shelf. Find the number of different selections that can be made by taking two books from each shelf (i.e. 8 books in all). Find also the number of different selections that can be made by taking eight books from each shelf (i.e. 32 books in all.)

(b) Eight cards each have a single digit written on them. The digits are 2, 2, 4, 5, 7, 7, 7, 7 respectively. Find the number of different 7-digit numbers that can be formed by placing seven of the cards side by side.

* * * * *

Problem 5. A team in a particular sport consists of 1 goalkeeper, 4 defenders, 2 midfielders and 4 attackers. A certain club has 3 goalkeepers, 8 defenders, 5 midfielders and 6 attackers.

(a) How many different teams can be formed by the club?

One of the midfielders in the club is the brother of one of the attackers in the club.

(b) How many different teams can be formed which include exactly one of the two brothers?

The two brothers leave the club. The club manager decides that one of the remaining midfielders can play either as a midfielder or a defender.

(c) How many different teams can now be formed by the club?

* * * * *

Problem 6. A group of 12 people consists of 6 married couples.

(a) The group stands in a line.
  (i) Find the number of different possible orders.
  (ii) Find the number of different possible orders in which each man stands next to his wife.

(b) The group stands in a circle.
  (i) Find the number of different possible arrangements.
  (ii) Find the number of different possible arrangements if men and women alternate.
  (iii) Find the number of different possible arrangements if each man stands next to his wife and men and women alternate.

* * * * *

Problem 7 ( ). A delegation of four students is to be selected from five badminton players, m floorball players, where $m > 3$, and six swimmers to attend the opening ceremony of the 2017 National Games. A pair of twins is among the floorball players. The delegation is to consist of at least one player from each sport.

(a) Show that the number of ways to select the delegation in which neither of the twins is selected is $k(m-2)(m+6)$, where k is an integer to be determined.

(b) Given that the number of ways to select a delegation in which neither of the twins is selected is more than twice the number of ways to select a delegation which includes exactly one of the twins, find the least value of m.

The pair of twins, one badminton player, one swimmer and two teachers, have been selected to attend a welcome lunch at the opening ceremony. Find the number of ways in which the group can be seated at a round table with distinguishable seats if the pair of twins is to be seated together and the teachers are separated.

Assignment A11

Problem 1. Find the number of different arrangements of seven letters in the word ADVANCE. Find the number of these arrangements which begin and end with "A" and in which "C" and "D" are always together.

Find the number of 4-letter code words that can be made from the letters of the word ADVANCE, using

(a) neither of the "A"s,

(b) both of the "A"s.

Solution. Tally of letters: 2 "A"s, 1 "D", 1 "V", 1 "N", 1 "C", 1 "E" (7 total, 6 distinct)

$$\text{Number of different arrangements} = \frac{7!}{2!} = 2520.$$

Since both "A"s are at the extreme ends, we are effectively finding the number of arrangements of the word "DVNCE" such that "C" and "D" are always together.

Let "C" and "D" be one unit. Altogether, there are 4 units. Hence,

$$\text{Required number of arrangements} = 4! \cdot 2 = 48.$$

Part (a). Without both "A"s, there are only 5 available letters to form the code words. This gives ${}^5C_4$ ways to select the 4 letters of the code word. Since each of the 5 remaining letters are distinct, there are 4! possible ways to arrange each word. This gives ${}^5C_4{\cdot}4! = 120$ such code words.

Part (b). With both "A"s included, we need another 2 letters from the 5 non-"A" letters. This gives ${}^5C_2$ ways to select the 4 letters of the code word. Since the 2 non-"A" letters are distinct, but the "A"s are repeated, there are $\frac{4!}{2!}$ possible ways to arrange each code word. This gives ${}^5C_2 \cdot \frac{4!}{2!} = 120$ such code words.

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Problem 2. A box contains 8 balls, of which 3 are identical (and so are indistinguishable from one another) and the other 5 are different from each other. 3 balls are to be picked out of the box; the order in which they are picked out does not matter. Find the number of different possible selections of 3 balls.

Solution. Note that there are 6 distinct balls in the box.

Case 1: *No identical balls chosen.* No. of selections = ${}^6C_3$

Case 2: *2 identical balls chosen.* No. of selections = ${}^5C_1$

Case 3: *3 identical balls chosen.* No. of selections = ${}^3C_3$

Hence, the total number of selections is given by ${}^6C_3 + {}^5C_1 + {}^3C_3 = 26$.

* * * * *

Problem 3. The management board of a company consists of 6 men and 4 women. A chairperson, a secretary and a treasurer are chosen from the 10 members of the board. Find the number of ways the chairperson, the secretary and the treasurer can be chosen so that

(a) they are all women,

(b) at least one is a woman and at least one is a man.

The 10 members of the board sit at random around a round table. Find the number of ways that

(c) the chairperson, the secretary and the treasurer sit in three adjacent places.

(d) the chairperson, the secretary and the treasurer are all separated from each other by at least one other person.

(Extension) What if the seats around the table are numbered? Try parts (c) and (d) again.

Solution.

Part (a). Since there are 4 women and 3 distinct roles, the required number of ways is given by ${}^{4}P_3 = 24$.

Part (b). Note that the number of ways that all three positions are men is given by ${}^{6}P_3$, while the number of ways to choose without restriction is given by ${}^{10}P_3$. Hence, the required number of ways is given by ${}^{10}P_3 - {}^{6}P_3 - 24 = 576$.

Part (c). Consider the three positions as one unit. This gives 8 units altogether. There are hence $(8-1)! \cdot 3! = 30240$ ways.

Part (d). Seat the seven other people first. There are $(7-1)!$ ways to do so. Then, slot in the three positions in the 7 slots. There are ${}^{7}C_3 \cdot 3!$ ways to do so. Hence, the required number of ways is given by $(7-1)! \cdot {}^{7}C_3 \cdot 3! = 151200$.

Extension. Since the seats are numbered, the number of ways scales up by the number of seats, i.e. 10. Hence, the number of ways becomes 302400 and 1512000.

A12 Probability

Tutorial A12

Problem 1. A and B are two independent events such that $\mathrm{P}(A) = 0.2$ and $\mathrm{P}(B) = 0.15$. Evaluate the following probabilities.

(a) $\mathrm{P}(A \mid B)$,

(b) $\mathrm{P}(A \cap B)$,

(c) $\mathrm{P}(A \cup B)$.

Solution.
Part (a). Since A and B are independent, $\mathrm{P}(A \mid B) = \mathrm{P}(A) = 0.2$.
Part (b). Since A and B are independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = 0.2 \cdot 0.15 = 0.03$.
Part (c). $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.2 + 0.15 - 0.03 = 0.32$.

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Problem 2. Two events A and B are such that $\mathrm{P}(A) = \frac{8}{15}$, $\mathrm{P}(B) = \frac{1}{3}$ and $\mathrm{P}(A \mid B) = \frac{1}{5}$. Calculate the probabilities that

(a) both events occur,

(b) only one of the two events occurs,

(c) neither event occurs.

Determine if event A and B are mutually exclusive or independent.

Solution.
Part (a).

$$\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A \mid B) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}.$$

Part (b).

$$\begin{aligned}\mathrm{P}(\text{only one occurs}) &= \mathrm{P}(A \cup B) - \mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - 2\mathrm{P}(A \cap B) \\ &= \frac{8}{15} + \frac{1}{3} - 2\left(\frac{1}{15}\right) = \frac{11}{15}.\end{aligned}$$

Part (c).

$$\mathrm{P}(\text{neither occurs}) = 1 - \mathrm{P}(\text{at least one occurs}) = 1 - \left(\frac{1}{15} - \frac{11}{15}\right) = \frac{1}{5}.$$

Since $\mathrm{P}(A) = \frac{8}{15} \neq \frac{1}{5} = \mathrm{P}(A \mid B)$, it follows that A and B are not independent. Also, since $\mathrm{P}(A \cap B) = \frac{1}{15} \neq 0$, the two events are also not mutually exclusive.

Problem 3. Two events A and B are such that $\mathrm{P}(A) = \mathrm{P}(B) = p$ and $\mathrm{P}(A \cup B) = \frac{5}{9}$.

(a) Given that A and B are independent, find a quadratic equation satisfied by p.

(b) Hence, find the value of p and the value of $\mathrm{P}(A \cap B)$.

Solution.

Part (a). Since A and B are independent, we have $\mathrm{P}(A \mid B) = \mathrm{P}(A) = p$. Hence,

$$p = \mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)}{\mathrm{P}(B)} = \frac{p + p - 5/9}{p} = 2 - \frac{5}{9p}$$
$$\implies 9p^2 = 18p - 5 \implies 9p^2 - 18p + 5 = 0.$$

Part (b). Observe that $9p^2 - 18p + 5 = (3p - 1)(3p - 5)$. Thus, $p = \frac{1}{3}$. Note that $p \neq \frac{5}{3}$ since $0 < p \leq 1$.

Since A and B are independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

* * * * *

Problem 4. Two players A and B regularly play each other at chess. When A has the first move in a game, the probability of A winning that game is 0.4 and the probability of B winning that game is 0.2. When B has the first move in a game, the probability of B winning that game is 0.3 and the probability of A winning that game is 0.2. Any game of chess that is not won by either player ends in a draw.

(a) Given that A and B toss a fair coin to decide who has the first move in a game, find the probability of the game ending in a draw.

(b) To make their games more enjoyable, A and B agree to change the procedure for deciding who has the first move in a game. As a result of their new procedure, the probability of A having the first move in any game is p. Find the value of p which gives A and B equal chances of winning each game.

Solution.

Part (a).

$$\begin{aligned}\mathrm{P}(\text{draw}) &= \mathrm{P}(A \text{ first})\mathrm{P}(\text{draw} \mid A \text{ first}) + \mathrm{P}(B \text{ first})\mathrm{P}(\text{draw} \mid B \text{ first}) \\ &= 0.5 \cdot (1 - 0.4 - 0.2) + 0.5 \cdot (1 - 0.3 - 0.2) = 0.45.\end{aligned}$$

Part (b). Observe that

$$\begin{aligned}\mathrm{P}(A \text{ wins}) &= \mathrm{P}(A \text{ first})\mathrm{P}(A \text{ wins} \mid A \text{ first}) + \mathrm{P}(B \text{ first})\mathrm{P}(A \text{ wins} \mid B \text{ first}) \\ &= p \cdot 0.4 + (1 - p) \cdot 0.2 = 0.2p + 0.2\end{aligned}$$

and

$$\begin{aligned}\mathrm{P}(B \text{ wins}) &= \mathrm{P}(A \text{ first})\mathrm{P}(B \text{ wins} \mid A \text{ first}) + \mathrm{P}(B \text{ first})\mathrm{P}(B \text{ wins} \mid B \text{ first}) \\ &= p \cdot 0.2 + (1 - p) \cdot 0.3 = -0.1p + 0.3\end{aligned}$$

Consider $\mathrm{P}(A \text{ wins}) = \mathrm{P}(B \text{ wins})$. Then $0.2p + 0.2 = -0.1p + 0.3 \implies p = \frac{1}{3}$.

Problem 5. Two fair dices are thrown, and events A, B and C are defined as follows:

- A: the sum of the two scores is odd,
- B: at least one of the two scores is greater than 4,
- C: the two scores are equal.

Find, showing your reasons clearly in each case, which two of these three events are

(a) mutually exclusive,

(b) independent.

Find also $\mathrm{P}(C \mid B)$, making your method clear.

Solution.

Part (a). Let the scores of the first and second die be p and q respectively. Suppose A occurs. Then p and q are of different parities (e.g. p even $\implies$ q odd). Thus, p and q cannot be equal. Hence, C cannot occur, whence A and C are mutually exclusive.

Part (b). Let the scores of the first and second die be p and q respectively. Observe that p is independent of q, and vice versa. Hence, the parity of q is not affected by the parity of p. Thus, $\mathrm{P}(A) = \mathrm{P}(p \text{ even})\mathrm{P}(q \text{ odd}) + \mathrm{P}(p \text{ odd})\mathrm{P}(q \text{ even}) = \frac{3}{6} \cdot \frac{3}{6} + \frac{3}{6} \cdot \frac{3}{6} = \frac{1}{2}$.

We also have $\mathrm{P}(B) = 1 - \mathrm{P}(\text{neither } p \text{ nor } q \text{ is greater than } 4) = 1 - \left(\frac{4}{6}\right)^2 = \frac{20}{36}$.

$p \backslash q$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

We now consider $\mathrm{P}(A \cap B)$. From the table of outcomes above, it is clear that $\mathrm{P}(A \cap B) = \frac{10}{36} = \mathrm{P}(A)\mathrm{P}(B)$. Hence, A and B are independent.

* * * * *

Problem 6. For events A and B, it is given that $\mathrm{P}(A) = 0.7$, $\mathrm{P}(B) = 0.6$ and $\mathrm{P}(A \mid B') = 0.8$. Find

(a) $\mathrm{P}(A \cap B')$,

(b) $\mathrm{P}(A \cup B)$,

(c) $\mathrm{P}(B' \mid A)$.

For a third event C, it is given that $\mathrm{P}(C) = 0.5$ and that A and C are independent.

(d) Find $\mathrm{P}(A' \cap C)$.

(e) Hence, find an inequality satisfied by $\mathrm{P}(A' \cap B \cap C)$ in the form

$$p \leq \mathrm{P}\big(A' \cap B \cap C\big) \leq q,$$

where p and q are constants to be determined.

Solution.

Part (a).

$$\mathrm{P}\big(A \cap B'\big) = \mathrm{P}\big(B'\big)\mathrm{P}\big(A \mid B'\big) = (1 - 0.6) \cdot 0.8 = 0.32.$$

Part (b).

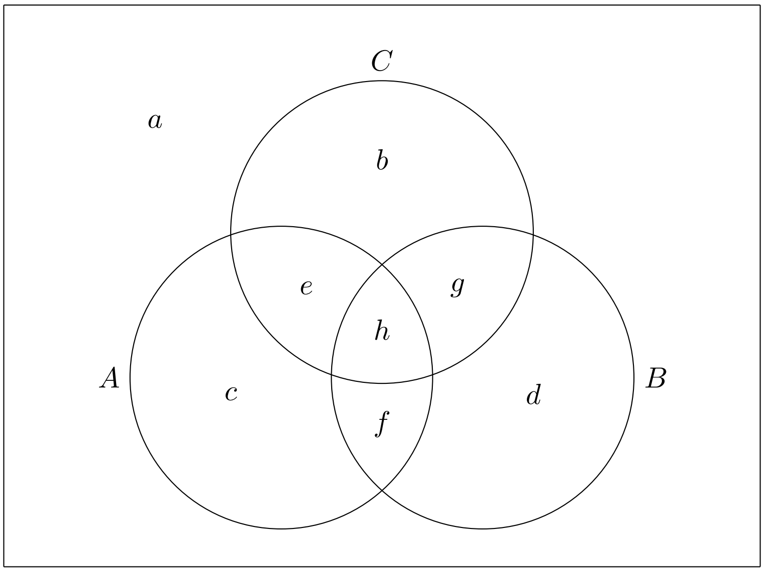
$$\begin{aligned}\mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \big[\mathrm{P}(A) - \mathrm{P}\big(A \cap B'\big)\big] \\ &= 0.7 + 0.6 - (0.7 - 0.32) = 0.92.\end{aligned}$$

Part (c).

$$\mathrm{P}\big(B' \mid A\big) = \frac{\mathrm{P}(B' \cap A)}{\mathrm{P}(A)} = \frac{0.32}{0.7} = \frac{16}{35}.$$

Part (d). Since A and C are independent, $\mathrm{P}(A \cap C) = \mathrm{P}(A)\mathrm{P}(C)$. Hence, $\mathrm{P}(A' \cap C) = \mathrm{P}(C) - \mathrm{P}(A \cap C) = 0.5 - 0.7 \cdot 0.5 = 0.15$.

Part (e). Consider the following Venn diagram.



Note that $\mathrm{P}(A' \cap B \cap C) = g$. Firstly, from part (d), we have $b + g = \mathrm{P}(A' \cap C) = 0.15$. Hence, $g \leq 0.15$. Secondly, from part (b), we have $a + b = 1 - \mathrm{P}(A \cup B) = 1 - 0.92 = 0.08$. Hence, $b \leq 0.08 \implies g \geq 0.07$. Lastly, we know that $\mathrm{P}(A' \cap B) = \mathrm{P}(A \cup B) - \mathrm{P}(A) = 0.92 - 0.7 = 0.22$. Hence, $d + g = 0.22 \implies g \leq 0.22$.

Thus, $0.07 \leq g \leq 0.15$, whence $0.07 \leq \mathrm{P}(A' \cap B \cap C) \leq 0.15$.

* * * * *

Problem 7. Camera lenses are made by two companies, A and B. 60% of all lenses are made by A and the remaining 40% by B. 5% of the lenses made by A are faulty. 7% of the lenses made by B are faulty.

(a) One lens is selected at random. Find the probability that

  (i) it is faulty,

  (ii) it was made by A, given that it is faulty.

(b) Two lenses are selected at random. Find the probability that both were made by A, given that exactly one is faulty.

(c) Ten lenses are selected at random. Find the probability that exactly two of them are faulty.

Solution.

Part (a).

Part (a)(i).

$$\mathrm{P}(\text{faulty}) = \mathrm{P}(A \cup \text{faulty}) + \mathrm{P}(B \cup \text{faulty}) = 0.6 \cdot 0.05 + 0.4 \cdot 0.07 = 0.058.$$

Part (a)(ii).

$$\mathrm{P}(A \mid \text{faulty}) = \frac{\mathrm{P}(A \cap \text{faulty})}{\mathrm{P}(\text{faulty})} = \frac{0.6 \cdot 0.05}{0.058} = \frac{15}{19}.$$

Part (b).

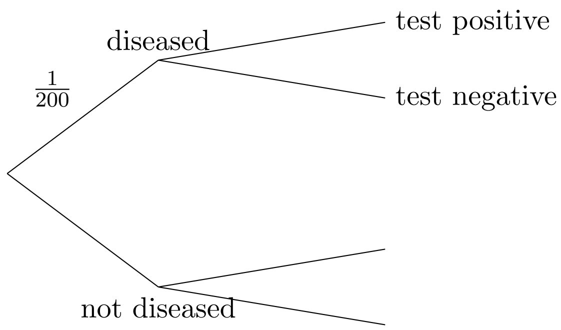
$$\mathrm{P}(\text{both } A \mid \text{one faulty}) = \frac{\mathrm{P}(\text{both } A \cup \text{one faulty})}{\mathrm{P}(\text{one faulty})} = \frac{[0.6 \cdot 0.05] \cdot [0.6 \cdot (1 - 0.05)]}{0.058 \cdot (1 - 0.058)} = \frac{1425}{4553}.$$

Part (c).

$$\mathrm{P}(\text{two faulty}) = 0.058^2(1 - 0.058)^8 \cdot \frac{10!}{2!8!} = 0.0939 \text{ (3 s.f.)}$$

* * * * *

Problem 8. A certain disease is present in 1 in 200 of the population. In a mass screening programme a quick test of the disease is used, but the test is not totally reliable. For someone who does have the disease there is a probability of 0.9 that the test will prove positive, whereas for someone who does not have the disease there is a probability of 0.02 that the test will prove positive.



(a) One person is selected at random and test.

  (i) Copy and complete the tree diagram, which illustrates one application of the test.

  (ii) Find the probability that the person has the disease and the test is positive.

  (iii) Find the probability that the test is negative.

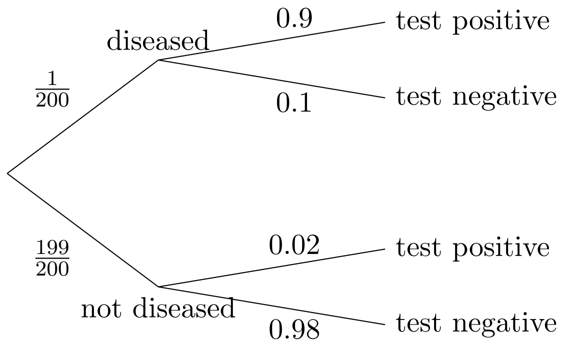
  (iv) Given that the test is positive, find the probability that the person has the disease.

(b) People for whom the test proves positive are recalled and re-tested. Find the probability that a person has the disease if the second test also proves positive.

Solution.

Part (a).

Part (a)(i).



Part (a)(ii).

$$P(\text{diseased} \cap \text{positive}) = \frac{1}{200} \cdot 0.9 = 0.0045.$$

Part (a)(iii).

$$P(\text{negative}) = \frac{1}{200} \cdot 0.1 + \frac{199}{200} \cdot 0.98 = 0.9756.$$

Part (a)(iv).

$$P(\text{diseased} \mid \text{positive}) = \frac{P(\text{diseased} \cap \text{positive})}{P(\text{positive})} = \frac{0.0045}{1 - 0.9756} = 0.184.$$

Part (b).

$$\begin{aligned}\text{Required probability} &= \frac{P(\text{diseased} \cap \text{both positive})}{P(\text{both positive})} \\ &= \frac{P(\text{diseased} \cap \text{both positive})}{P(\text{diseased} \cap \text{both positive}) + P(\text{not diseased} \cap \text{both positive})} \\ &= \frac{1/200 \cdot 0.9^2}{1/200 \cdot 0.9^2 + 199/200 \cdot 0.02^2} = \frac{2025}{2224}.\end{aligned}$$

* * * * *

Problem 9. In a probability experiment, three containers have the following contents.

- A jar contains 2 white dice and 3 black dice.
- A white box contains 5 red balls and 3 green balls.
- A black box contains 4 red balls and 3 green balls.

One die is taken at random from the jar. If the die is white, two balls are taken from the white box, at random and without replacement. If the die is black, two balls are taken from the black box, at random and without replacement. Events W and M are defined as follows:

- W: A white die is taken from the jar.
- M: One red ball and one green ball are obtained.

Show that $P(M \mid W) = \frac{15}{28}$.

Find, giving each of your answers as an exact fraction in its lowest terms,

(a) $P(M \cap W)$,

(b) $P(W \mid M)$,

(c) $P(W \cup M)$.

All the dice and balls are now placed in a single container, and four objects are taken at random, each object being replaced before the next one is taken. Find the probability that one object of each colour is obtained.

Solution. Since W has occurred, both red and green balls must come from the white box. Note that there are two ways for M to occur: first a red then a green, or first a green then a red. Hence, $P(M \mid W) = \frac{5}{8} \cdot \frac{3}{7} + \frac{3}{8} \cdot \frac{5}{7} = \frac{15}{28}$ as desired.

Part (a).

$$P(M \cup W) = P(W)P(M \mid W) = \frac{2}{5} \cdot \frac{15}{28} = \frac{3}{14}.$$

Part (b). Let B represent the event that a black die is taken from the jar. Then

$$\begin{aligned} P(M) &= P(M \cap W) + P(M \cap B) = P(M \cap W) + P(B)P(M \mid B) \\ &= \frac{3}{14} + \frac{3}{5}\left(\frac{4}{7} \cdot \frac{3}{6} + \frac{3}{7} \cdot \frac{4}{6}\right) = \frac{39}{70}. \end{aligned}$$

Hence, $P(W \mid M) = \frac{P(W \cap M)}{P(M)} = \frac{3/14}{39/70} = \frac{5}{13}$.

Part (c).

$$P(W \cup M) = P(W) + P(M) - P(W \cap M) = \frac{2}{5} + \frac{39}{70} - \frac{3}{14} = \frac{26}{35}.$$

Note that the container has 2 white objects, 3 black objects, 9 red objects and 6 green objects, for a total of 20 objects. The probability that one object of each colour is taken is thus given by

$$\frac{2}{20} \cdot \frac{3}{20} \cdot \frac{9}{20} \cdot \frac{6}{20} \cdot 4! = \frac{243}{5000}.$$

* * * * *

Problem 10. A man writes 5 letters, one each to A, B, C, D and E. Each letter is placed in a separate envelope and sealed. He then addresses the envelopes, at random, one each to A, B, C, D and E.

(a) Find the probability that the letter to A is in the correct envelope and the letter to B is in an incorrect envelope.

(b) Find the probability that the letter to A is in the correct envelope, given that the letter to B is in an incorrect envelope.

(c) Find the probability that both the letters to A and B are in incorrect envelopes.

Solution.

Part (a).

$$\mathrm{P}(A \text{ correct} \cap B \text{ incorrect}) = \frac{1}{5} \times \frac{3}{4} = \frac{3}{20}.$$

Part (b).

$$\mathrm{P}(A \text{ correct} \mid B \text{ incorrect}) = \frac{\mathrm{P}(A \text{ correct} \cap B \text{ incorrect})}{\mathrm{P}(B \text{ incorrect})} = \frac{3/20}{4/5} = \frac{3}{16}.$$

Part (c).

$$\begin{aligned}\mathrm{P}(A \text{ incorrect} \cap B \text{ incorrect}) &= \mathrm{P}(B \text{ incorrect})\mathrm{P}(A \text{ incorrect} \mid B \text{ incorrect}) \\ &= \frac{4}{5}\left(1 - \frac{3}{16}\right) = \frac{13}{20}.\end{aligned}$$

* * * * *

Problem 11. A bag contains 4 red counters and 6 green counters. Four counters are drawn at random from the bag, without replacement. Calculate the probability that

(a) all the counters drawn are green,

(b) at least one counter of each colour is drawn,

(c) at least two green counters are drawn,

(d) at least two green counters are drawn, given that at least one counter of each colour is drawn.

State with a reason whether the events "at least two green counters are drawn" and "at least one counter of each colour is drawn" are independent.

Solution.

Part (a).

$$\mathrm{P}(\text{all green}) = \frac{{}^6C_4}{10!/(4!\,6!)} = \frac{1}{14}.$$

Part (b).

$$\mathrm{P}(\text{one of each colour}) = 1 - \mathrm{P}(\text{all green}) - \mathrm{P}(\text{all red}) = 1 - \frac{1}{14} - \frac{{}^4C_4}{10!/(4!\,6!)} = \frac{97}{105}.$$

Part (c).

$$\mathrm{P}(\text{at least 2 green}) = 1 - \mathrm{P}(\text{no green}) - \mathrm{P}(\text{one green}) = 1 - \frac{1}{210} - \frac{{}^6C_1 \cdot {}^4C_3}{10!/(4!\,6!)} = \frac{37}{42}.$$

Part (d).

$$\mathrm{P}(\text{at least 2 green} \mid \text{one of each colour}) = \frac{{}^6C_3 \cdot {}^4C_1 + {}^6C_2 \cdot {}^4C_2}{10!/(4!\,6!) - {}^6C_4 - {}^4C_4} = \frac{85}{97}.$$

Since $\mathrm{P}(\text{at least 2 green}) = \frac{37}{42} \neq \frac{85}{97} = \mathrm{P}(\text{at least 2 green} \mid \text{one of each colour})$, the two events are not independent.

Problem 12. A group of fifteen people consists of one pair of sisters, one set of three brothers and ten other people. The fifteen people are arranged randomly in a line.

(a) Find the probability that the sisters are next to each other.

(b) Find the probability that the brother are not all next to one another.

(c) Find the probability that either the sisters are next to each other or the brothers are all next to one another or both.

(d) Find the probability that the sisters are next to each other given that the brothers are not all next to one another.

Solution.

Part (a). Let the two sisters be one unit. There are hence 14 units altogether, giving $14! \cdot 2!$ arrangements with the restriction. Since there are a total of 15! arrangements without the restriction, the required probability is $\frac{14! \cdot 2!}{15!} = \frac{2}{15}$.

Part (b). Consider the case where all brothers are next to one another. Counting the brothers as one unit gives 13 units altogether. There are hence $13! \cdot 3!$ arrangements with this restriction. Since there are a total of 15! arrangements without the restriction, the probability that all three brothers are not together is given by $\frac{13! \cdot 3!}{15!} = \frac{34}{35}$.

Part (c). Consider the case where both the sisters are adjacent, and all three brothers are next to one another. Counting the sisters as one unit, and counting the brothers as one unit gives 12 units altogether. There are hence $12! \cdot 2! \cdot 3!$ arrangements with this restriction. Since there are a total of 15! arrangements without the restriction, we have

$$\mathrm{P}(\text{sisters together} \cap \text{brothers together}) = \frac{12! \cdot 2! \cdot 3!}{15!} = \frac{2}{455}.$$

Hence,

$$\begin{aligned} &\mathrm{P}(\text{sisters together} \cup \text{brothers together}) \\ &= \mathrm{P}(\text{sisters together}) + \mathrm{P}(\text{brothers together}) - \mathrm{P}(\text{sisters together} \cap \text{brothers together}) \\ &= \frac{2}{15} + \left(1 - \frac{1}{35}\right) - \frac{2}{455} = \frac{43}{273}. \end{aligned}$$

Part (d). Note that

$$\begin{aligned} &\mathrm{P}(\text{sisters together} \cap \text{brothers not together}) \\ &= \mathrm{P}(\text{sisters together}) - \mathrm{P}(\text{sisters together} \cap \text{brothers together}) \\ &= \frac{2}{15} - \frac{2}{455} = \frac{176}{1365}. \end{aligned}$$

Hence, the required probability can be calculated as

$$\begin{aligned} \mathrm{P}(\text{sisters together} \mid \text{brothers not together}) &= \frac{\mathrm{P}(\text{sisters together} \cap \text{brothers not together})}{\mathrm{P}(\text{brothers not together})} \\ &= \frac{176/1365}{34/35} = \frac{88}{663}. \end{aligned}$$

Self-Practice A12

Problem 1. Two events A and B are such that $P(A) = 0.6$, $P(B) = 0.3$, $P(A \mid B) = 0.2$. Calculate the probabilities that

(a) both events occur,

(b) at least one of the two events occurs,

(c) exactly one of the events occur.

* * * * *

Problem 2. For events A and B, it is given that $P(A) = 0.7$, $P(B \mid A') = 0.8$, $P(A \mid B') = 0.88$. Find

(a) $P(B \cap A')$,

(b) $P(A' \cap B')$,

(c) $P(A \cap B)$.

* * * * *

Problem 3. A group of student representatives is to be chosen from three schools, R, S and T. The group is to consist of 10 students and is chosen from a set of 15 students consisting of 3 from R, 4 from S and 8 from T. Find the probability that the group consists of

(a) students from S and T only,

(b) at least one student from each school.

* * * * *

Problem 4. A box contains 25 apples, of which 20 are red and 5 are green. Of the red apples, 3 contain maggots and of the green apples, 1 contains maggots. Two apples are chosen at random from the box. Find, in any order,

(a) the probability that both apples contain maggots.

(b) the probability that both apples are red and at least one contains maggots.

(c) the probability that at least one apple contains maggots, given that both apples are red.

(d) the probability that both apples are red given that at least one apple is red.

* * * * *

Problem 5. A bag contains 15 tokens that are indistinguishable apart from their colours. 2 of the tokens are blue and the rest are either red or green. Participants are required to draw the tokens randomly, one at a time, from the bag without replacement.

(a) Given that the probability that a participant draws 2 red tokens on the first 2 draws is $1/35$, show that there are 3 red tokens in the bag.

(b) Find the probability that a participant draws a red or green token on the second draw.

Events A and B are defined as follows.

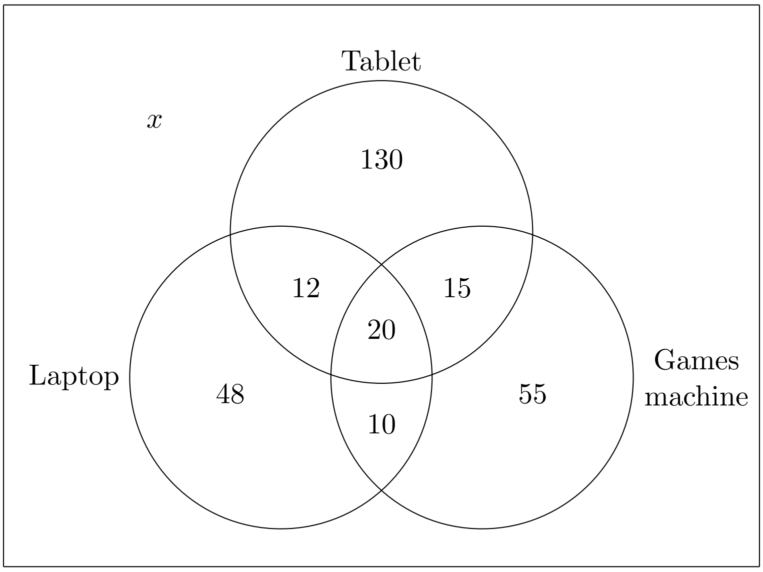
- A: A participant draws his/her second red token on the third draw.
- B: A participant draws a blue token on the second draw.

(c) Find $\mathrm{P}(A \cup B)$.

(d) Determine if A and B are independent events.

* * * * *

Problem 6.



A group of students is asked whether they own any of a laptop, a tablet and a games machine. The numbers owning different combinations are shown in the Venn diagram. The number of students owning none of these is x. One of the students is chosen at random.

- L is the event that the student owns a laptop.
- T is the event that the student owns a tablet.
- G is the event that the student owns a game machine.

(a) Write down expressions for $\mathrm{P}(L)$ and $\mathrm{P}(G)$ in terms of x. Given that L and G are independent, show that $x = 10$.

Using this value of x, find

(b) $\mathrm{P}(L \cup T)$,

(c) $\mathrm{P}(T \cap G')$,

(d) $\mathrm{P}(L \mid G)$.

Two students from the whole group are chosen at random.

(e) Find the probability that both of these students each owns exactly two out of the three items (laptop, tablet, games machine).

* * * * *

Problem 7. A group of students takes an examination in Science. A student who fails the examination at the first attempt is allowed one further attempt. For a randomly chosen student, the probability of passing the examination at the first attempt is p. If the student fails the examination at the first attempt, the probability of passing at the second attempt is 0.3 more than the probability of passing the examination at the first attempt.

(a) Show that the probability that a randomly chosen student passes the examination is $0.3 + 1.7p - p^2$.

Find the value of p such that the probability that a randomly chosen student passes the examination on the first attempt given that the student passes is 0.6.

Two students are randomly chosen.

(b) (i) Find the probability that one passes the examination on the first attempt and the other passes the examination on the second attempt, leaving your answer in terms of p.

(ii) Find the value of p such that the value of the probability in part (i) is maximum.

* * * * *

Problem 8. In Haha College, 70% of the students watch the show *Jogging Man* and 60% of the students watch the show *Voice of Me*. 40% of those who do not watch the show *Voice of Me* watch the show *Jogging Man*. Find the probability that a student chosen at random from the college

(a) watches both shows,

(b) watches exactly one show,

(c) watches the show *Voice of Me* given that the student does not watch the show *Jogging Man*.

State, with a reason, whether the events ‘watches *Jogging Man*’ and ‘watches *Voice of Me*’ are independent.

* * * * *

Problem 9. For events A and B, it is given that $\mathrm{P}(A) = 2/3$ and $\mathrm{P}(B) = 1/2$.

(a) State an inequality satisfied by $\mathrm{P}(A \cap B)$.

It is given further that A and B are independent. Find

(b) $\mathrm{P}(A \cap B)$,

(c) $\mathrm{P}(A' \cup B)$.

* * * * *

Problem 10 ( ). A fast food restaurant gives away a free action figure for every child’s meal bought. There are five different action figures and each figure is equally likely to be given away with a child’s meal. A customer intends to collect all five different figures by buying child’s meals.

(a) Find the probability that the first 4 child's meals bought by the customer all had different action figures.

(b) Two of the five action figures are X and Y. Find the probability that the first 4 action figures obtained result in the customer having at least one X or one Y or both.

(c) Find the probability that the first 4 child's meals bought by the customer had exactly two different action figures.

(d) At a certain stage, the customer collected 4 of the five action figures. Given that the probability of the customer completing the set by at most n meals is larger than 0.95, find the least value of n.

Assignment A12

Problem 1.

(a) Events A and B are such that $\mathrm{P}(A) = 0.4$, $\mathrm{P}(B) = 0.3$ and $\mathrm{P}(A \cup B) = 0.5$.

(i) Determine whether A and B are mutually exclusive.

(ii) Determine whether A and B are independent.

(b) In a competition, 2 teams (A and B) will play each other in the best of 3 games. That is, the first team to win 2 games will be the winner and the competition will end. In the first game, both teams have equal chances of winning. In subsequent games, the probability of team A winning team B given that team A won in the previous game is p and the probability of team A winning team B given that team A lost in the previous game is $\frac{1}{3}$.

(i) Illustrate the information with an appropriate tree diagram.

(ii) Find the value of p such that team A has equal chances of winning and losing the competition.

Solution.

Part (a).

Part (a)(i). Note that

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) = 0.4 + 0.3 - 0.5 = 0.2.$$

Since $\mathrm{P}(A \cap B) = 0.2 \neq 0$, A and B are not mutually exclusive.

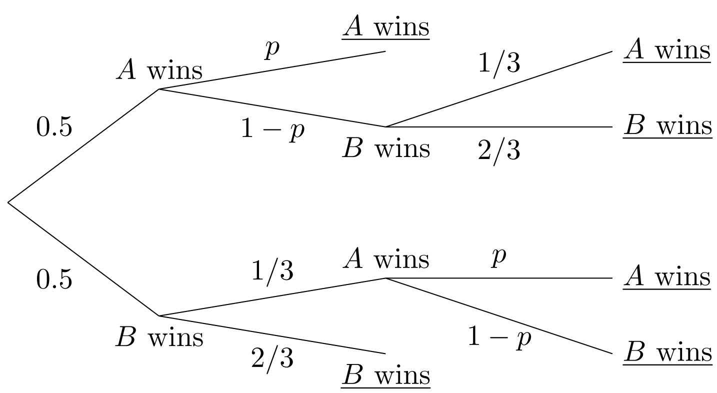
Part (a)(ii). Note that

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{0.2}{0.3} = \frac{2}{3}.$$

Since $\mathrm{P}(A) = 0.4 \neq \frac{2}{3} = \mathrm{P}(A \mid B)$, A and B are not independent.

Part (b).

Part (b)(i).



Part (b)(ii). Consider

$$\mathrm{P}(A \text{ wins competition}) = \left[\frac{1}{2} \cdot p\right] + \left[\frac{1}{2} \cdot (1-p) \cdot \frac{1}{3}\right] + \left[\frac{1}{2} \cdot \frac{1}{3} \cdot p\right] = \frac{p}{2} + \frac{1}{6} = \frac{1}{2}.$$

We hence need $p = \frac{2}{3}$ for A to have equal chances of winning and losing.

Problem 2. A Personal Identification Number (PIN) consists of 4 digits in order, where each digit ranges from 0 to 9. Susie has difficulty remembering her PIN. She tries to remember her PIN and writes down what she thinks it is. The probability that the first digit is correct is 0.8 and the probability that the second digit is correct is 0.86. The probability that the first two digits are both correct is 0.72. Find

(a) the probability that the second digit is correct given that the first digit is correct,

(b) the probability that the first digit is correct, and the second digit is incorrect,

(c) the probability that the second digit is incorrect given that the first digit is incorrect.

Solution. Let $1D$ be the event that the first digit is correct, and $2D$ be the event that the second digit is correct. We have $\mathrm{P}(1D) = 0.8$, $\mathrm{P}(2D) = 0.86$, and $\mathrm{P}(1D \cap 2D) = 0.72$.

Part (a).

$$\mathrm{P}(2D \mid 1D) = \frac{\mathrm{P}(2D \cap 1D)}{\mathrm{P}(1D)} = \frac{0.72}{0.8} = 0.9.$$

Part (b).

$$\mathrm{P}\big(1D \cap 2D'\big) = \mathrm{P}(1D) - \mathrm{P}(1D \cap 2D) = 0.8 - 0.72 = 0.08.$$

Part (c).

$$\mathrm{P}\big(2D' \mid 1D'\big) = \frac{\mathrm{P}(2D' \cap 1D')}{\mathrm{P}(1D')} = \frac{1 - \mathrm{P}(1D \cup 2D)}{1 - \mathrm{P}(1D)}$$
$$= \frac{1 - [\mathrm{P}(1D) + \mathrm{P}(2D) - \mathrm{P}(1D \cap 2D)]}{1 - \mathrm{P}(1D)} = \frac{1 - (0.8 + 0.86 - 0.72)}{1 - 0.8} = 0.3.$$

* * * * *

Problem 3. An international tour group consists of the following seventeen people: a pair of twin sisters and their boyfriends, all from Canada; three policewomen from China; a married couple and their two daughters from Singapore, and a large family from Indonesia, consisting of a man, his wife, his parents and his two sons.

Four people from the group are randomly chosen to play a game. Find the probability that

(a) the four people are all of different nationalities,

(b) the four people are all the same gender,

(c) the four people are all of different nationalities, given that they are all the same gender.

Solution.

TALLY	Male	Female	SUBTOTAL
Canada	2	2	4
China	0	3	3
Singapore	1	3	4
Indonesia	4	2	6
SUBTOTAL	7	10	17

Part (a).

$$\mathrm{P}(\text{all different nationalities}) = \frac{4}{17} \cdot \frac{3}{16} \cdot \frac{4}{15} \cdot \frac{6}{14} \cdot 4! = \frac{72}{595}.$$

Part (b).

$$\mathrm{P}(\text{all same gender}) = \frac{{}^{7}C_4 + {}^{10}C_4}{{}^{17}C_4} = \frac{7}{68}.$$

Part (c).

$$\mathrm{P}(\text{all different nationalities} \mid \text{all female}) = \frac{2}{17} \cdot \frac{3}{16} \cdot \frac{3}{15} \cdot \frac{2}{14} \cdot 4! = \frac{9}{595}$$

Note that P(all different nationalities | all male) since there are no males from China, whence

$$\begin{aligned}
&\mathrm{P}(\text{all different nationalities} \mid \text{all same gender}) \\
&= \frac{\mathrm{P}(\text{all different nationalities} \cap \text{all same gender})}{\mathrm{P}(\text{all same gender})} = \frac{9/595 + 0}{7/68} = \frac{36}{245}.
\end{aligned}$$

A14A Discrete Random Variables

Tutorial A14A

Problem 1. An unbiased die is in the form of a regular tetrahedron and has its faces numbered 1, 2, 3, 4. When the die is thrown on to a horizontal table, the number on the fact in contact with the table is noted. Two such dice are thrown and the score X is found by multiplying these numbers together. Obtain the probability distribution of X. Find the values of

(a) $\mathrm{P}(X > 8)$,

(b) $\mathrm{E}(X)$,

(c) $\mathrm{Var}(X)$.

Solution. The following table displays all possible outcomes.

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

Hence, the probability distribution is

x	1	2	3	4	6	8	9	12	16
$\mathrm{P}(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

Part (a).

$$\mathrm{P}(X > 8) = \mathrm{P}(X = 9) + \mathrm{P}(X = 12) + \mathrm{P}(X = 16) = \frac{1}{16} + \frac{2}{16} + \frac{1}{16} = \frac{1}{4}.$$

Part (b). Using G.C., $\mathrm{E}(X) = 6.25$.
Part (c). Using G.C., $\mathrm{Var}(X) = (4.14578)^2 = 17.2$.

* * * * *

Problem 2. A computer can give independent observations of a random variable X with probability distribution given by $\mathrm{P}(X = 0) = \frac{3}{4}$ and $\mathrm{P}(X = 2) = \frac{1}{4}$. It is programmed to output a value for the random variable Y defined by $Y = X_1 + X_2$, where X_1 and X_2 are two observations of X.

Tabulate the probability distribution of Y and show that $\mathrm{E}(Y) = 1$.

The random variable T is defined by $T = Y^2$. Find $\mathrm{E}(T)$ and show that $\mathrm{Var}(T) = \frac{63}{4}$.

Solution. Quite clearly, we have

$$\mathrm{P}(Y = 0) = \left(\frac{3}{4}\right)^2 = \frac{9}{16}, \quad \mathrm{P}(Y = 2) = 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{3}{8}, \quad \mathrm{P}(Y = 4) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

Thus, the probability distribution of Y is given by

y	0	2	4
$P(Y=y)$	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

Thus,

$$E(Y) = 0\left(\frac{9}{16}\right) + 2\left(\frac{3}{8}\right) + 4\left(\frac{1}{16}\right) = 1.$$

Note that $E(T) = E(Y^2)$ and $E(T^2) = E(Y^4)$. Hence,

$$E(T) = 0^2\left(\frac{9}{16}\right) + 2^2\left(\frac{3}{8}\right) + 4^2\left(\frac{1}{16}\right) = \frac{5}{2}$$

and

$$E(T^2) = 0^4\left(\frac{9}{16}\right) + 2^4\left(\frac{3}{8}\right) + 4^4\left(\frac{1}{16}\right) = 22.$$

Thus,

$$\mathrm{Var}(T) = E(T^2) - E(T)^2 = 22 - \left(\frac{5}{2}\right)^2 = \frac{63}{4}.$$

* * * * *

Problem 3. The discrete random variable X takes values -1, 0, 1 with probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ respectively. The variable $\bar{X}$ is the mean of a random sample of 3 values of X (i.e. X_1, X_2 and X_3 are independent random variables).

Tabulate the probability distribution of $\bar{X}$, and use your values to calculate $\mathrm{Var}(\bar{X})$. Hence, verify that $\mathrm{Var}(\bar{X}) = \frac{1}{3}\mathrm{Var}(X)$ in this case.

Solution. By symmetry, we have $P(\bar{X} = -n) = P(\bar{X} = n)$. Now, notice that the only way to get a total score of 3 is to have $X_1 = X_2 = X_3 = 1$. Thus,

$$P(\bar{X} = 1) = P(\bar{X} = -1) = \left(\frac{1}{4}\right)^3 = \frac{1}{64}.$$

Similarly, the only way to get a total score of 2 is to have two 1's and one 0. Thus,

$$P\left(\bar{X} = \frac{2}{3}\right) = P\left(\bar{X} = -\frac{2}{3}\right) = \binom{3}{1}\left(\frac{1}{4}\right)^2\left(\frac{1}{2}\right) = \frac{3}{32}.$$

Now note that there are two ways to achieve a total score of 1: have two 1's and one -1, or have two 0's and one 1. This gives

$$P\left(\bar{X} = \frac{1}{3}\right) = P\left(\bar{X} = -\frac{1}{3}\right) = \binom{3}{1}\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right) + \binom{3}{1}\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right) = \frac{15}{64}.$$

Lastly, by the complement principle, we have

$$P(\bar{X} = 0) = 1 - 2\left(\frac{1}{64} + \frac{3}{32} + \frac{15}{64}\right) = \frac{5}{16}.$$

Hence, the probability distribution of X is given by

$\bar{x}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$P(\bar{X}=\bar{x})$	$\frac{1}{64}$	$\frac{3}{32}$	$\frac{15}{64}$	$\frac{5}{16}$	$\frac{15}{64}$	$\frac{3}{32}$	$\frac{1}{64}$

We now calculate $\operatorname{Var}(\bar{X})$. Observe that the means of X and $\bar{X}$ are 0 by symmetry. Hence,

$$\operatorname{Var}(\bar{X}) = \operatorname{E}(\bar{X}^2) = 2\left[1^2\left(\frac{1}{64}\right) + \left(\frac{2}{3}\right)^2\left(\frac{3}{32}\right) + \left(\frac{1}{3}\right)^2\left(\frac{15}{64}\right)\right] = \frac{1}{6}.$$

Now, note that

$$\operatorname{Var}(X) = \operatorname{E}(X^2) = 2\left[1^2\left(\frac{1}{4}\right)\right] = \frac{1}{2}.$$

Thus,

$$\operatorname{Var}(\bar{X}) = \frac{1}{3}\operatorname{Var}(X).$$

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Problem 4. The probability of obtaining a head when a particular type of coin is tossed is p. The random variable X is the number of heads obtained when three such coins are tossed.

(a) Draw up a table showing the probability distribution of X.

(b) Prove that $\operatorname{E}\left(\frac{1}{3}X\right) = p$.

(c) Given that $p = \frac{1}{3}$, and denoting by E the event that $X > 1$, find the probability that in 100 throws of the three coins, E will not occur more than 30 times.

Solution.

Part (a). Observe that

$$\operatorname{P}(X = n) = \binom{3}{n} p^n (1-p)^n.$$

Hence, the probability distribution of X is given by

x	0	1	2	3
$\operatorname{P}(X = x)$	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3

Part (b). Note that

$$\operatorname{E}(X) = \sum_{n=0}^{3} n\binom{3}{n} p^n (1-p)^{3-n}.$$

Differentiating with respect to p, we get

$$0 = \sum_{n=0}^{3} \binom{3}{n} \left[np^{n-1}(1-p)^{3-n} - (3-n)p^n(1-p)^{3-n-1}\right].$$

Rearranging, we have

$$\left(\frac{1}{p} + \frac{1}{1-p}\right) \underbrace{\sum_{n=0} \binom{3}{n} np^n (1-p)^{3-n}}_{\operatorname{E}(X)} = \frac{3}{1-p} \underbrace{\sum_{n=0} \binom{3}{n} p^n (1-p)^{3-n}}_{1}.$$

Thus,

$$\operatorname{E}(X) = \frac{\frac{3}{1-p}}{\frac{1}{p} + \frac{1}{1-p}} = 3p \implies \operatorname{E}\left(\frac{1}{3}X\right) = \frac{1}{3}\operatorname{E}(X) = \frac{1}{3}(3p) = p.$$

Part (c). Note that

$$\mathrm{P}(E) = \mathrm{P}(X > 1) = \mathrm{P}(X = 2) + \mathrm{P}(X = 3) = 3\left(\frac{1}{3}\right)^2\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3}\right)^3 = \frac{7}{27}.$$

Now, observe that

$$\mathrm{P}(\#E = n) = \binom{100}{n}\left(\frac{7}{27}\right)^n\left(1 - \frac{7}{27}\right)^{100-n}.$$

Thus,

$$\mathrm{P}(\#E \leq 30) = \sum_{n=0}^{30}\binom{100}{n}\left(\frac{7}{27}\right)^n\left(1 - \frac{7}{27}\right)^{100-n} = 0.851.$$

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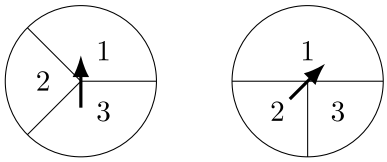
Problem 5.



A circular card is divided into 3 sectors 1, 2, 3 and having angles 135°, 90° and 135° respectively. On a second circular card, sectors scoring 1, 2, 3 have angles 180°, 90° and 90° respectively (see diagram). Each card has a pointer pivoted at its centre. After being set in motion, the pointers come to rest independently in random positions. Find the probability that

(a) the score on each card is 1,

(b) the score on at least one of the cards is 3.

The random variable X is the larger of the two scores if they are different, and their common value if they are the same. Show that $\mathrm{P}(X = 2) = \frac{9}{32}$.

Show that $\mathrm{E}(X) = \frac{75}{32}$ and find $\mathrm{Var}(X)$.

Solution.

Part (a). Clearly,

$$\mathrm{P}(\text{both scores are } 1) = \frac{135}{360} \cdot \frac{180}{360} = \frac{3}{16}.$$

Part (b). Likewise,

$$\mathrm{P}(\text{one score is } 3) = \frac{135}{360} + \frac{90}{360} - \frac{135}{360} \cdot \frac{90}{360} = \frac{17}{32}.$$

Observe that the event $X = 1$ is equivalent to both scores being 1, whence we have $\mathrm{P}(X = 1) = \frac{3}{16}$ from part (a). From part (b), we also have $\mathrm{P}(X = 3) = \frac{17}{32}$. Thus,

$$\mathrm{P}(X = 2) = 1 - \mathrm{P}(X = 1) - \mathrm{P}(X = 3) = 1 - \frac{3}{16} - \frac{17}{32} = \frac{9}{32}.$$

Note that

$$\mathrm{E}(X) = 1\left(\frac{3}{16}\right) + 2\left(\frac{9}{32}\right) + 3\left(\frac{17}{32}\right) = \frac{75}{32}$$

and

$$\mathrm{E}\big(X^2\big) = 1^2\left(\frac{3}{16}\right) + 2^2\left(\frac{9}{32}\right) + 3^2\left(\frac{17}{32}\right) = \frac{195}{32}.$$

Thus,

$$\mathrm{Var}(X) = \mathrm{E}\big(X^2\big) - \mathrm{E}(X)^2 = \frac{195}{32} - \left(\frac{75}{32}\right)^2 = \frac{615}{1024}.$$

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Problem 6. Alfred and Bertie play a game, each starting with cash amounting to \$100. Two dice are thrown. If the total score if 5 or more, then Alfred pays $\$x$, where $0 < x \leq 8$, to Bertie. If the total score if 4 or less, then Bertie pays $\$(x+8)$ to Alfred.

(a) Show that the expectation of Alfred's cash after the first game is $\$\frac{1}{3}(304 - 2x)$.

(b) Find the expectation of Alfred's cash after six games.

(c) Find the value of x for the game to be fair.

(d) Given that $x = 3$, find the variance of Alfred's cash after the first game.

Solution.

Part (a). Note that

$$\mathrm{P}(\text{score} < 5) = \frac{3+2+1}{6^2} = \frac{1}{6} \implies \mathrm{P}(\text{score} \geq 5) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Let $\$a_n$ be the expectation of Alfred's cash after n games. Suppose Alfred and Bertie play one more game (i.e. $n+1$ total games). Then

$$a_{n+1} = \frac{5}{6}(a_n - x) + \frac{1}{6}(a_n + x + 8) = a_n + \frac{2}{3}(2 - x).$$

a_n is in AP with common difference $\frac{2}{3}(2-x)$ and is thus given by

$$a_n = a_0 + n\left[\frac{2}{3}(2-x)\right] = 100 + \frac{2n}{3}(2-x).$$

Hence, the expectation of Alfred's cash after the first game is

$$a_1 = 100 + \frac{2(1)}{3}(2-x) = \frac{1}{3}(304 - 2x).$$

Part (b). The expectation of Alfred's cash after six games is

$$a_6 = 100 + \frac{2(6)}{3}(2-x) = 108 - 4x.$$

Part (c). For the game to be fair, $a_0 = a_1 = a_2 = \cdots$, i.e. the common difference is 0. Hence, $x = 2$.

Part (d). Let the random variable X be Alfred's cash after one game. Since the payouts are unaffected by a_0, we take $a_0 = 0$. When $x = 3$, $\mathrm{E}(X) = -\frac{2}{3}$. Hence,

$$\mathrm{Var}(X) = \frac{5}{6}\left(3 - \frac{2}{3}\right)^2 + \frac{1}{6}\left(3 + 8 + \frac{2}{3}\right)^2 = \frac{245}{9}.$$

Problem 7. A random variable X has the probability distribution given in the following table.

x	2	3	4	5
$\mathrm{P}(X = x)$	p	$\frac{2}{10}$	$\frac{3}{10}$	q

(a) Given that $\mathrm{E}(X) = 4$, find p and q.

(b) Show that $\mathrm{Var}(X) = 1$.

(c) Find $\mathrm{E}(|X - 4|)$.

(d) Ten independent observations of X are taken. Find the probability that the value 3 is obtained at most three times.

Solution.

Part (a). We have

$$\mathrm{E}(X) = 2p + 3\left(\frac{2}{10}\right) + 4\left(\frac{3}{10}\right) + 5q = 4 \implies 2p + 5q = 2.2.$$

Additionally, we know that the probabilities must sum to 1:

$$p + \frac{2}{10} + \frac{3}{10} + q = 1 \implies p + q = 0.5.$$

We hence get a system of two linear equations. Solving, we have $p = 1/10$ and $q = 2/5$.

Part (b). Note that

$$\mathrm{E}(X^2) = 2^2\left(\frac{1}{10}\right) + 3^2\left(\frac{2}{10}\right) + 4^2\left(\frac{3}{10}\right) + 5^2\left(\frac{2}{5}\right) = 17.$$

Thus,

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 17 - 4^2 = 1.$$

Part (c).

x	0	1	2
$\mathrm{P}(\lvert X - 4\rvert = x)$	$\frac{3}{10}$	$\frac{2}{5} + \frac{2}{10}$	$\frac{1}{10}$

Hence,

$$\mathrm{E}(|X - 4|) = 0\left(\frac{3}{10}\right) + 1\left(\frac{2}{5} + \frac{2}{10}\right) + 2\left(\frac{2}{10}\right) = 0.8.$$

Part (d). Observe that the probability that we get exactly n 3's is given by

$$\mathrm{P}(n \text{ 3's}) = \binom{10}{n}\left(\frac{2}{10}\right)^n\left(1 - \frac{2}{10}\right)^{10-n}.$$

Hence, the required probability is

$$\text{Required probability} = \sum_{n=0}^{3}\binom{10}{n}\left(\frac{2}{10}\right)^n\left(1 - \frac{2}{10}\right)^{10-n} = 0.879.$$

Self-Practice A14A

Problem 1. An unbiased disc has a single dot marked on one side and two dots marked on the other side. The disc and an unbiased die are thrown, and the random variable X is the sum of the number of dots showing on the disc and on the top of the die.

(a) Tabulate the probability distribution of X.

(b) Show that $\mathrm{P}(X \geq 4 \mid X \leq 7) = 8/11$.

(c) Write down $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = 19/6$.

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Problem 2. The discrete random variable X denotes the number of "sixes" showing when two ordinary fair dice are thrown. Tabulate the probability distribution of X.

Two dice are thrown repeatedly. Find the probability that, in 5 throws, the result $X = 2$ occurs at least once.

The dice are thrown n times. Find the least value of n such that

$$\mathrm{P}(X = 2 \text{ occurs at least once in the } n \text{ throws}) > 0.9.$$

* * * * *

Problem 3. A writer who writes articles for a magazine finds that his proposed articles sometimes need to be revised before they are accepted for publication. The writer finds that the number of days, X, spent in revising a randomly chosen article can be modelled by the following discrete probability distribution.

x	0	1	2	4
$\mathrm{P}(X = x)$	0.8	0.1	0.05	0.05

Calculate $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

The writer prepares a series of 15 articles for the magazine. Find the expected value of the total time required for revisions to these articles.

The writer regards articles that need no revisions (i.e. $X = 0$) or which need only minor revisions (i.e. $X = 1$) as 'successful' articles, and those requiring major revisions (i.e. $X = 2$) or complete replacement (i.e. $X = 4$) as 'failures'. Assuming independence, find the probability that there will be fewer than 3 'failures' in the 15 articles in the series.

* * * * *

Problem 4. In a game, 2 red balls and 8 blue balls are placed in a bottle. The bottle is shaken and Mary draws 3 balls at random without replacement. The number of red balls that the draws is denoted by R. Find the probability distribution of R, and show that $\mathrm{P}(R \geq 1) = 8/15$.

Show that the expectation of R is $3/5$ and find the variance of R.

Mary scores 4 points for each red ball that she draws. The balls are now replaced in the bottle and the bottle is shaken again. John draws 3 balls at random and without replacement. He scores 1 point for each blue ball that he draws. Mary's score is denoted by M and John's score is denoted by J. Find the expectation and variance of $M - J$.

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Problem 5 (🌶). A fair cubical die has three faces marked with a '1', two faces marked with a '2' and one face marked with a '3'.

(a) Calculate the expectation and variance of the score obtained when this die is thrown once.

(b) Deduce the expectation and variance of the score obtained in one throw of a second cubical die, which has one face marked '1', two faces marked '2' and three faces marked '3'.

(c) Two of the first type of die and one of the second type are thrown together, and X denotes the total score obtained. Denoting the expectation and variance of X by μ and σ^2 respectively, show that $\sigma^2 = 5/3$ and $\mathrm{P}(|X - \mu| > 2\sigma) = 1/18$.

Assignment A14A

Problem 1. On a long train journey, a statistician is invited by a gambler to play a dice game. The game uses two ordinary dice which the statistician is to throw.

If the total score is 12, the statistician is paid \$6 by the gambler. If the total score is 8, the statistician is paid \$3 by the gambler. However, if both or either dice show a 1, the statistician pays the gambler \$2. The game is considered a draw if none of the 3 scenarios occur.

Let $\$X$ be the amount paid to the statistician by the gambler after the dice are thrown once.

(a) Determine the probability that

(i) $X = 6$,

(ii) $X = 3$,

(iii) $X = -2$.

(b) Find the expected value of X and show that, if the statistician played the game 100 times, his expected loss would be \$2.78, to the nearest cent.

(c) Find the amount $\$a$ that the \$6 would have to be changed to in order to make the game unbiased.

Solution.

Part (a).

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

From the table of outcomes, we clearly have

Part (a)(i).

$$\mathrm{P}(X = 6) = \frac{1}{36}.$$

Part (a)(ii).

$$\mathrm{P}(X = 3) = \frac{5}{36}.$$

Part (a)(iii).

$$\mathrm{P}(X = -2) = \frac{11}{36}.$$

Part (b). We have

$$\mathrm{E}(X) = (6)\left(\frac{1}{36}\right) + (3)\left(\frac{5}{36}\right) + (-2)\left(\frac{11}{36}\right) = -\frac{1}{36}.$$

Thus, the expected value of X after 100 games is

$$\mathrm{E}(X_1 + X_2 + \cdots + X_{100}) = -\frac{1}{36} \cdot 100 = -2.78.$$

Part (c). Replacing \$6 with \$$a$, the expected value of X becomes

$$\mathrm{E}(X) = (a)\left(\frac{1}{36}\right) + (3)\left(\frac{5}{36}\right) + (-2)\left(\frac{11}{36}\right) = \frac{1}{36}(a-7).$$

For the game to be unbiased, $\mathrm{E}(X) = 0$. Hence, $a = 7$.

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Problem 2. Four rods of length 1, 2, 3, and 4 units are placed in a bag from which one rod is selected at random. The probability of selecting a rod of length l is kl.

(a) Find the value of k.

(b) Show that the expected value of X, the length of the selected rod, is 3 units and find the variance of X.

After a rod has been selected it is not replaced. The probabilities of selection for each of the three rods that remain are in the same ratio as they were before the first selection. A second rod is now selected from the bag. Let Y be the length of this rod.

(c) Show that $16\mathrm{P}(Y = 1 \mid X = 2) = 9\mathrm{P}(Y = 2 \mid X = 1)$.

(d) Show that $\mathrm{P}(X + Y = 3) = 17/370$.

Solution.

Part (a). The sum of probabilities must be 1. Hence,

$$1k + 2k + 3k + 4k = 1 \implies k = \frac{1}{10}.$$

Part (b). We have

$$\mathrm{E}(X) = 1\left(\frac{1}{10}\right) + 2\left(\frac{2}{10}\right) + 3\left(\frac{3}{10}\right) + 4\left(\frac{4}{10}\right) = 3.$$

Also,

$$\mathrm{E}(X^2) = 1^2\left(\frac{1}{10}\right) + 2^2\left(\frac{2}{10}\right) + 3^2\left(\frac{3}{10}\right) + 4^2\left(\frac{4}{10}\right) = 10.$$

Thus,

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 10 - 3^2 = 1.$$

Part (c). Consider $\mathrm{P}(Y = 1 \mid X = 2)$. Since the rod of length 2 has already been chosen, we are left with the rods of length 1, 3, and 4. Thus,

$$\mathrm{P}(Y = 1 \mid X = 2) = \frac{1}{1+3+4} = \frac{1}{8}.$$

Consider $\mathrm{P}(Y = 2 \mid X = 1)$. Since the rod of length 1 has already been chosen, we are left with the rods of length 2, 3, and 4. Thus,

$$\mathrm{P}(Y = 2 \mid X = 1) = \frac{2}{2+3+4} = \frac{2}{9}.$$

Thus,

$$16\mathrm{P}(Y = 1 \mid X = 2) = 16\left(\frac{1}{8}\right) = 2 = 9\left(\frac{2}{9}\right) = 9\mathrm{P}(Y = 2 \mid X = 1).$$

Part (d). For $X+Y=3$, either $X=1$, $Y=2$ or $X=2$, $Y=1$. Thus,

$$\mathrm{P}(X+Y=3)=\mathrm{P}(X=1,Y=2)+\mathrm{P}(X=2,Y=1)=\left(\frac{1}{10}\right)\left(\frac{2}{9}\right)+\left(\frac{2}{10}\right)\left(\frac{1}{8}\right)=\frac{17}{360}.$$

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Problem 3. The random variable has the following probability distribution:

x	1	2	3
$\mathrm{P}(X=x)$	θ	2θ	$1-3\theta$

(a) It is given that $0<\theta<1/3$. Show that $\mathrm{E}(X)=3-4\theta$, and find $\mathrm{Var}(X)$ in terms of θ.

The random variable S is the sum of n independent values of X.

(b) Write down $\mathrm{E}(S)$ and $\mathrm{Var}(S)$ in terms of θ and n.

The random variable T is defined by $T=a+bS$. The values of a and b are such that $\mathrm{E}(T)=\theta$ for all θ in the interval $0<\theta<1/3$. Show that

(c) $a=3/4$ and $b=-1/4n$,

(d) $\mathrm{Var}(T)=\theta(3-8\theta)/8n$.

Solution.

Part (a). We have

$$\mathrm{E}(X)=1(\theta)+2(2\theta)+3(1-3\theta)=3-4\theta.$$

Also,

$$\mathrm{E}\big(X^2\big)=1^2(\theta)+2^2(2\theta)+3^2(1-3\theta)=9-18\theta.$$

Hence,

$$\mathrm{Var}(\theta)=\mathrm{E}\big(X^2\big)-(\mathrm{E}(X))^2=(9-18\theta)-(3-4\theta)^2=6\theta-16\theta^2.$$

Part (b). We have

$$\mathrm{E}(S)=\mathrm{E}(X_1+X_2+\cdots+X_n)=n\mathrm{E}(X)=n\,(3-4\theta)\,.$$

Also,

$$\mathrm{Var}(S)=\mathrm{Var}(X_1+X_2+\cdots+X_n)=n\,\mathrm{Var}(X)=n\left(6\theta-16\theta^2\right).$$

Part (c). We have

$$\mathrm{E}(T)=\mathrm{E}(a+bS)=a+b\mathrm{E}(S)=a+bn\,(3-4\theta)=(a+3bn)-(4bn)\,\theta.$$

Since $\mathrm{E}(T)=\theta$, we have the system

$$a=3bn=0,\qquad 4bn=1,$$

whence $b=-1/4n$. Substituting this into the first equation yields

$$a+3\left(-\frac{1}{4n}\right)n=0\implies a=\frac{3}{4}.$$

Part (d). We have

$$\mathrm{Var}(T)=\mathrm{Var}(a+bS)=\mathrm{Var}(bS)=b^2\,\mathrm{Var}(S)=\left(-\frac{1}{4n}\right)^2\left[n\left(6\theta-16\theta^2\right)\right]=\frac{\theta(3-8\theta)}{8n}.$$

A14B Special Discrete Random Variables

Tutorial A14B

Problem 1. For each of the following situations, determine whether it can be modelled by a binomial distribution, geometric distribution, a Poisson distribution, or neither of the mentioned.

(a) The number of heads obtained when a biased coin is tossed three times.

(b) The number of phone calls received in a randomly chosen hour.

(c) The number of accidents occurring in a factory in a randomly chosen week.

(d) The number of accidents until the first fatal accident at a traffic junction.

(e) The number of red balls obtained when 3 balls are chosen from a bag containing 4 red, 3 green and 3 white balls

- (i) with replacement;
- (ii) without replacement.

(f) The number of typing errors on a randomly chosen page in a draft of a novel.

(g) The number of seeds in a chosen packet of 12 seeds that fail to germinate.

(h) The number of throws of a die until a six is obtained.

Solution.

Part	Distribution
(a)	Binomial
(b)	Poisson
(c)	Poisson
(d)	Geometric
(e)(i)	Binomial
(e)(ii)	-
(f)	Poisson
(g)	Binomial
(h)	Geometric

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Problem 2. Explain why each of the following situations is not a good model for the proposed distribution.

(a) Using the Poisson distribution to model the number of cars sold at a particular car dealer in a randomly chosen year.

(b) Using the Binomial distribution to model the number of family members that will vote for Party A in the coming election.

(c) Using the Poisson distribution to model the number of people using a particular ATM during a randomly chosen day.

(d) Using the Geometric distribution to model the number of train trips for a particular train before the first breakdown.

Solution.

Part (a). Over the course of a year, the mean rate will likely not be uniform. For instance, the car dealer may only be open on weekdays, so the mean rate on weekdays is different from that on weekends.

Part (b). The probability that one will vote for Party A is not uniform.

Part (c). Over the course of a day, the mean rate will likely not be uniform. For instance, the mean rate at night will be less than the mean rate in the afternoon.

Part (d). The trials are not independent. For instance, wear and tear from previous trials will affect the probability that the next train will break down.

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Problem 3. Calculate the probability that in a group of ten people,

(a) none has his or her birthday on a Saturday,

(b) at least two have their birthdays on Saturday,

(c) more than two but at most five have their birthdays on Saturday.

(d) less than four have their birthdays on other days except Saturday,

Find also the mean number of people whose birthday falls on Saturday.

Solution. Let X be the number of people with a birthday on Saturday. Note that $X \sim \mathrm{B}(10, 1/7)$.

Part (a). $\mathrm{P}(X = 0) = 0.214$.

Part (b). $\mathrm{P}(X \geq 2) = 1 - \mathrm{P}(X \leq 1) = 0.429$.

Part (c). $\mathrm{P}(2 < X << 5) = \mathrm{P}(X = 3) + \mathrm{P}(X = 4) + \mathrm{P}(X = 5) = 0.161$.

Part (d). $\mathrm{P}(X > 6) = 1 - \mathrm{P}(X \leq 6) = 9.77 \times 10^{-5}$.

Since $n = 10$ and $p = 1/7$, the expected value of X is

$$\mathrm{E}(X) = np = \frac{10}{7}.$$

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Problem 4. In a binomial experiment, the mean number of successful trials is 24 and the variance is 20. Find the number of trials conducted and the probability of success for each trial.

Solution. Let n be the number of trials and p be the probability of success of each trial. We have

$$\mu = np = 24 \quad \text{and} \quad \sigma^2 = np(1-p) = 20.$$

Thus,

$$p = 1 - \frac{20}{np} = \frac{1}{6} \implies n = \frac{24}{p} = 144.$$

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Problem 5. If $Y \sim \mathrm{Po}(2.5)$, state the expected value, μ, and the standard deviation, σ, of Y. Use your GC to evaluate the following correct to 3 significant figures.

(a) $\mathrm{P}(Y = 3)$

(b) $\mathrm{P}(Y > 4.5)$

(c) $\mathrm{P}(Y \leq 5)$

(d) $\mathrm{P}(6 < Y < 10)$

(e) $\mathrm{P}(Y \text{ is } 0 \text{ or } 1)$

(f) $\mathrm{P}(|Y - \mu| < \sigma)$

Solution. Since Y follows a Poisson distribution, $\mu = \sigma^2 = 2.5$, whence $\sigma = \sqrt{2.5}$.
Part (a). $\mathrm{P}(Y = 3) = 0.214$.
Part (b). $\mathrm{P}(Y > 4.5) = 1 - \mathrm{P}(Y \leq 4) = 0.109$.
Part (c). $\mathrm{P}(Y \leq 5) = 0.958$.
Part (d). $\mathrm{P}(6 < Y < 10) = \mathrm{P}(Y \leq 9) - \mathrm{P}(Y \leq 6) = 0.0139$.
Part (e). $\mathrm{P}(Y \text{ is } 0 \text{ or } 1) = \mathrm{P}(Y = 0) + \mathrm{P}(Y = 1) = 0.287$.
Part (f). $\mathrm{P}(|Y - \mu| < \sigma) = \mathrm{P}\big(2.5 - \sqrt{2.5} < Y < 2.5 + \sqrt{2.5}\big) = \mathrm{P}(1 \leq Y \leq 4) = 0.809$.

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Problem 6. Epple Company manufactures many E-phones. It is known that 1% of the E-phones manufactured are defective.

(a) A random sample of n phones was selected. Using an algebraic method, find the smallest value of n such that the probability that there is at least one defective phone in the sample is more than 0.95.

(b) A carton, which consists of 24 E-phones, will be rejected if there are at least two defective phones. Show that the probability that a randomly chosen carton is being rejected is 0.0239.

Solution.
Part (a). Let X be the number of defective phones in the sample. Then $X \sim \mathrm{B}(n, 0.01)$. Consider $\mathrm{P}(X \geq 1) \geq 0.95$:

$$\mathrm{P}(X \geq 1) \geq 0.95 \implies \mathrm{P}(X = 0) = 0.99^n \leq 0.05 \implies n \geq 298.1.$$

Since $n \in \mathbb{N}$, the least n is 299.
Part (b). Take $n = 24$. Then

$$\mathrm{P}(X \geq 2) = 1 - \mathrm{P}(X \leq 1) = 0.0239.$$

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Problem 7. In an opinion poll before an election, a sample of 30 voters is obtained.

(a) The number of voters in the sample who support the Alliance Party is denoted by A. State, in context, what must be assumed for A to be well modelled by a binomial distribution.

Assume now that A has the distribution $\mathrm{B}(30, p)$.

(b) Given that $p = 0.15$, find $\mathrm{P}(A = 3 \text{ or } 4)$.

(c) For an unknown value of p, it is given that $\mathrm{P}(A = 15) = 0.06864$ correct to 5 decimal places. Show that p satisfies an equation of the form $p(1-p) = k$, where k is a constant to be determined. Hence, find the value of p to a suitable degree of accuracy, given that $p < 0.5$.

Solution.

Part (a). Votes must be made independently, and the probability of voting A is the same for all voters.

Part (b). $\mathrm{P}(A = 3 \text{ or } 4) = \mathrm{P}(A = 3) + \mathrm{P}(A = 4) = 0.373$.

Part (c). Note that $k = p(1-p) = \operatorname{Var}(A)$ is a constant. Consider $\mathrm{P}(A = 15)$:

$$\mathrm{P}(A = 15) = \binom{30}{15} p^{15}(1-p)^{15} = 0.06864 \implies k = p(1-p) = \sqrt[15]{\frac{0.06864}{\binom{30}{15}}} = 0.23790.$$

Expanding $k = p(1-p)$ into a quadratic in p, we have

$$p^2 - p + 0.23790 = 0 \implies p = 0.390 \text{ or } 0.610.$$

Since $p < 0.5$, we take $p = 0.390$.

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Problem 8. In a large population, the proportion having blood group A is 35%. Specimens of blood from the first five people attending a clinic are to be tested. It can be assumed that these five people are a random sample from the population. The random variable X denotes the number of people in the sample we are found to have blood group A.

(a) Find $\mathrm{P}(X \leq 2)$, correct to 3 decimal places.

(b) Three such samples of five people are taken. Find

  (i) the probability that each of these three samples has more than two people with blood group A,

  (ii) the probability that one of these three samples has exactly one person with blood group A, another has exactly two people with blood group A, and the remaining sample has more than two people with blood group A.

(c) Ten such samples of five people were taken. Find the probability that seven samples have exactly one person with blood group A.

Solution. Note that $X \sim \mathrm{B}(5, 0.35)$.

Part (a). $\mathrm{P}(X \leq 2) = 0.76483$ (5 s.f.) $= 0.765$ (3 d.p.).

Part (b).

Part (b)(i). The required probability is given by

$$[\mathrm{P}(X > 2)]^3 = [1 - \mathrm{P}(X \leq 2)]^3 = 0.0130.$$

Part (b)(ii). The required probability is given by

$$3!\,[\mathrm{P}(X = 1)\mathrm{P}(X = 2)\mathrm{P}(X > 2)] = 0.148.$$

Part (c). Note that $\mathrm{P}(X = 1) = 0.31239$. Let Y be the number of samples with exactly one person with blood group A. Then $Y \sim \mathrm{B}(10, 0.31239)$. Hence, $\mathrm{P}(Y = 7) = 0.0113$.

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Problem 9. Every student in Sunny Junior College owns a graphic calculator (GC). The probability of a student carrying a GC to school is 0.98. Assume that the number of students who carries a GC to school follows a binomial distribution.

(a) (i) Given that the probability that more than m students, in a random sample of 30 students, carry a GC to school is at most 0.99, find the least value of m.

(ii) Give a reason why in real life, the number of students who carries a GC to school may not follow a binomial distribution.

The latest operating system (OS) of the GC is required for the installation of a new application. On average, 3 out of 4 students have the latest OS in their GC. A class of 26 students is to report to their mathematics tutor, Mr Ng, to install the new application.

(b) Find the probability that the 15th student who reports to Mr Ng is the 9th student whose GC has the latest OS while the last student is the 10th student without the latest OS.

Solution.

Part (a).

Part (a)(i). Let X be the number of students who bring a GC to school. Then $X \sim \mathrm{B}(30, 0.98)$. Consider $\mathrm{P}(X > m) \leq 0.99$. Using GC, the least m is 27.

Part (a)(ii). The probability that a student brings their GC to school is not the same for all students, as different students may have different timetables.

Part (b). Number the students in the order in which they report to Mr Ng.

- Students 1–14: 8 students have the latest OS installed, remaining 6 students do not.
- Student 15: Has the latest OS installed.
- Students 16–25: 10 students have the latest OS installed, remaining 3 students do not.
- Student 26: Does not have the latest OS installed.

The probability of this happening is given by

$$\left[\binom{14}{8}\left(\frac{3}{4}\right)^8\left(\frac{1}{4}\right)^6\right]\left[\frac{3}{4}\right]\left[\binom{10}{3}\left(\frac{3}{4}\right)^7\left(\frac{1}{4}\right)\right]\left[\frac{1}{4}\right] = 0.00344.$$

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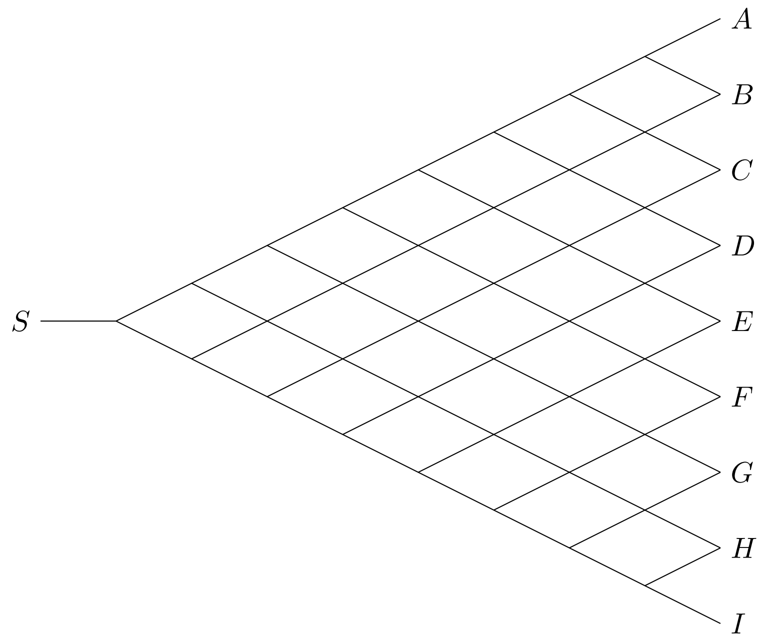
Problem 10. In a computer game, a bug moves from left to right through a network of connected paths. The bug starts at S and, at each junction, randomly takes the left fork with probability p or the right fork with probability q, where $q = 1 - p$. The forks taken at each junction are independent. The bug finishes its journey at one of the 9 endpoints labelled A–I (see diagram below).



(a) Show that the probability that the bug finishes its journey at D is $56p^5q^3$.

(b) Given that the probability that the bug finishes its journey at D is greater than the probability at any one of the other endpoints, find exactly the possible range of values of p.

In another version of the game, the probability that, at each junction, the bug takes the left fork is $0.9p$, the probability that the bug takes the right fork is $0.9q$ and the probability that the bug is swallowed up by a 'black hole' is 0.1.

(c) Find the probability that, in this version of the game, the bug reaches one of the endpoints A–I, without being swallowed up by a black hole.

Solution.

Part (a). Relabel each endpoint from A–I to 0–8. Let the random variable X be the end-point that the bug ends up at. Clearly, to reach endpoint i, the bug must take i right forks and $8-i$ left forks. Hence, $X \sim \mathrm{B}(8, q)$ and the probability that the bug reaches endpoint 3 (i.e. endpoint D) is

$$\mathrm{P}(X = 3) = \binom{8}{3} q^3 (1-q)^{8-3} = 56p^5q^3.$$

Part (b). Since X follows a binomial distribution, it suffices to find the range of values of p that satisfy $\mathrm{P}(X = 2) < \mathrm{P}(X = 3) > \mathrm{P}(X = 4)$.

Case 1: $\mathrm{P}(X = 2) < \mathrm{P}(X = 3)$. Note that $\mathrm{P}(X = 2) = \binom{8}{2} q^2(1-q)^{8-2} = 28p^6q^2$.

$$\mathrm{P}(X = 2) < \mathrm{P}(X = 3) \implies 28p^6q^2 < 56p^5q^3 \implies 28p < 56(1-p) \implies p < \frac{2}{3}.$$

Case 2: $\mathrm{P}(X = 3) > \mathrm{P}(X = 4)$. Note that $\mathrm{P}(X = 4) = \binom{8}{4} q^4(1-q)^{8-4} = 70p^4q^4$.

$$\mathrm{P}(X = 3) > \mathrm{P}(X = 4) \implies 56p^5q^3 > 70p^4q^4 \implies 56p > 70(1-p) \implies p > \frac{5}{9}.$$

Hence, $5/9 < p < 2/3$.

Part (c). Note that the bug most take a total of 8 forks. Since the probability of not getting swallowed by a black hole at each fork is 0.9, the desired probability is clearly $0.9^8 = 0.430$ (3 s.f.).

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Problem 11. The number of injuries, X, sustained by workers in a factory per week follows a Poisson distribution with standard deviation σ. Given that $3\mathrm{P}(X=2) = 16\mathrm{P}(X=4)$ determine the value of σ and hence state the mean of X.

(a) Find the probability that, in a randomly chosen week, there is at least one injury.

(b) Assuming that a month consists of four weeks, find the probability that, in a randomly chosen month, there are less than 4 injuries.

(c) Calculate the probability that there will be more than 1 but less than 4 injuries in each of two consecutive weeks.

Solution. We have

$$3\mathrm{e}^{-\mu}\frac{\mu^2}{2!} = 3\mathrm{P}(X=2) = 16\mathrm{P}(X=4) = 16\mathrm{e}^{-\mu}\frac{\mu^4}{4!} \implies \mu^2 = \sigma = 2.25 \implies \mu = 1.5.$$

Thus, $X \sim \mathrm{Po}\,(1.5)$.

Part (a). $\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X=0) = 0.777$.

Part (b). Let Y be the number of injuries in a month. Then $Y \sim \mathrm{Po}\,(6)$. Hence, $\mathrm{P}(Y < 4) = \mathrm{P}(Y \leq 3) = 0.151$.

Part (c). The required probability is given by

$$[\mathrm{P}(1 < X < 4)]^2 = [\mathrm{P}(X=2) + \mathrm{P}(X=3)]^2 = 0.142.$$

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Problem 12. During a weekday, heavy lorries pass a census point P on a village high street independently and at random times. The mean rate for westward travelling lorries is 2 in any 30-minute period and for eastward travelling lorries is 3 in any 30-min period. Find the probability

(a) that there will be no lorries passing P in a given 10-min period,

(b) that at least one lorry from each direction will pass P in a given 10-minute period,

(c) more than 2 westward travelling lorries will pass P between the time 1410 and 1445,

(d) that there will be exactly 4 lorries passing P in a given 20-minutes period

(e) at least 2 eastward travelling lorries passing P in a period of 20 minutes given that there are exactly 4 lorries passing P at that time.

Solution. Let W_k and E_k be the number of westward and eastward travelling lorries passing P in a k-minute period. Then

$$W_k \sim \mathrm{Po}\left(\frac{k}{15}\right) \quad \text{and} \quad E_k \sim \mathrm{Po}\left(\frac{k}{10}\right).$$

Part (a). $\mathrm{P}(W_{10} = 0)\mathrm{P}(E_{10} = 0) = 0.189$.

Part (b). $\mathrm{P}(W_{10} \geq 1)\mathrm{P}(E_{10} \geq 1) = [1 - \mathrm{P}(W_{10} = 0)]\,[1 - \mathrm{P}(E_{10} = 0)] = 0.308$.

Part (c). $\mathrm{P}(W_{35} \geq 2) = 1 - \mathrm{P}(W_{35} \leq 2) = 0.413$.

Part (d). Note that $W_{20} + E_{20} \sim \mathrm{P}\left(\frac{20}{15} + \frac{20}{10}\right) = \mathrm{P}\left(\frac{10}{3}\right)$. Thus, $\mathrm{P}(W_{20} + E_{20} = 4) = 0.184$.

Part (e). The desired probability is given by

$$1 - \frac{\mathrm{P}(W_{20} = 4)\mathrm{P}(E_{20} = 0) + \mathrm{P}(W_{20} = 3)\mathrm{P}(E_{20} = 1)}{\mathrm{P}(W_{20} + E_{20} = 4)} = 0.821.$$

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Problem 13. A car rental company has n cars which may be hired on a daily basis. The demand for cars in a day follows a Poisson distribution with variance 1.5.

(a) For $n = 5$, for any one day,

 (i) find the probability that less than 3 cars are hired.

 (ii) find the probability that all the cars are hired.

(b) The probability that the demand for cars being met on any day is at least 0.95. Find the least value of n.

(c) The probability that no car is rented out on k consecutive days is less than 0.01. Find the least value of k.

(d) The probability that there are less than two cars rented out on h consecutive days is less than 0.005. Find the least value of h.

Solution. Let X be the number of cars hired in a given day. Then $X \sim \mathrm{Po}\,(1.5)$.

Part (a).

Part (a)(i). $\mathrm{P}(X < 3) = \mathrm{P}(X \leq 2) = 0.809$ (3 s.f.).

Part (a)(ii). $\mathrm{P}(X \geq 5) = 1 - \mathrm{P}(X \leq 4) = 0.0186$.

Part (b). Consider $\mathrm{P}(X \leq n) \geq 0.95$. Using G.C., the least n is 4.

Part (c). Consider $\mathrm{P}(X = 0)^k \leq 0.01$. Using G.C., the least k is 4.

Part (d). Consider $\mathrm{P}(X < 2)^h = \mathrm{P}(X \leq 1)^h \leq 0.005$. Using G.C., the least h is 10.

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Problem 14. A randomly chosen doctor in general practices sees, on average, one case of a broken nose per year and each case is independent of other similar cases.

(a) Regarding a month as a twelfth part of a year,

 (i) show that the probability that, between them, three such doctors see no cases of a broken nose in a period of one month is 0.779, correct to three significant figures,

 (ii) find the variance of the number of cases seen by three such doctors in a period of six months.

(b) Find the probability that, between them, three such doctors see at least three cases in one year.

(c) Find the probability that, of three such doctors, one sees three cases and the other two see no cases in one year.

Solution. Let $X_{t,n}$ be the number of cases of a broken nose seen by n doctors in t months. Then $X_{t,n} \sim \text{Po}\,(tn/12)$.

Part (a).

Part (a)(i). The required probability is

$$\text{P}(X_{1,3} = 0) = 0.779 \text{ (3 s.f.)}.$$

Part (a)(ii). Since $X_{t,n}$ follows a Poisson distribution, $\mu = \sigma^2$. Hence,

$$\text{Var}(X_{6,3}) = \frac{(6)(3)}{12} = \frac{3}{2}.$$

Part (b). The required probability is

$$\text{P}(X_{12,3} \geq 3) = 1 - \text{P}(X_{12,3} \leq 2) = 0.577 \text{ (3 s.f.)}.$$

Part (c). The required probability is

$$^3C_1 \text{P}(X_{12,1} = 3)\text{P}(X_{12,2} = 0) = 0.0249 \text{ (3 s.f.)}.$$

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Problem 15. During the winter in New York, the probability that snow will fall on any given day is 0.1. Taking November 1st as the first day of winter and assuming independence from day to day, find the probability that the first snow of winter will fall in New York on the last day of November (30th).

Given that no snow has fallen at New York during the whole of November, a teacher decides not to wait any longer to book a skiing holiday. The teacher decides to book for the earliest date for which the probability that snow will have fallen on, or before, that date is at least 0.9. Find the date of the booking.

Solution. The probability that the first snow of winter will fall on 30 November is given by

$$(0.9)^{29}(0.1) = 0.00471 \text{ (3 s.f.)}.$$

Let n be the number of days after 30 November. Consider $\text{P}(X \leq n) \geq 0.9$, where $X \sim \text{Geo}\,(0.1)$. Using G.C., the least n is 22. Hence, the date of the booking is 22 December.

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Problem 16. A salesman sells goods by telephone. The probability that any particular call achieves a sale is 1/12. The salesman continues to make calls until one call achieves a sale.

(a) State one assumption need for this to be modelled by a Geometric distribution.

(b) Given that a Geometric distribution is used to model this, find the probability that the call achieves a sale

  (i) is the fifth call made,

  (ii) does not occur in the first five calls.

(c) The salesman uses 5 minutes for each call, find the expected amount of time he has to spend to reach his first sale.

Solution.

Part (a). The calls must be independent.

Part (b). Let X be the number of calls made under the salesman achieves a sale. Then $X \sim \text{Geo}(1/12)$.

Part (b)(i). $\text{P}(X = 5) = 0.0588$ (3 s.f.).

Part (b)(ii). $\text{P}(X > 5) = (1 - 1/12)^5 = 0.647$.

Part (c). Note that $\text{E}(X) = 1/p = 12$. Hence, the expected amount of time he has to spend to reach his first sale is $5 \times 12 = 60$ minutes.

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Problem 17. If $X \sim \text{B}(n, p)$ and $Y \sim \text{Geo}(p)$, explain why $\text{P}(Y = n) \leq \text{P}(X = 1)$.

Solution. Both $X = 1$ and $Y = n$ represent the event that there is exactly one success in n trials. However, the event $Y = n$ has the added restriction that the success must come on the last trial, whereas the event $X = 1$ has no such restriction; the success can occur in any of the n trials. Hence, the event $Y = n$ is a subset of the event $X = 1$, thus $\text{P}(Y = n) \leq \text{P}(X = 1)$, with equality only when $n = 1$.

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Problem 18. Serious delays on a certain railway line occurs at random, at an average rate of one per week. Show that the probability of at least 4 serious delays occurring during a particular 4-week period is 0.567, correct to 3 decimal places.

Taking a year to consist of thirteen 4-week periods, find the probability that, in a particular year, there are at least ten of these 4-week periods during which at least 4 serious delays occur.

Given that the probability of at least n serious delays occurring in a period of 6 weeks is greater than 0.795, find the largest possible integer value of n.

Solution. Let the number of serious delays in k weeks be $X_k \sim \text{Po}(k)$. We have

$$\text{P}(X_4 \geq 4) = 1 - \text{P}(X_4 \leq 3) = 0.56653 = 0.567 \text{ (3 d.p.)}.$$

Let Y be the number of 4-week periods during which at least 4 serious delays occur. Note that $Y \sim \text{B}(13, 0.56653)$. Hence,

$$\text{P}(Y \geq 10) = 1 - \text{P}(Y \leq 9) = 0.115 \text{ (3 s.f.)}.$$

Consider $\text{P}(X_6 \geq n) \geq 0.795$. Using G.C., the greatest value of n is 4.

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Problem 19. The demand for XO pies in a confectionary shop may be taken to follow a Poisson distribution with a mean of 0.4 pies per hour. The shop opens for 5 days in a week and does business for 8 hours per day.

(a) Find the probability that the demand for XO pies is at least 3 in a day.

(b) Find the probability that there is one day with demand for XO pies of at least 3, and another two days with demand 0.

(c) Find the probability that there is at most one day with zero demand for XO pies in a week.

(d) Given that the demand for XO pies is exactly 3 on a particular day, what is the probability that this occurred within the first hour of business.

Solution. Let the number of XO pies demanded in k hours be $X_k \sim \mathrm{Po}\,(0.4k)$.

Part (a). $\mathrm{P}(X_8 \geq 3) = 1 - \mathrm{P}(X_8 \leq 2) = 0.620$ (3 s.f.).

Part (b). Note that $\mathrm{P}(X_8 = 0) = 0.40762$. The required probability is

$$^3C_1 \mathrm{P}(X_8 \geq 3)\mathrm{P}(X_8 = 0) = 0.00309.$$

Part (c). Let the number of days in a week where there is 0 demand for XO pies be $Y \sim \mathrm{B}\,(5, 0.40762)$. Then $\mathrm{P}(Y \leq 1) = 0.985$ (3 s.f.).

Part (d). If all three pies for the day were sold within the first hour of business, then no pies were sold in the remaining seven hours. Hence, the required probability is

$$\frac{\mathrm{P}(X_1 = 3)\mathrm{P}(X_1 = 0)^7}{\mathrm{P}(X_8 = 3)} = 0.00195.$$

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Problem 20. Given the climate of the country and duration of transportation, the probability of a strawberry from a particular orchard turning rotten is believed to be 0.15. In a fruits wholesale centre where strawberries from that orchard are sold, they are packed and sold in trays of 20.

(a) Show that the probability that there are at most 5 rotten strawberries in a tray is 0.933.

(b) Find, to 3 decimal places, the probability that there are exactly 3 rotten strawberries in 2 randomly selected trays.

A cold desserts hawker bought 60 trays of strawberries from the wholesaler centre. Using a suitable approximation, find the probability that there are at least 4 trays with more than 5 rotten strawberries in each tray.

Solution. Let X_k be the number of rotten strawberries in k trays. We have $X_k \sim \mathrm{B}\,(20k, 0.15)$.

Part (a). $\mathrm{P}(X_1 \leq 5) = 0.933$ (3 s.f.).

Part (b). $\mathrm{P}(X_2 = 3) = 0.0816$ (3 s.f.).

Note that $\mathrm{P}(X_1 > 5) = 1 - \mathrm{P}(X_1 \leq 5) = 0.067308$. Let Y be the number of trays with more than 5 rotten strawberries. We can approximate Y using a Poisson distribution since $n = 60$ is large and $p = 0.067308$ is small. We hence have $Y \sim \mathrm{Po}\,(np) = \mathrm{Po}\,(4.0385)$. Then $\mathrm{P}(Y \geq 4) = 1 - \mathrm{P}(Y \leq 3) = 0.574$ (3 s.f.).

Self-Practice A14B

Problem 1. A crossword puzzle is published in The Times each day of the week, except Sunday. A man is able to complete, on average, 8 out of 10 of the crossword puzzles.

(a) Find the expected value and the standard deviation of the number of completed crossword puzzles in a given week.

(b) Show that the probability that he will complete at least 5 in a given week is 0.655.

(c) Given that he completes the puzzle on Monday, find the probability that he will complete at least 4 in the rest of the week.

(d) Find the probability that, in a period of 4 weeks, he completes 4 or less in only one of the 4 weeks.

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Problem 2. There is a lift on the ground floor of an old 50-storey building. This lift serves only the first 25 floors with an average number of breakdowns of 2 per week. On the 25th floor, there is another lift serving the 26th to the 50th floor with an average number of breakdowns of 0.5 per week, independent of the other lift. Find, correct to 3 decimal places, the probability that

(a) both lifts do not break down in a particular week,

(b) there are not more than 2 breakdowns altogether in a particular day,

(c) in a period of 7 days, there are 6 days on which there are not more than 2 breakdowns altogether.

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Problem 3. The centre pages of the '8 days' magazine consist of 1 page of film and theatre reviews and 1 page of classified advertisements. The number of misprints in the reviews has a Poisson distribution with mean 2.3 and the number of misprints in the classified section has a Poisson distribution with variance 1.7.

Find the probabilities that, on the centre pages, there will be

(a) no misprints,

(b) 5 misprints.

Given that there are 5 misprints in the centre pages, find the probability that 2 of them occur in the classified section.

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Problem 4. The student resource centre has 4 rooms which can be booked for students' activities in a day at a time. Requests of the booking of a room take place independently with a mean of 4 requests per day.

(a) Find the probability that not all requests for the booking of a room can be met on any particular day.

(b) On any particular day during which more than 2 requests are received for the booking of a room, find the probability that all requests for the booking of a room can be met on that day.

(c) Find the probability that there are at most 3 days in a particular week (taking 1 week to be 5 schooling days) during which not all requests for the booking of a room for that day can be met.

(d) Find the least number of rooms that the student resource centre should have so that, on any particular day, the probability that a request for the booking of a room for that day has to be refused is less than 0.05.

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Problem 5. A jeweller sells rubies and diamonds. The average number of rubies and diamonds sold per week is 1.8 and 2.7 respectively.

(a) Find the probability that exactly two rubies are sold in a given week.

(b) Find the probability that exactly 4 diamonds are sold in a given two-week period.

(c) Find the probability that the total number of jewels sold in a given week is at least 4.

(d) Given that less than 3 jewels are sold in a given week, find the probability that the number of rubies sold is more than the diamonds sold.

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Problem 6. The number of printing errors, Y, in a page of a book follows a Poisson distribution with standard deviation σ. Given that $3\text{P}(Y = 2) = 16\text{P}(Y = 4)$, determine the value of σ and hence state the mean of Y.

(a) The probability that there are no errors in k consecutive pages is less than 0.01. Determine the least k value.

(b) For a book of 100 pages, find the probability that at least one page has at least four errors.

* * * * *

Problem 7. In order to be offered a scholarship, a candidate has to pass two rounds of interview (it is assumed that all interviewers' decisions are independent).

In the first round, there will be a panel of 10 interviewers and the probability of each interviewer passing a candidate is 0.9. The candidate fails to qualify for the second round if more than one interviewer decides not to pass him or her.

(a) Find the probability that a candidate passes the first round of interview.

In the second round, there will be a panel of 5 interviewers and the probability of each interviewer passing a candidate is 0.8. The candidate is offered a scholarship only if all interviews pass him or her in the second round.

(b) Show that the probability that the candidate is offered the scholarship is 0.241, correct to three significant figures.

(c) There are n candidates going for the interviews. Find the smallest n such that there is at least a 98% chance of 2 or more candidates being offered the scholarship.

Problem 8. The two most common types of disciplinary offences in a particular boy school is keeping long hair and failure to wear the school badge. The mean number of disciplinary offences recorded per day involving long hair is 1.12. Assuming that each school week consists of five school days, the mean number of disciplinary offences recorded per school week involving failure to wear the school badge is 4.2. The number of cases for each disciplinary offence is assumed to have an independent Poisson distribution.

(a) Find the probability that at most 9 cases of disciplinary offence are recorded in a given school week.

(b) In a school week in which there are more than 7 cases of disciplinary offence involving long hair, find the probability that at most 9 cases of disciplinary offence are recorded.

(c) Calculate the probability that on a Thursday in a particular school week, it is the third day in the school week in which the discipline master caught at least 4 students having long hair in a day (you may assume that Monday is the first day of school for a school week).

(d) Explain why the Poisson distribution may not be a good model for the number of disciplinary cases involving long hair, in a school year.

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Problem 9. In a small company, the employees send an average of 1.2 print jobs to the colour print and α print jobs to the laser printer per day. It is assumed that the print jobs are independent.

(a) Given that on 1 in 100 working days there are no print jobs for both printers, show that $\alpha = 3.41$ correct to 3 significant figures.

(b) Let E be the event that more than 3 print jobs were sent in on a working day. Find $\mathrm{P}(E)$. Hence, find the probability that the first occurrence of event E happens before the 5th working day.

(c) A typical working day consists of 8 hours of work. Find the probability that more than half of the total print jobs sent during a typical working day occurs within the first hour of work, given that there was a total of 3 print jobs for the day.

* * * * *

Problem 10. A firm investigated the number of employees suffering injuries whilst at work. The results recorded below were obtained for a 52-week period:

Number of employees injured in a week	0	1	2	3	4 or more
Number of weeks	31	17	3	1	0

Give reasons why one might expect this distribution to approximate to a Poisson distribution. Evaluate the mean and variance of the data and explain why this gives further evidence in favour of a Poisson distribution. Using the calculated value of the mean, find the theoretical frequencies of a Poisson distribution for the number of weeks in which 0, 1, 2, 3, 4 or more employees were injured.

Problem 11 (🌶). The Entrepreneur Club is in charge of selling the school's tee shirt. Based on the sales record of the club, it was found that the monthly demand for tee shirt size XS has a Poisson distribution with mean 2 and the monthly demand for tee shirt size XXL has a Poisson distribution with mean 3. The club kept a monthly stock of 3 and 4 for tee shirt sizes XS and XXL respectively.

(a) Calculate the probability that there is more than one XS size tee shirt being sold in a day, assuming there are 30 days in a month.

(b) Calculate the probability that the club will not meet the demand for either XS or XXL tee shirts in a month.

(c) Find the most probable number of XXL tee shirts sold in a month.

(d) Determine the least number of stock needed each month for the XS tee shirts in order to meet the demand with a probability of at least 0.95.

Part IX

Group B

B1 Graphs and Transformations I

Tutorial B1A

Problem 1. Without using a calculator, sketch the following graphs and determine their symmetries.

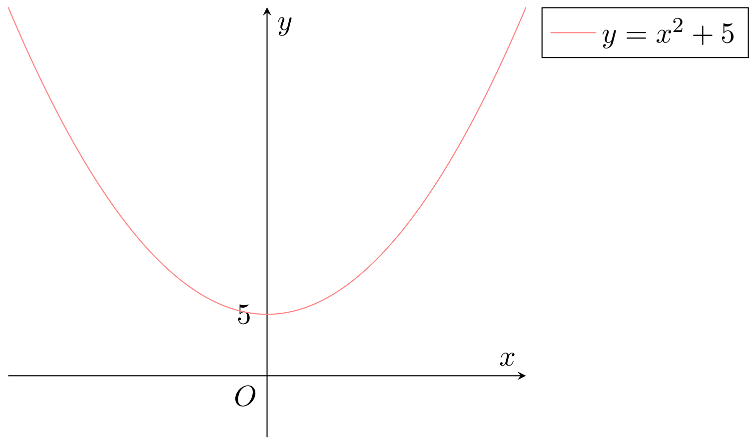
(a) $y = x^2 + 5$

(b) $y = 2x - x^3$

(c) $y = x^2 - 4x + 3$

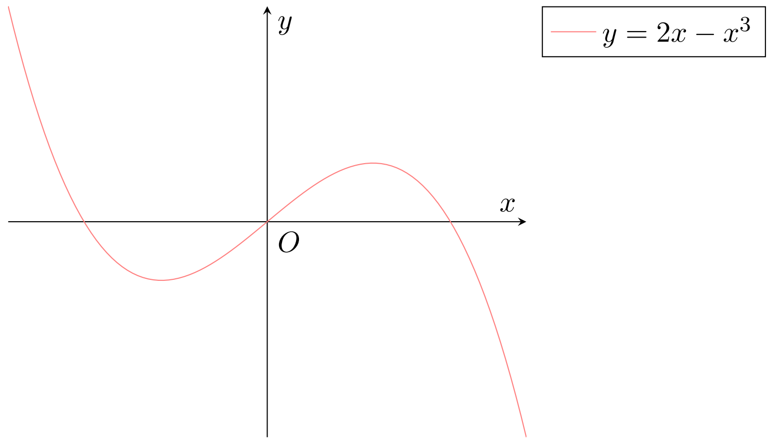
Solution.

Part (a).



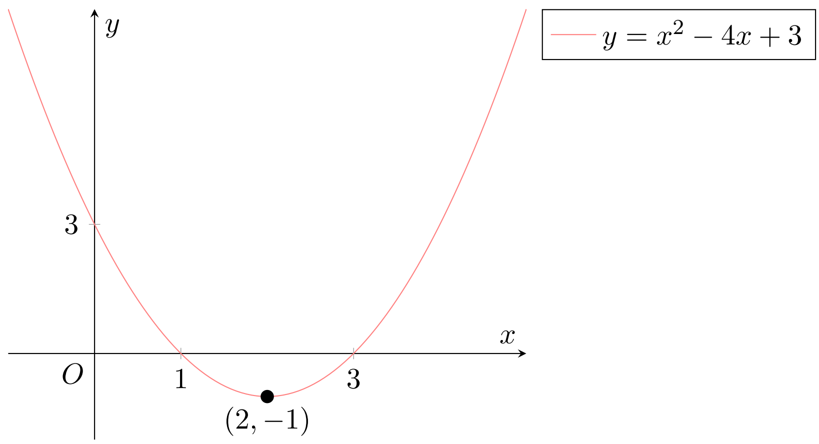
Symmetry: $x = 0$.

Part (b).



Symmetry: $(0, 0)$.

Part (c).



Symmetry: $x = 2$.

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Problem 2. Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

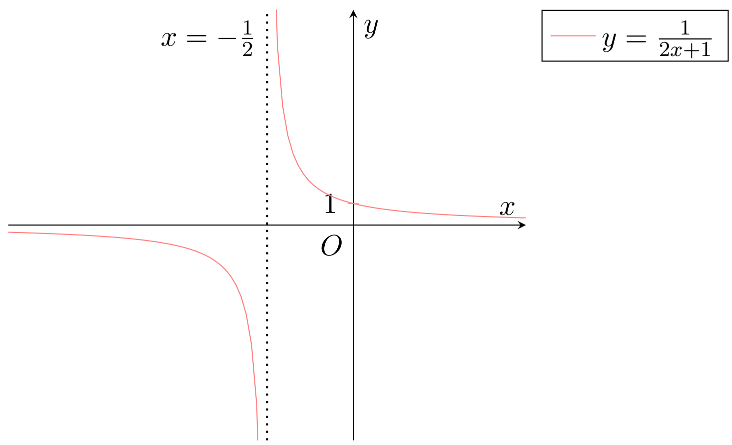
(a) $y = \frac{1}{2x+1}$

(b) $y = \frac{3x}{x-2}$

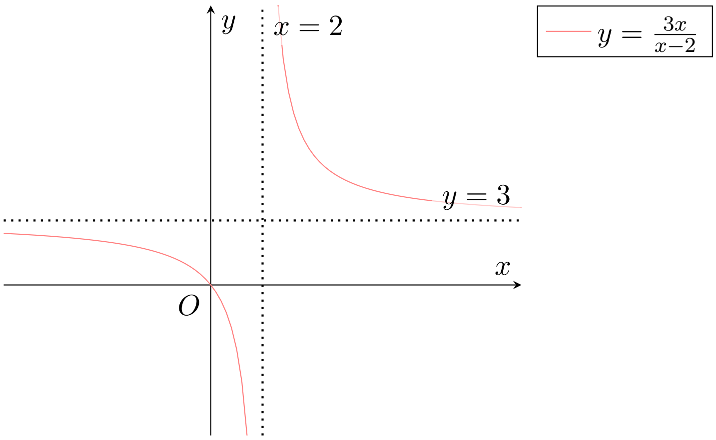
(c) $y = \frac{x^2+x-6}{2x-2}$

Solution.

Part (a).



Part (b).



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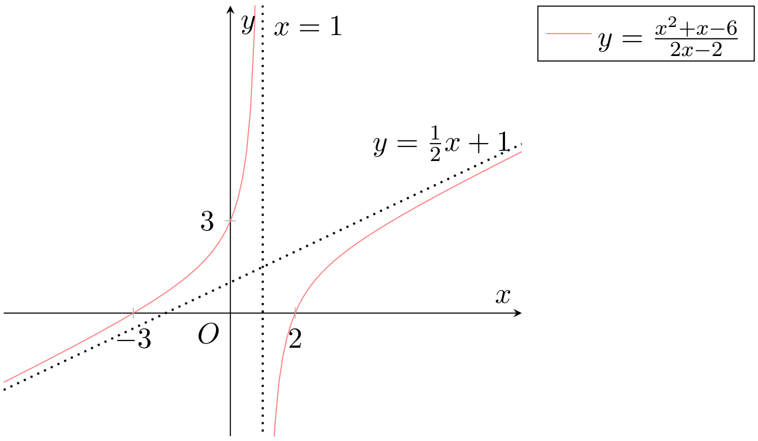

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Problem 3. Sketch the following graphs

(a) $x^2 + 2x + 2y + 4 = 0$

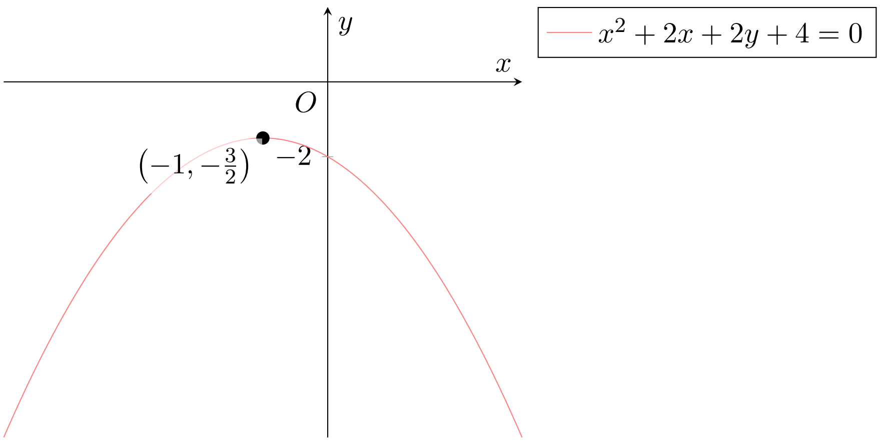
(b) $y^2 = x - 9$

(c) $y^2 = (x-2)^4 + 5$

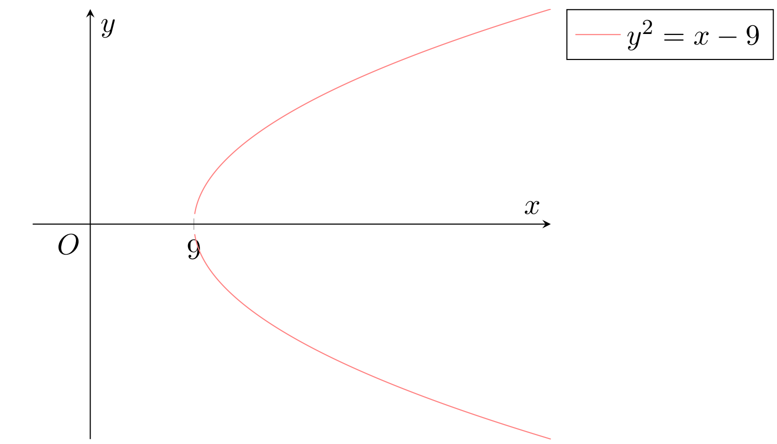
(d) $y = \tan\left(\frac{1}{2}x\right)$, $-2\pi \le x \le 2\pi$

Solution.

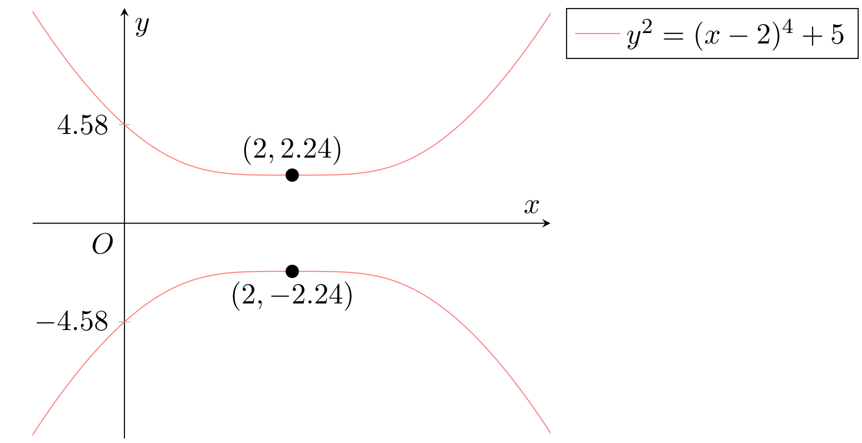
Part (a).



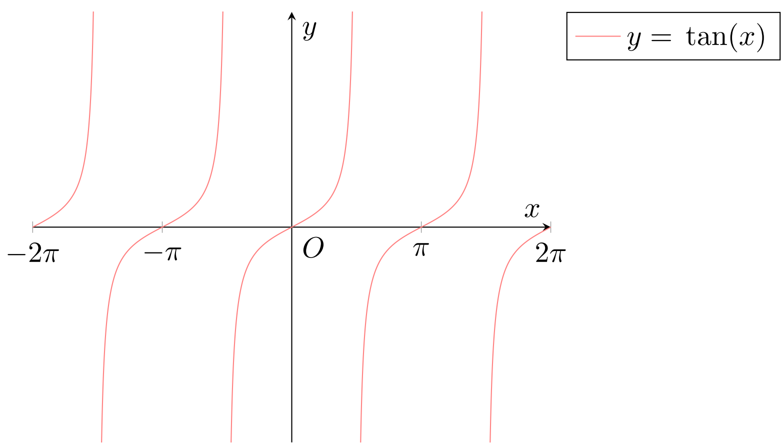
Part (b).



Part (c).



Part (d).



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Problem 4. Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

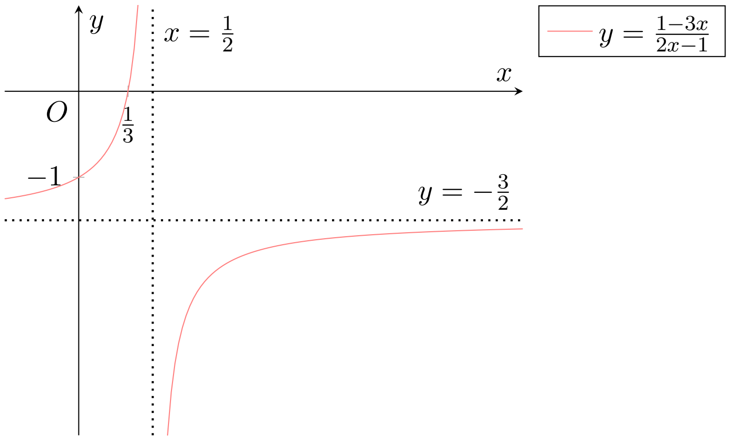
(a) $y = \frac{1-3x}{2x-1}$

(b) $y = \frac{ax}{x-a}$, $a < 0$

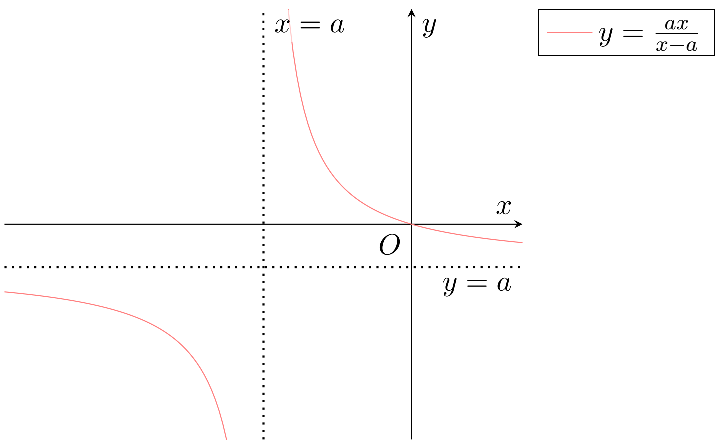
(c) $y = -\frac{b(x+3a)}{x+a}$, $a, b > 0$

Solution.

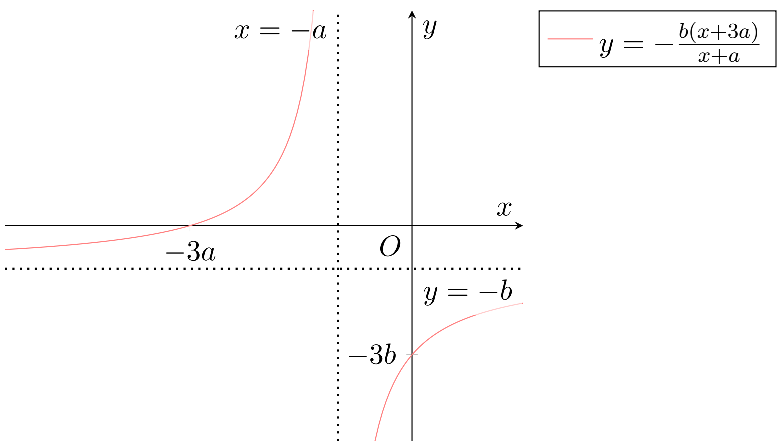
Part (a).



Part (b).



Part (c).



Problem 5. Sketch the following curves and find the coordinates of any turning points on the curves.

(a) $y = x + 2\sin x$, $0 \le x \le 2\pi$

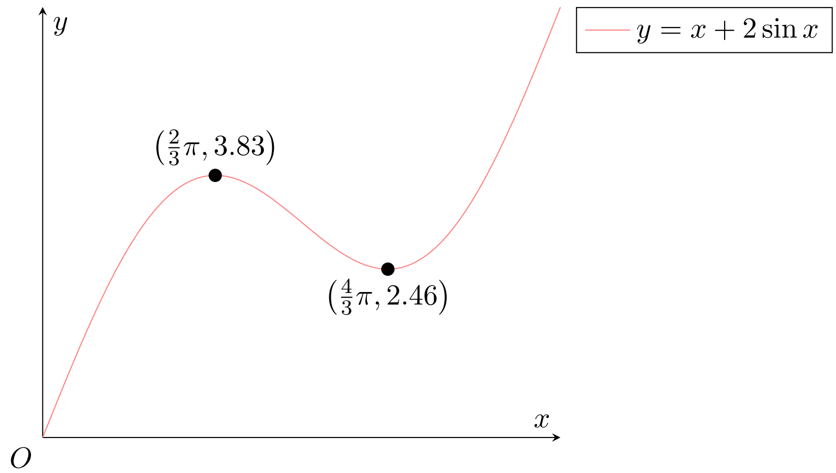
(b) $y = \frac{x}{\ln x}$, $x > 0$, $x \ne 1$

(c) $y = x\mathrm{e}^{-x}$

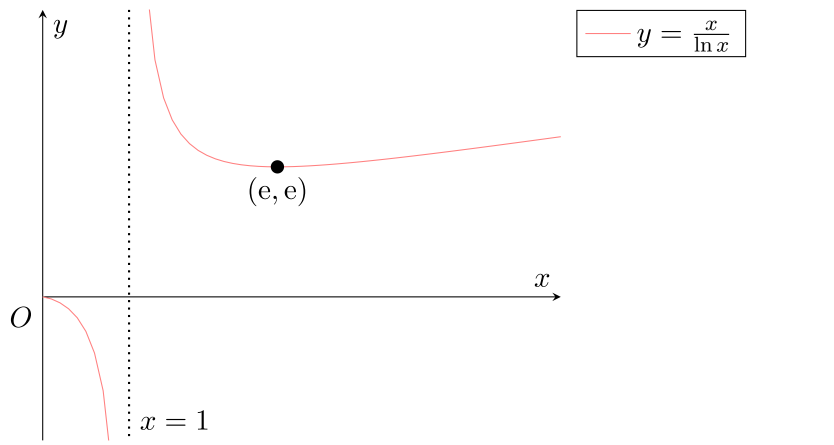
(d) $y = x\mathrm{e}^{-x^2}$

Solution.

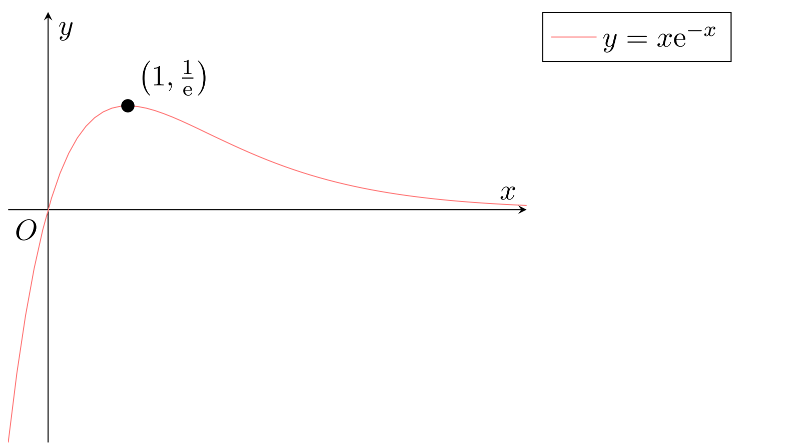
Part (a).



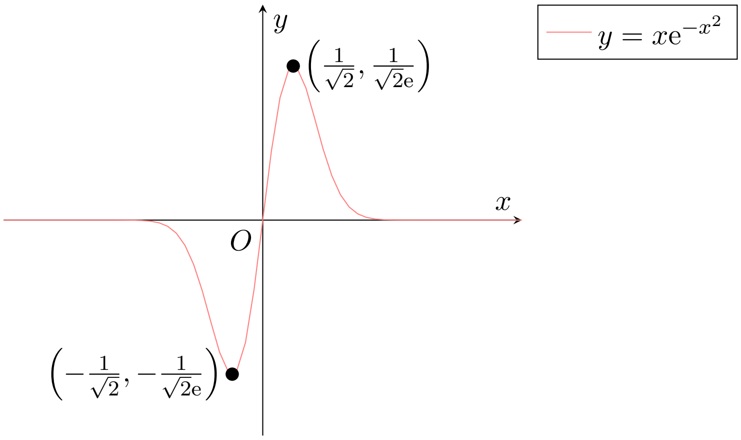
Part (b).



Part (c).



Part (d).



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Problem 6. The equation of a curve C is $y = 1 + \frac{6}{x-3} - \frac{24}{x+3}$.

(a) Explain why $y = 1$ and $x = 3$ are asymptotes to the curve.

(b) Find the coordinates of the points where C meets the axes.

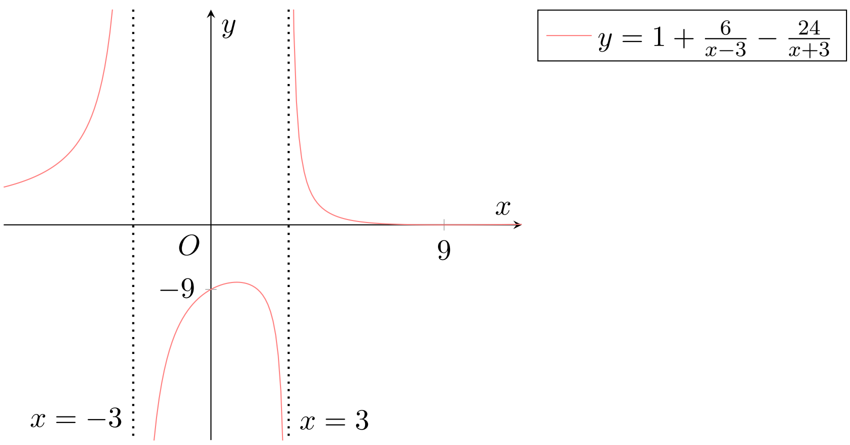
(c) Sketch C.

Solution.

Part (a). As $x \to \pm\infty$, $y \to 1$. Hence, $y = 1$ is an asymptote to C. As $x \to 3^{\pm}$, $y \to \pm\infty$. Hence, $x = 3$ is an asymptote to C.

Part (b). When $x = 0$, $y = -9$. When $y = 0$, $x = 9$. Hence, C meets the axes at $(0, -9)$ and $(9, 0)$.

Part (c).



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Problem 7. The curve C has equation $y = \frac{ax^2+bx}{x+2}$, where $x \neq -2$. It is given that C has an asymptote $y = 1 - 2x$.

(a) Show (do not verify) that $a = -2$ and $b = -3$.

(b) Using an algebraic method, find the set of values that y can take.

(c) Sketch C, showing clearly the positions of any axial intercept(s), asymptote(s) and stationary point(s).

(d) Deduce that the equation $x^4+2x^3+2x^2+3x=0$ has exactly one real non-zero root.

Solution.

Part (a).

$$y=\frac{ax^2+bx}{x+2}=\frac{(ax+b-2a)(x+2)-2(b-2a)}{x+2}=ax+b-2a-\frac{2(b-2a)}{x+2}.$$

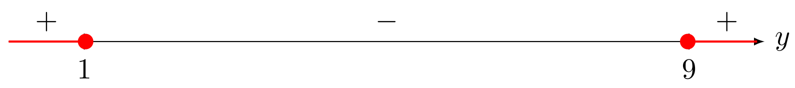
Since C has an asymptote $y=1-2x$, we have $a=-2$ and $b-2a=1$, whence $b=-3$.

Part (b).

$$y=\frac{-2x^2+-3x}{x+2}\implies y(x+2)=-2x^2-3x\implies 2x^2+(3+y)x+2y=0.$$

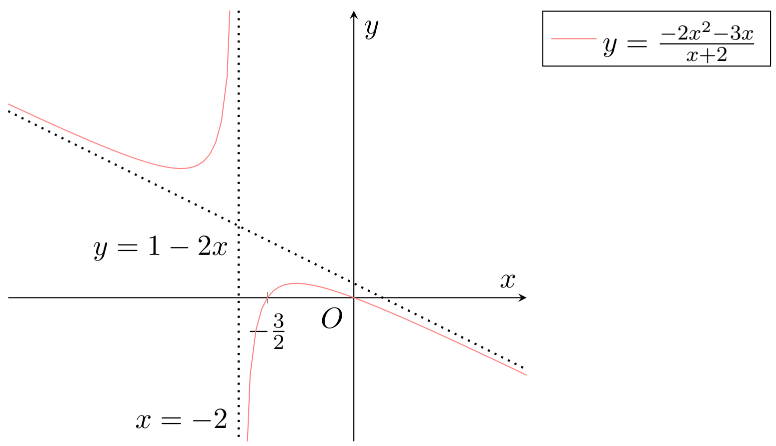
For all values that y can take on, there exists a solution to $2x^2+(3+y)x+2y=0$. Hence, $\Delta\geq 0$.

$$(3+y)^2-4(2)(2y)\geq 0\implies y^2-10y+9\geq 0\implies (y-1)(y-9)\geq 0.$$



Thus, $\{y\in\mathbb{R}\colon y\leq 1 \text{ or } y\geq 9\}$.

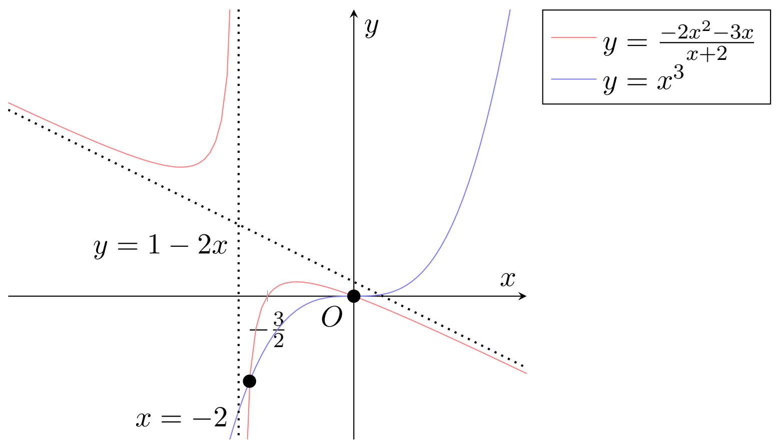
Part (c).



Part (d). Observe that

$$x^4+2x^3+2x^2+3x=0\implies x^3(x+2)=-2x^2-3x\implies x^3=\frac{-2x^2-3x}{x+2}.$$

This motivates us to plot $y=x^3$ and $y=\frac{-2x^2-3x}{x+2}$ on the same graph.



We thus see that $y = x^3$ intersects $y = \frac{-2x^2-3x}{x+2}$ twice, with one intersection point being the origin. Thus, there is only one real non-zero root to $x^4 + 2x^3 + 2x^2 + 3x = 0$.

* * * * *

Problem 8. The curve C is defined by the equation $y = \frac{x}{x^2-5x+4}$.

(a) Write down the equations of the asymptotes.

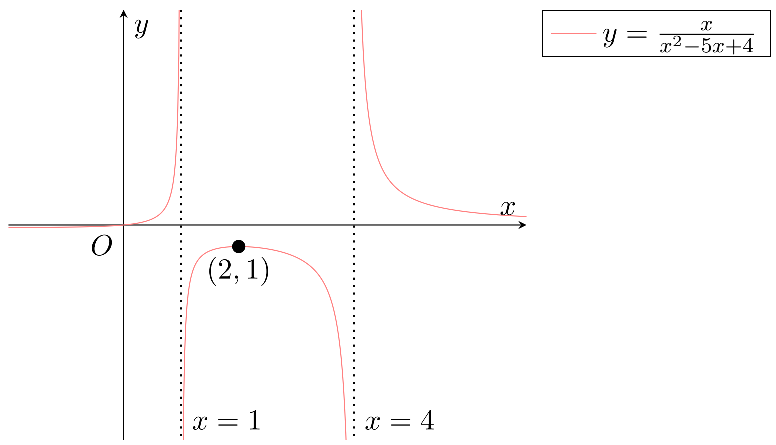
(b) Sketch C, indicating clearly the axial intercept(s), asymptote(s) and turning point(s).

(c) Find the positive value k such that the equation $\frac{x}{x^2-5x+4} = kx$ has exactly 2 distinct real roots.

Solution.

Part (a). As $x \to \pm\infty$, $y \to 0$. Hence, $y = 0$ is an asymptote. Observe that $x^2 - 5x + 4 = (x-1)(x-4)$. Hence, $x = 1$ and $x = 4$ are also asymptotes.

Part (b).



Part (c). Note that $x = 0$ is always a root of $\frac{x}{x^2-5x+4} = kx$. We thus aim to find the value of k such that $\frac{x}{x^2-5x+4} = kx$ has only one non-zero root.

We observe that if $k > 0$, $y = kx$ will intersect with $y = \frac{x}{x^2-5x+4}$ at least twice: before $x = 1$ and after $x = 4$. In order to have only one non-zero root, we must force the intersection point that comes before $x = 1$ to be at the origin $(0,0)$. Hence, k is tangential to C at $(0,0)$, thus giving $k = \mathrm{d}C/\mathrm{d}x|_{x=0}$.

$$k = \left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=0} = \left.\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x^2-5x+4}\right)\right|_{x=0} = \left.\frac{3x^2-10x+4}{(x^2-5x+4)^2}\right|_{x=0} = \frac{1}{4}.$$

Tutorial B1B

Problem 1. Without using a calculator, sketch the following graphs of conics.

(a) $y^2 - 4x = 12$

(b) $(x+1)^2 + y^2 = 4$

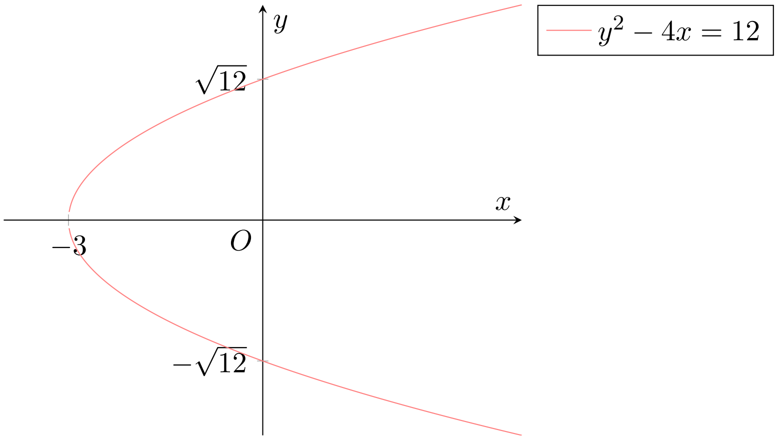
(c) $\frac{(x-3)^2}{9} + \frac{y^2}{2} = 1$

(d) $4x^2 + y^2 = 4$

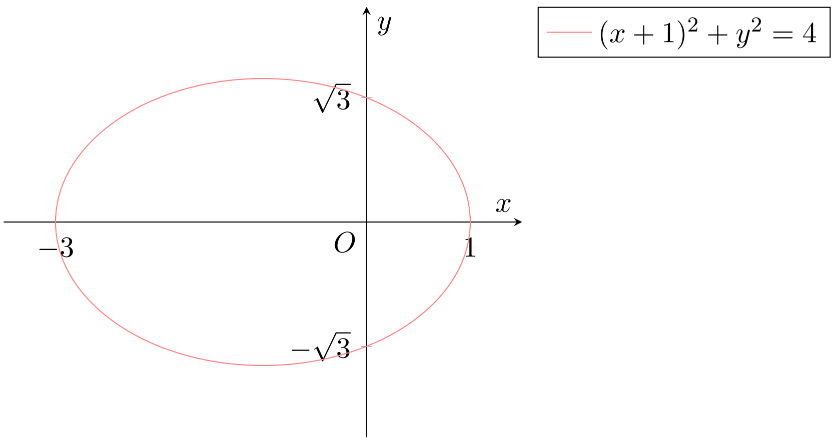
(e) $8y^2 - 2x^2 = 16$

Solution.

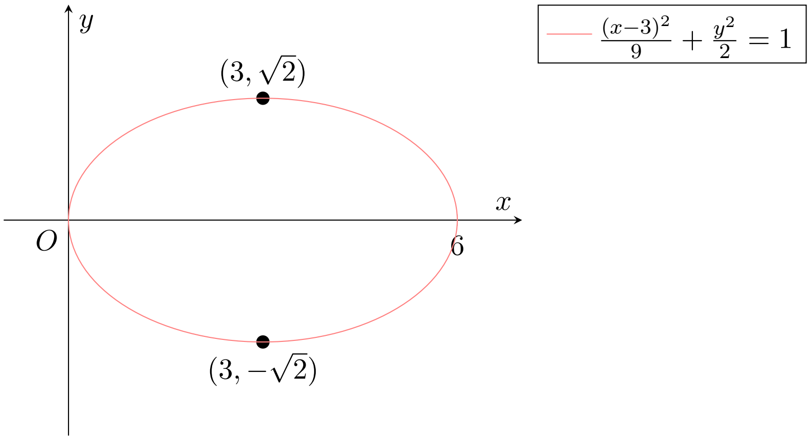
Part (a).



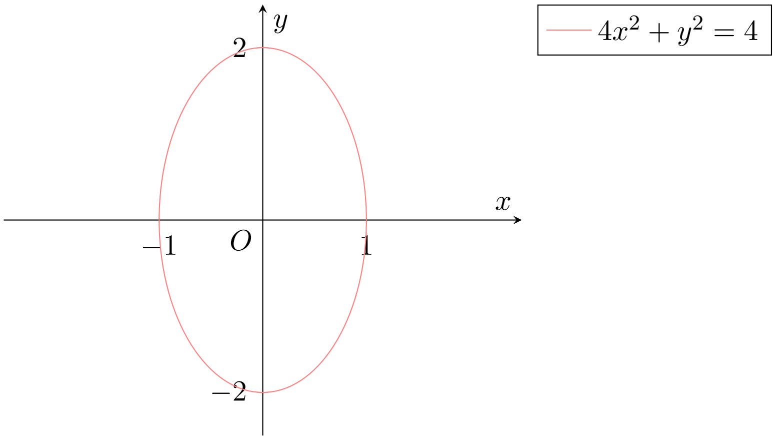
Part (b).



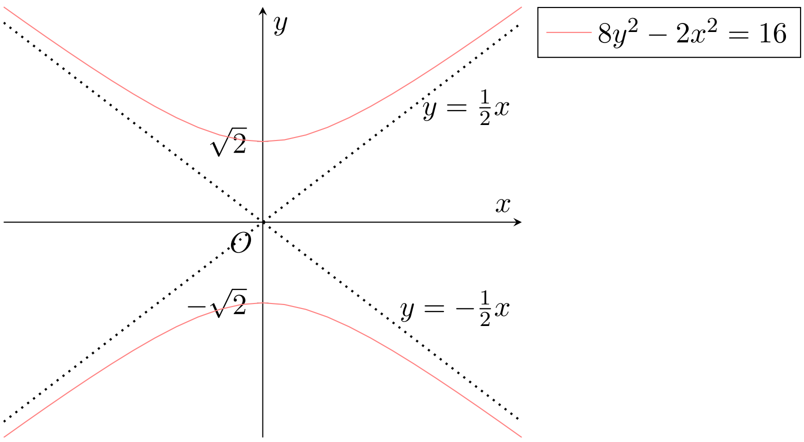
Part (c).



Part (d).



Part (e).



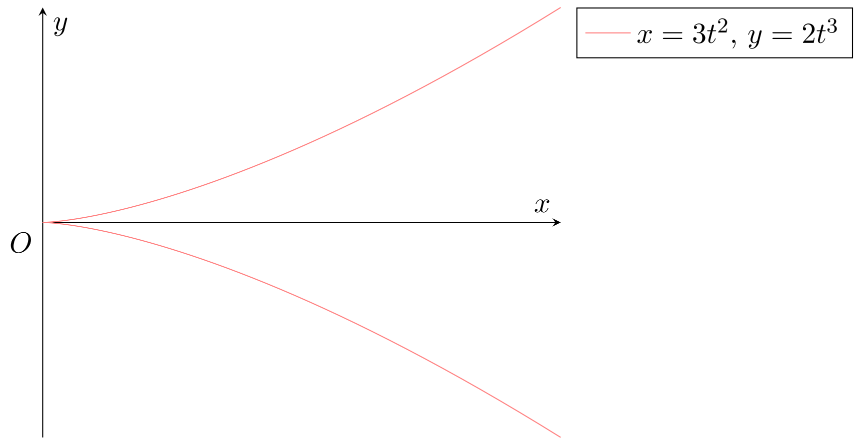
Problem 2. Sketch the curves defined by the following parametric equations, indicating the coordinates of any intersection with the axes.

(a) $x = 3t^2$, $y = 2t^3$

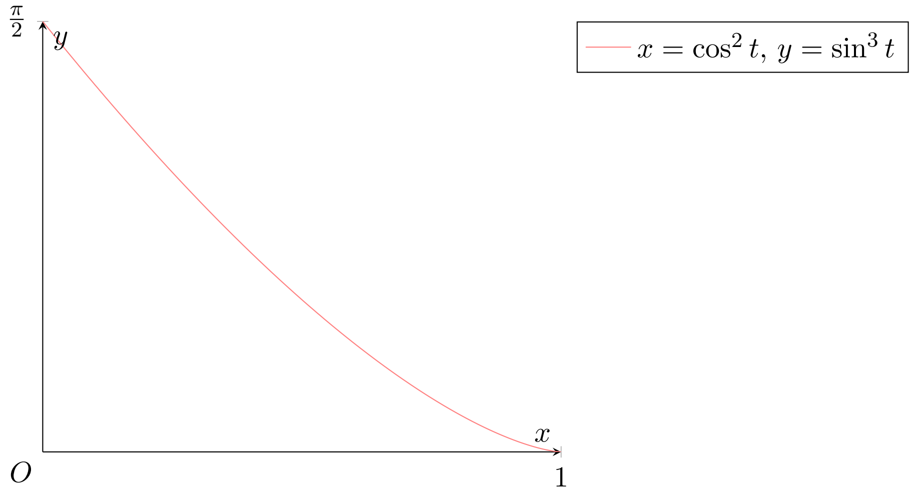
(b) $x = \cos^2 t$, $y = \sin^3 t$, $0 \le t \le \frac{\pi}{2}$

Solution.

Part (a).



Part (b).



Problem 3. Without using a calculator, sketch the following graphs of conics.

(a) $y^2 + 4y + x = 0$

(b) $x^2 + y^2 - 4x - 4y = 0$

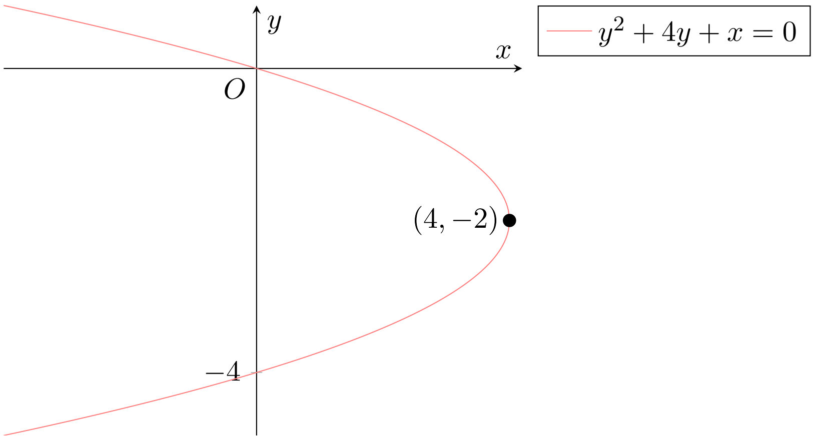
(c) $x^2 + 4y^2 - 2x - 24y + 33 = 0$

(d) $4x^2 - y^2 - 8x + 4y = 1$

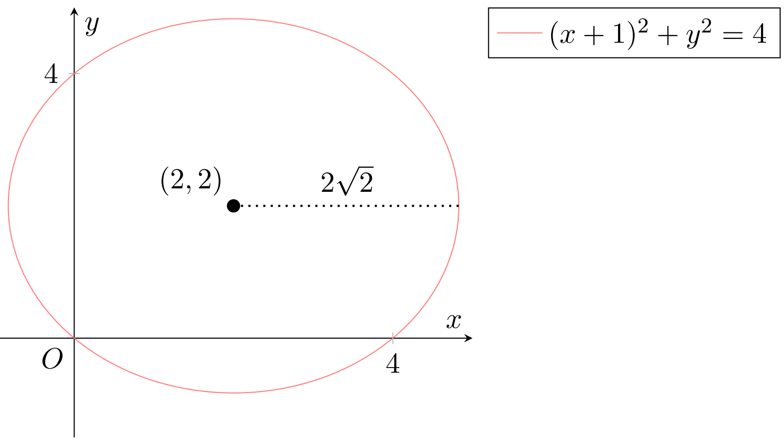
(e) $x = -\sqrt{17 - y^2}$

Solution.

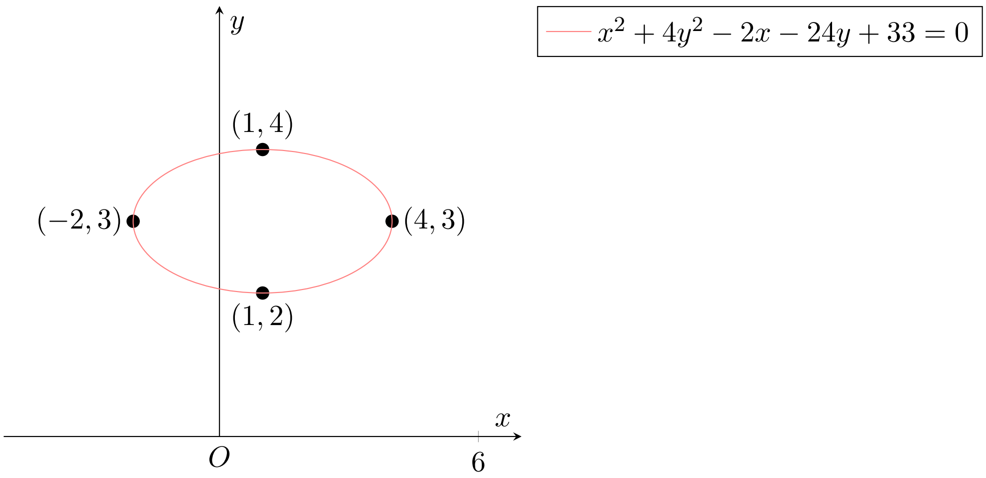
Part (a).



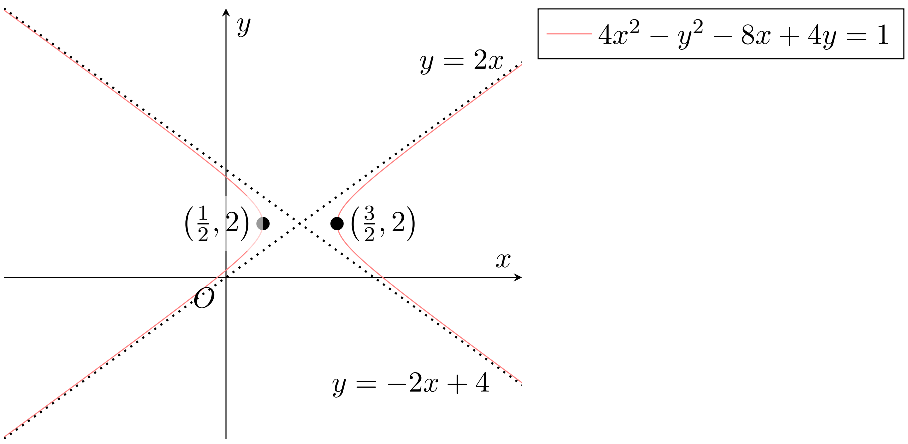
Part (b).



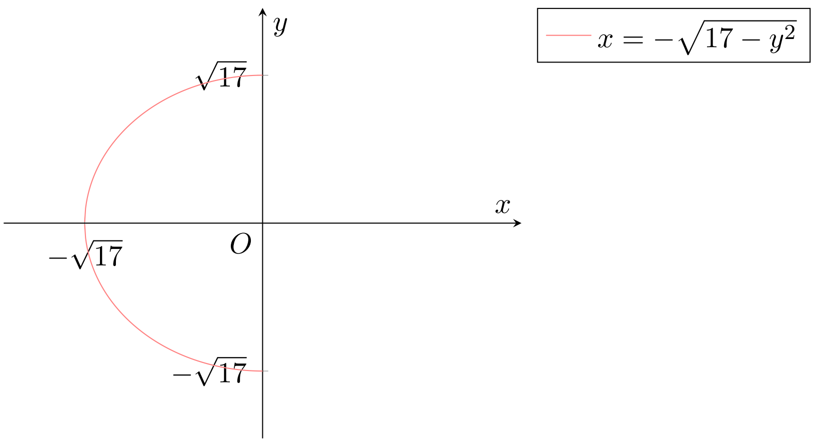
Part (c).



Part (d).



Part (e).



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Problem 4. Sketch the curves defined by the following parametric equations. Find also their respective Cartesian equations.

(a) $x = 4t + 3,\ y = 16t^2 - 9,\ t \in \mathbb{R}$

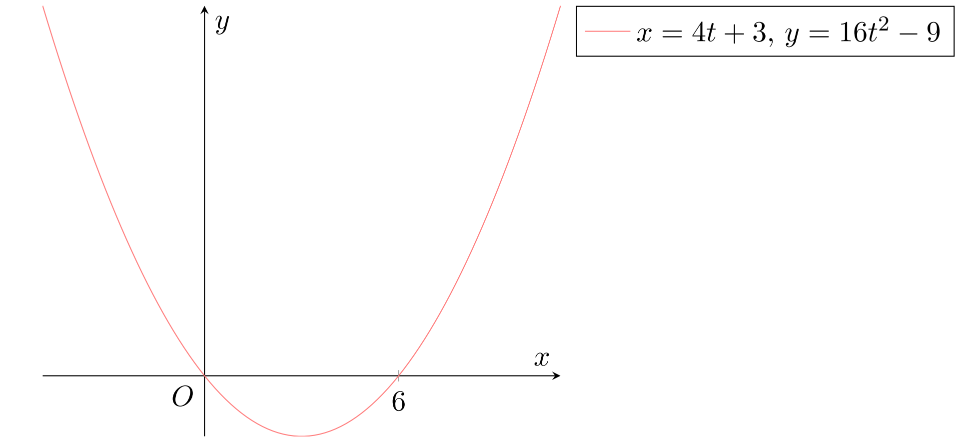
(b) $x = t^2,\ y = 2\ln t,\ t \geq 1$

(c) $x = 1 + 2\cos\theta$, $y = 2\sin\theta - 1$, $0 \le \theta \le \frac{\pi}{2}$

(d) $x = t^2$, $y = \frac{2}{t}$, $t \ne 0$

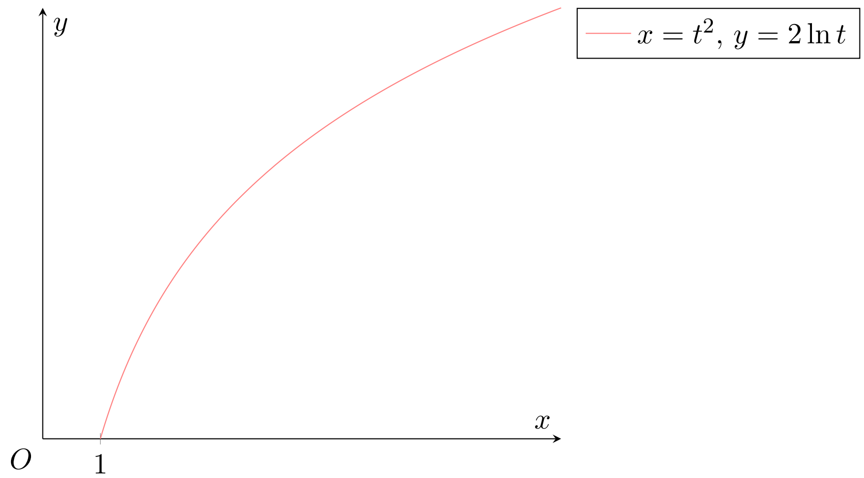
Solution.

Part (a).



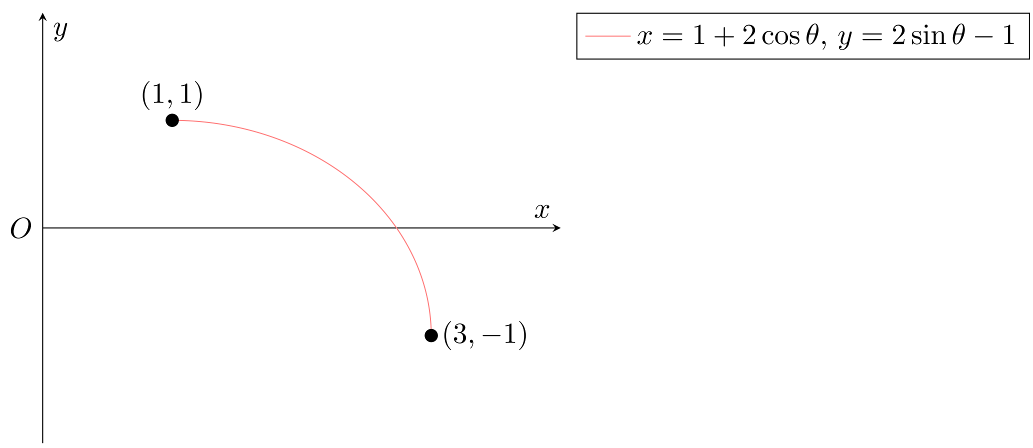
Since $x = 4t + 3$, we have $t = \frac{1}{4}(x - 3)$. Thus, $y = 16\left(\frac{1}{4}(x-3)\right)^2 - 9 = (x-3)^2 - 9$.

Part (b).



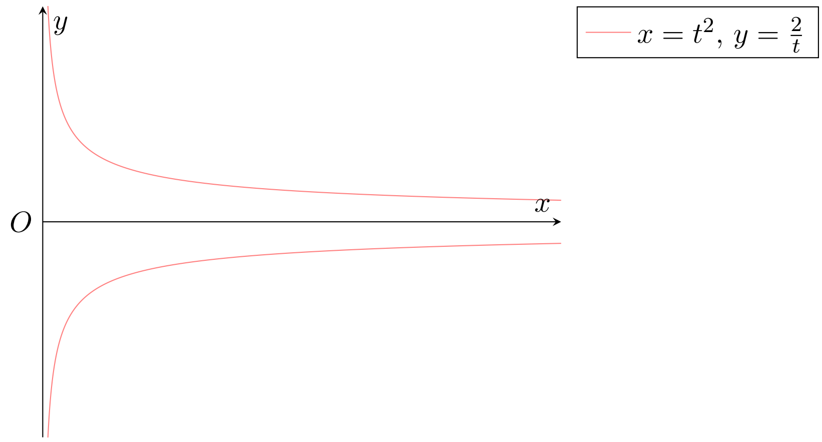
Since $x = t^2$ and $t \ge 1 > 0$, we have $t = \sqrt{x}$. Thus, $y = 2\ln(t) = 2\ln(\sqrt{x}) = \ln(x)$.

Part (c).



We have $2\cos\theta = x-1$ and $2\sin\theta = y+1$. Squaring both equations and adding them, we obtain $4\cos^2\theta + 4\sin^2\theta = (x-1)^2 + (y+1)^2$, which simplifies to $(x-1)^2+(y+1)^2 = 4$.

Part (d).



Since $x = t^2$, we have $t = \pm\sqrt{x}$. Hence, $y = \pm\frac{2}{\sqrt{x}}$.

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Problem 5. The curve C_1 has equation $y = \frac{x-2}{x+2}$. The curve C_2 has equation $\frac{x^2}{6} + \frac{y^2}{3} = 1$.

(a) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersections with the axes and the equations of any asymptotes.

(b) Show algebraically that the x-coordinates of the points of intersection of C_1 and C_2 satisfy the equation $2(x-2)^2 = (x+2)^2\left(6-x^2\right)$.

(c) Use your calculator to find these x-coordinates.

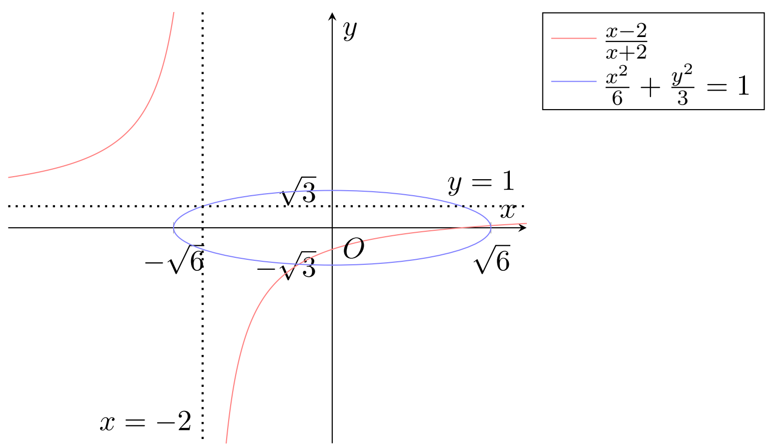
Another curve is defined parametrically by

$$x = \sqrt{6}\cos\theta,\ y = \sqrt{3}\sin\theta,\ -\pi \le \theta \le \pi.$$

(d) Find the Cartesian equation of this curve and hence determine the number of roots to the equation $\sqrt{3}\sin\theta = \frac{\sqrt{6}\cos\theta - 2}{\sqrt{6}\cos\theta+2}$ for $-\pi \le \theta \le \pi$.

Solution.

Part (a).



Part (b). From C_1, we have $y(x+2) = x-2$. Hence,

$$y^2(x+2)^2 = (x-2)^2.$$

From C_2, we have $x^2 + 2y^2 = 6$. Hence,

$$y^2 = \frac{6-x^2}{2}.$$

Putting both equations together, we have

$$(x-2)^2 = \frac{\left(6-x^2\right)(x+2)^2}{2} \implies 2(x-2)^2 = \left(6-x^2\right)(x+2)^2.$$

Part (c). The x-coordinates are $x = -0.515$ or $x = 2.45$.

Part (d). Since $x = \sqrt{6}\cos\theta$ and $y = \sqrt{3}\sin\theta$, we have $x^2 = 6\cos^2\theta$ and $2y^2 = 6\sin^2\theta$. Adding both equations together, we have

$$x^2 + 2y^2 = 6\cos^2\theta + 6\sin^2\theta = 6 \implies \frac{x^2}{6} + \frac{y^2}{3} = 1.$$

This is the equation that gives C_1. We further observe that the equation $\sqrt{3}\sin\theta = \frac{\sqrt{6}\cos\theta - 2}{\sqrt{6}\cos\theta + 2}$ simplifies to $y = \frac{x-2}{x+2}$. This is the equation that gives C_2. Since there are two intersections between C_1 and C_2, there are thus two roots to the equation $\sqrt{3}\sin\theta = \frac{\sqrt{6}\cos\theta - 2}{\sqrt{6}\cos\theta + 2}$.

Self-Practice B1

Problem 1. The equations of the curves C_1 and C_2 are given by $y = \frac{2x+1}{x-3}$ and $3(x-1)^2 + 4y^2 = 12$ respectively. Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes.

* * * * *

Problem 2. The curve C has equation $y = \frac{x^2-4x}{x^2-9}$.

(a) Express y in the form $P + \frac{Q}{x-3} + \frac{R}{x+3}$, where P, Q and R are constants.

(b) Sketch C, showing clearly the asymptotes and the coordinates of the points of intersection with the axes.

* * * * *

Problem 3. The curve C has the equation $y = \frac{x^2+px-q}{x+r}$. It is given that C has a vertical asymptote at $x = -3$ and intersects the x-axis at $x = -2$ and $x = 1$.

(a) Determine the values of p, q and r.

(b) State the equation of other asymptote(s).

(c) Prove, using an algebraic method, that y cannot lie between two values which are to be determined.

(d) Hence, sketch C, labelling clearly the axial intercepts, asymptotes and the coordinates of any turning points.

(e) Deduce the number of roots of the equation $3x^4 + 3x^3 - 6x^2 - x - 3 = 0$.

* * * * *

Problem 4. The curve C has equation $y^2 = 5x^2 + 4$.

(a) Sketch C, indicating clearly the axial intercepts, the equations of the asymptotes and the coordinates of the stationary points.

(b) Hence by inserting a suitable graph, determine the range of values of h, where h is a positive constant, such that the equation $5x^2 + 4 = h^2\left(1 - x^2\right)$ has no real roots.

* * * * *

Problem 5. The curve C has equation $y = \frac{mx^2+2x+m}{x}$, where m is a non-zero constant.

(a) Find the range of values of m for C to cut the x-axis at two distinct points.

(b) For $m = \frac{1}{2}$, find the equations of the asymptotes of C.

(c) Hence, sketch the curve C for $m = \frac{1}{2}$: indicating clearly the asymptotes, any turning points and axial intercepts.

(d) By drawing a sketch of another suitable curve in the same diagram as your sketch of C, deduce the number of real roots of the equation $x^2 + 4x + 1 = -2xe^x$.

Problem 6. The curve C_1 has equation $\frac{(x-1)^2}{4} = \frac{y^2}{9} + 4$.

Sketch C_1, making clear the main relevant features, and state the set of values that x can take.

Another curve C_2 is defined by the parametric equations

$$x = \frac{2}{t^2+1}, \quad y = 3\sqrt{t}\ln t, \quad t > 1.$$

Use a non-graphical method to determine the set of possible values of x. Sketch the curve C_2, labelling all axial intercepts and asymptotes (if any) clearly.

Hence, **without solving the equation**, state the number of real roots to the equation

$$9\left(\frac{2}{t^2+1} - 1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144,$$

explaining your reason(s) clearly.

Given that $k > 0$, state the smallest integer value of k such that the equation

$$9\left(\frac{2}{t^2+1} + k - 1\right)^2 = 4\left(3\sqrt{t}\ln t\right)^2 + 144,$$

has exactly one real root which is positive.

* * * * *

Problem 7 (🌶).

(a) An ellipse of equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $0 < b < a$, has two points called foci $F_1(-c, 0)$ and $F_2(c, 0)$. The definition of the ellipse is such that for every point P on the ellipse, the sum of the distance of P to F_1 and F_2 is always a constant k.

   (i) By considering a suitable point on the ellipse, determine the value of k in terms of a and/or b. By considering another suitable point on the ellipse, find c in terms of a and b.

(b) A hyperbola with equation $(y-h)^2 - 1 = \frac{1}{4}(x-k)^2$ has $y = \frac{1}{2}x + \frac{3}{2}$ as one of its asymptotes, and the point $(1, 3)$ is on the hyperbola. Find the values of h and k.

Assignment B1A

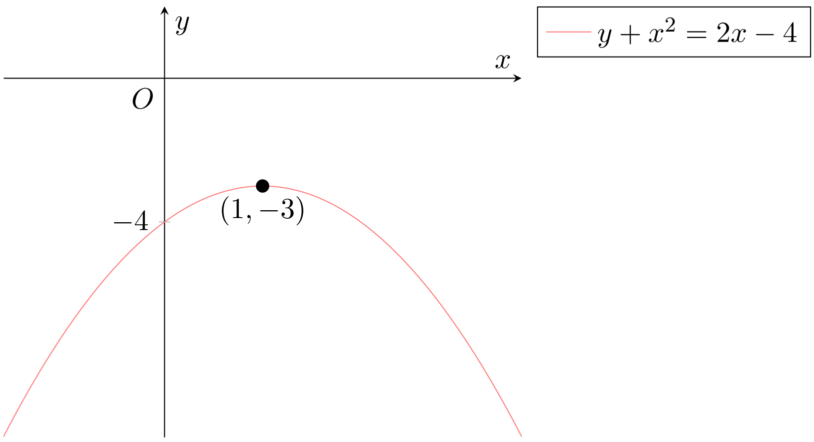
Problem 1. Sketch clearly labelled diagrams of each of the following curves, giving exact values of axial intercepts, stationary points and equations of asymptotes, if any.

(a) $y + x^2 = 2x - 4$

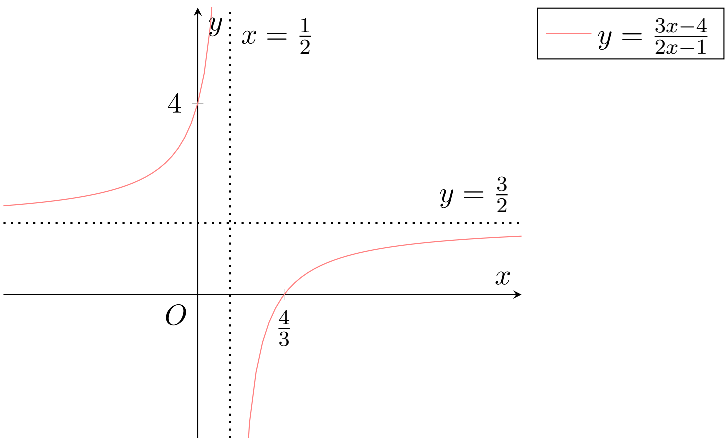
(b) $y = \frac{3x-4}{2x-1}$

Solution.

Part (a).



Part (b).



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Problem 2. On separate diagrams, sketch the graphs of

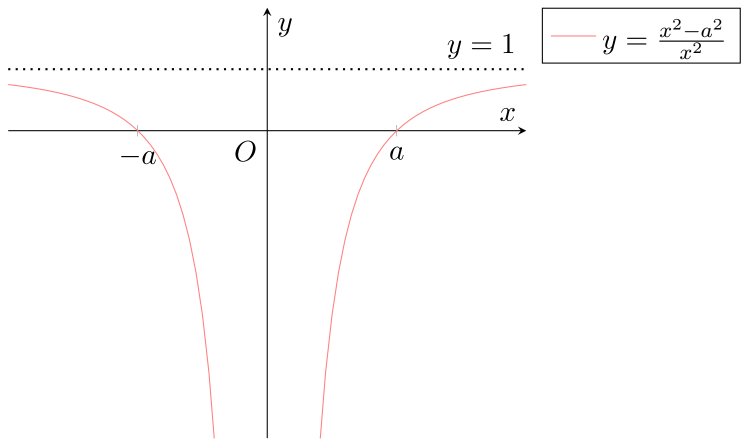
(a) $y = \frac{x^2-a^2}{x^2}$, $a > 0$

(b) $y = \frac{x-1}{2x(x+3)}$

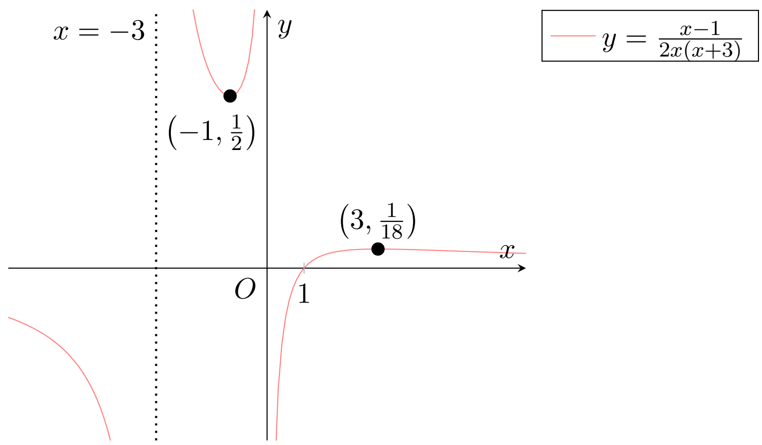
Indicate clearly the coordinates of axial intercepts, stationary points and equations of asymptotes, if any.

Solution.

Part (a).



Part (b).



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Problem 3. The curve C has equation $y = \frac{ax^2+bx-2}{x+4}$, where a and b are constants. It is given that $y = 2x - 5$ is an asymptote of C.

(a) Find the values of a and b.

(b) Sketch C.

(c) Using an algebraic method, find the set of values that y cannot take.

(d) By drawing a sketch of another suitable curve in the same diagram as your sketch of C in part (b), deduce the number of distinct real roots of the equation $x^3+6x^2+3x-2 = 0$.

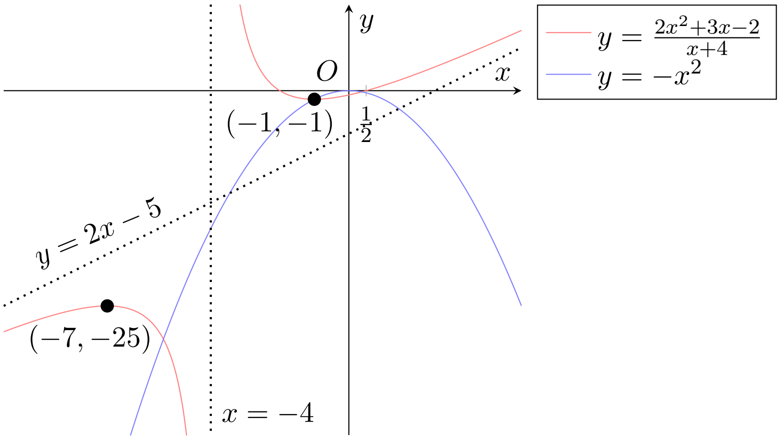
Solution.

Part (a). Since $y = 2x - 5$ is an asymptote of C, $\frac{ax^2+bx-2}{x+4}$ can be expressed in the form $2x - 5 + \frac{k}{x+4}$, where k is a constant.

$$\frac{ax^2 + bx - 2}{x + 4} = 2x-5+\frac{k}{x + 4} \implies ax^2+bx-2 = (2x-5)(x+4)+k = 2x^2+3x-20+k.$$

Comparing coefficients of x^2, x and constant terms, we have $a = 2$, $b = 3$ and $k = 18$.

Part (b).

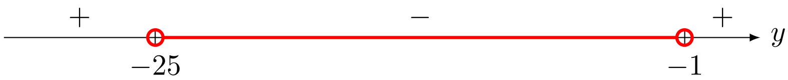


Part (c).

$$y = \frac{2x^2 + 3x - 2}{x + 4} \implies (x+4)y = 2x^2 + 3x - 2 \implies 2x^2 + (3 - y)x - (2 + 4y) = 0.$$

For values that y cannot take on, there exist no solutions to $2x^2 + (3-y)x - (2+4y) = 0$. Hence, $\Delta < 0$. Hence,

$$(3 - y)^2 - 4(2)\left(-(2 + 4y)\right) < 0 \implies y^2 + 26y + 25 < 0 \implies (y + 25)(y + 1) < 0.$$



Thus, the set of values that y cannot take is $\{y \in \mathbb{R} \colon -25 < y < -1\}$.

Part (d).

$$x^3 + 6x^2 + 3x - 2 = 0 \implies \frac{x^3 + 4x^2}{x + 4} + \frac{2x^2 + 3x - 2}{x + 4} = x^2 + \frac{2x^2 + 3x - 2}{x + 4} = 0$$
$$\implies \frac{2x^2 + 3x - 2}{x + 4} = -x^2.$$

Plotting $y = -x^2$ on the same digram, we see that there are 3 intersections between $y = x^2$ and C. Hence, there are 3 distinct real roots to $x^3 + 6x^2 + 3x - 2 = 0$.

Assignment B1B

Problem 1. Without using a calculator, sketch the graphs of the conics in parts (a), (b) and c.

(a) $3x^2 + 2y^2 = 6$

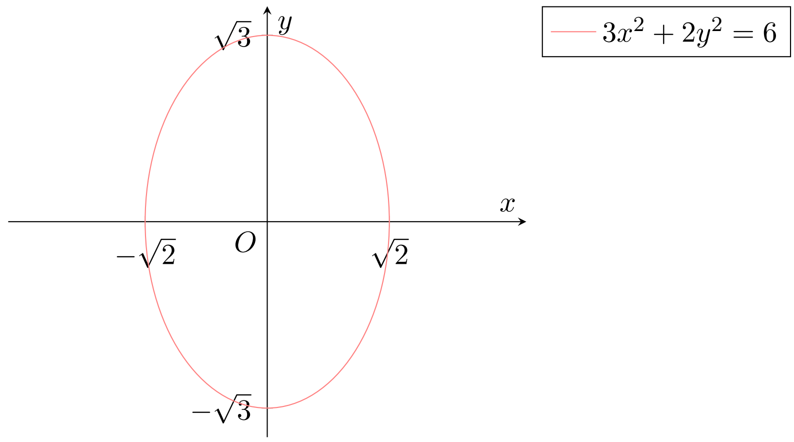
(b) $x^2 + y^2 + 4x - 2y - 20 = 0$

(c) $4(y-1)^2 - x^2 = 4$

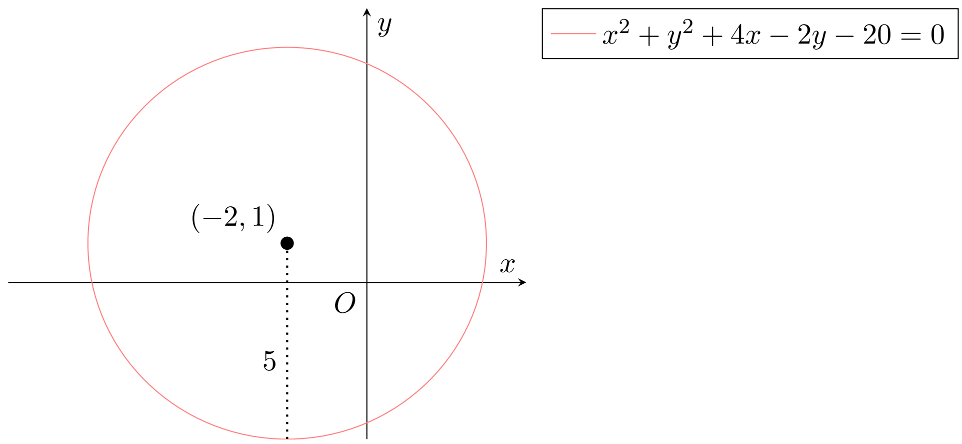
State a transformation that will transform the graph of (a) to a circle with centre $(0, 0)$ and radius $\sqrt{3}$.

Solution.

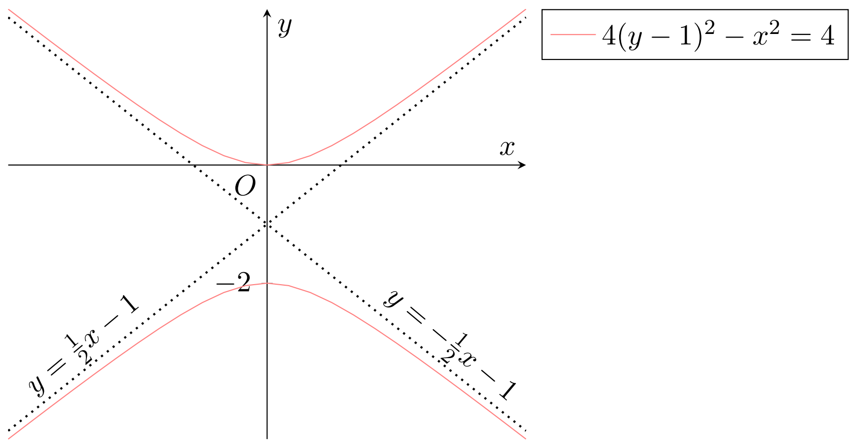
Part (a).



Part (b).



Part (c).



The transformation is $x \mapsto \sqrt{\frac{2}{3}}x$.

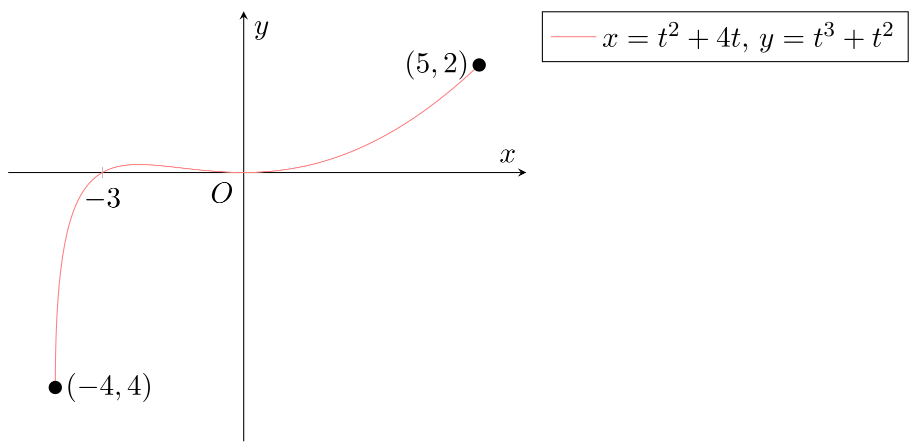
* * * * *

Problem 2. The curve C has parametric equations

$$x = t^2 + 4t,\ y = t^3 + t^2.$$

Sketch the curve for $-2 \leq t \leq 1$, stating the axial intercepts.

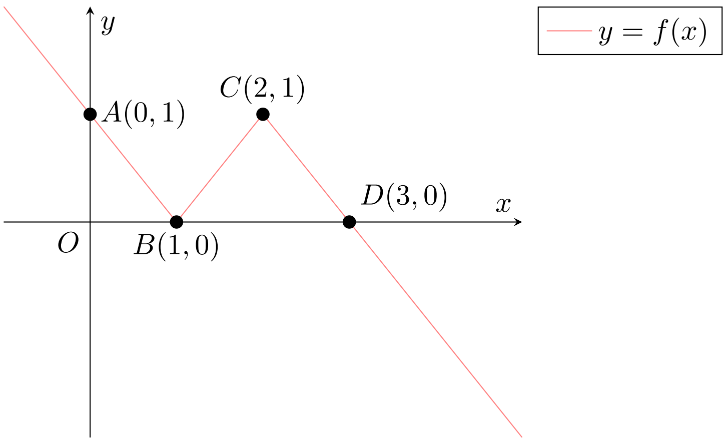
Solution.



B2 Graphs and Transformations II

Tutorial B2

Problem 1.



The graph of $y = f(x)$ is shown here. The points A, B, C and D have coordinates $(0, 1)$, $(1, 0)$, $(2, 1)$ and $(3, 0)$ respectively. Sketch, separately, the graphs of

(a) $y = f(2x)$

(b) $y = f(x + 3)$

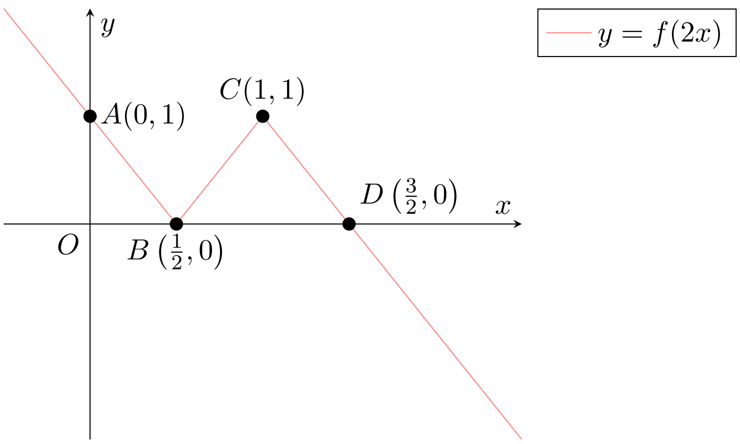
(c) $y = 1 - f(x)$

(d) $y = 3f\left(\frac{x}{2} - 1\right)$

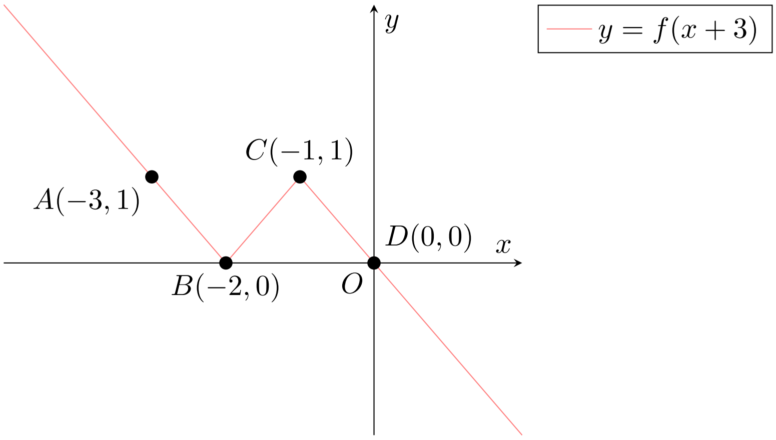
stating, in each case, the coordinates of the points corresponding to A, B, C and D.

Solution.

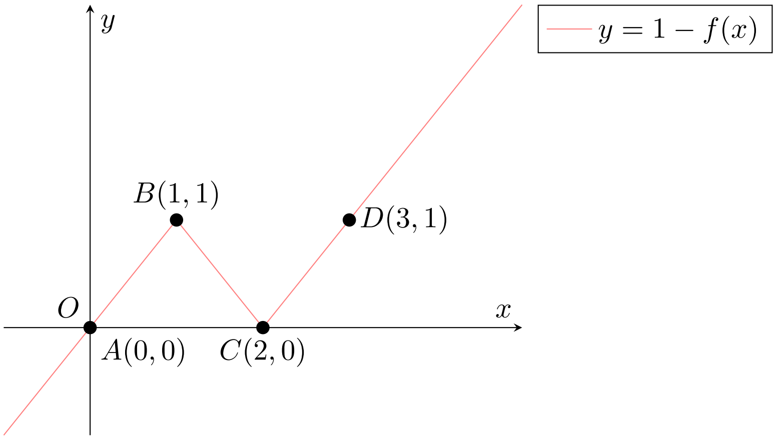
Part (a).



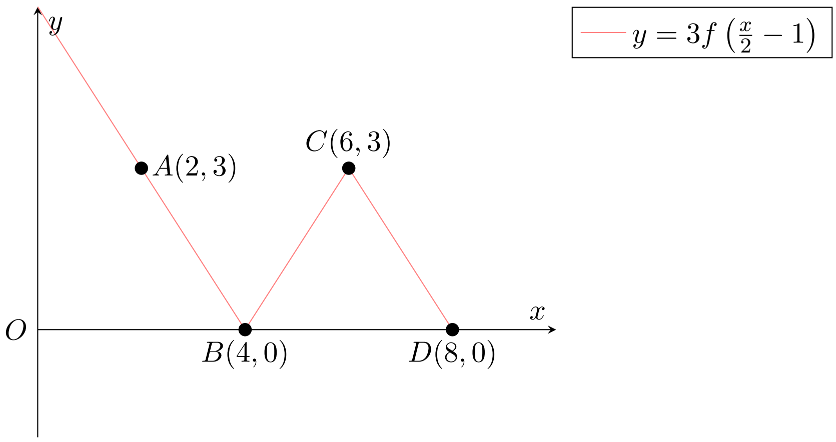
Part (b).



Part (c).



Part (d).



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Problem 2. Sketch, on a single clear diagram, the graphs of

(a) $y = x^2$

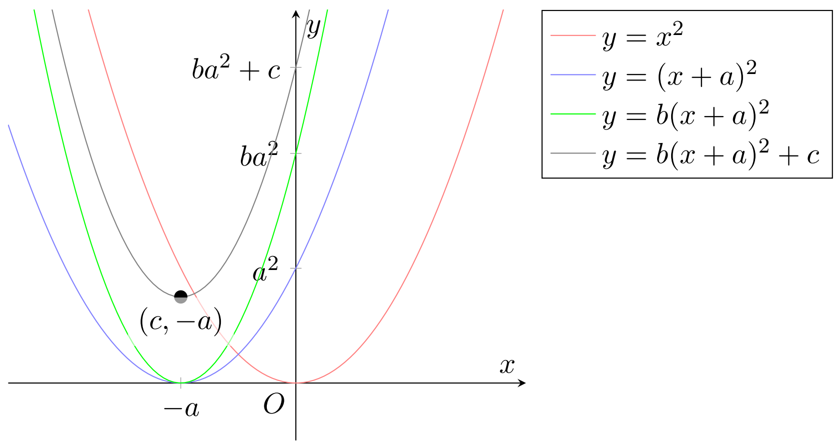
(b) $y = (x+a)^2$

(c) $y = b(x+a)^2$

(d) $y = b(x+a)^2 + c$

Assume constants $a > 0$, $c > 0$ and $b > 1$.

Solution.



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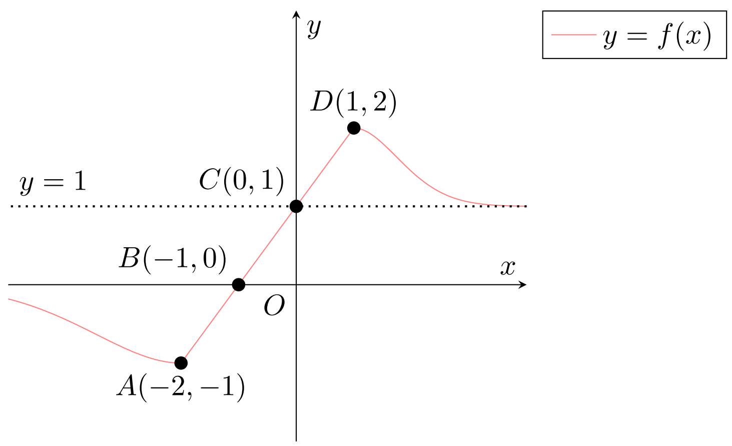
Problem 3. The graph below has equation $y = f(x)$. It has asymptotes $y = 1$ and $y = 0$, a maximum point at $D(1, 2)$, a minimum point at $A(-2, -1)$, cuts the x-axis at $B(-1.0)$ and cuts the y-axis at $C(0, 1)$.



Sketch on separate diagrams the graphs of the following curves, labelling each curve clearly, indicating the horizontal asymptotes and showing the coordinates of the points corresponding to the points A, B, C and D.

(a) $y = f(x+1)$

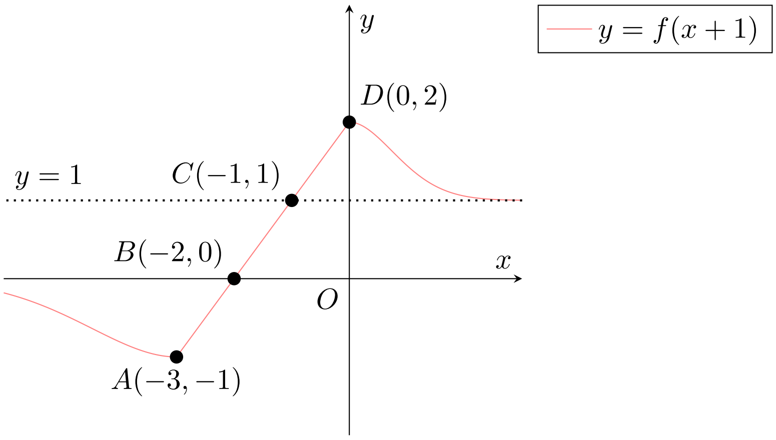
(b) $y = f\left(\frac{x}{2}\right)$

(c) $y = 2f(x) - 2$

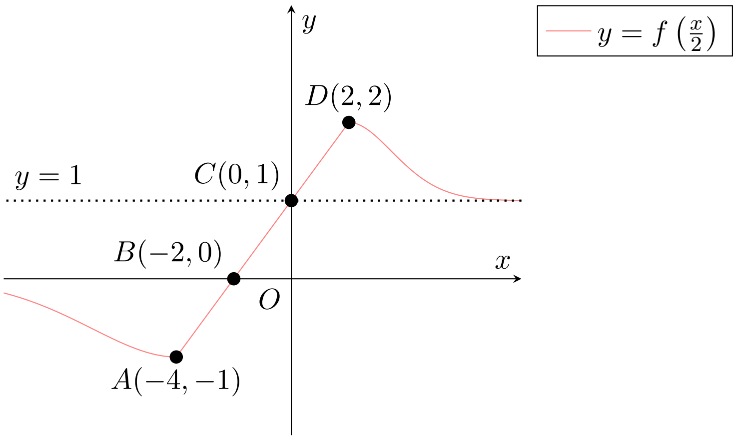
Find the number of solutions to the equation $f(x) = af(x)$ where $a \geq 2$.

Solution.

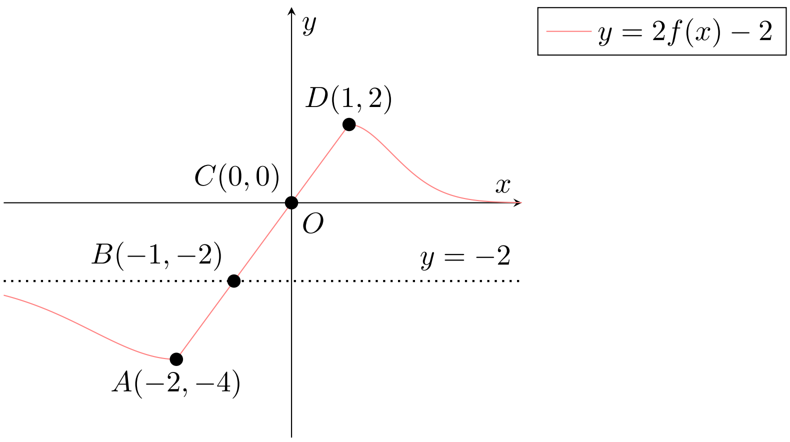
Part (a).



Part (b).



Part (c).



All points with a y-coordinate of 0 are invariant under the transformation $f(x) \mapsto af(x)$. Since there is only one such point $(B(-1,0))$, there is only 1 solution to the equation $f(x) = af(x)$, where $a \geq 2$.

Problem 4. The curve with equation $y = x^2$ is transformed by a translation of 2 units in the positive x-direction, followed by a stretch with scale factor $\frac{1}{2}$ parallel to the y-axis, followed by a translation of 6 units in the negative y-direction. Find the equation of the new curve in the form $y = f(x)$ and the exact coordinates of the points where this curve crosses the x- and y-axes.

Solution.

$$\begin{array}{ccc} y = x^3 & \xrightarrow{x \mapsto x-2} & y = (x-2)^3 \\ & & \big\downarrow y \mapsto 2y \\ 2(y+6) = (x-2)^3 & \xleftarrow{y \mapsto y+6} & 2y = (x-2)^3 \end{array}$$

Hence, $y = \frac{1}{2}(x-2)^3 - 6$

When $x = 0$, $y = -10$. When $y = 0$, $x = 2 + \sqrt[3]{12}$. Thus, the curve crosses the x-axis at $(2 + \sqrt[3]{12}, 0)$ and the y-axis at $(0, -10)$.

* * * * *

Problem 5. Find the values of the constants A and B such that $\frac{x^2-4x}{(x-2)^2} = A + \frac{B}{(x-2)^2}$ for all values of x except $x = 2$.

Hence, state a sequence of transformations by which the graph of $y = \frac{x^2-4x}{(x-2)^2}$ may be obtained from the graph of $y = \frac{1}{x^2}$.

Solution.

$$\frac{x^2-4x}{(x-2)^2} = \frac{(x-2)^2-4}{(x-2)^2} = 1 + \frac{-4}{(x-2)^2}.$$

Thus, $A = 1$ and $B = -4$.

$$\begin{array}{ccccc} y = \dfrac{1}{x^2} & \xrightarrow{x \mapsto x-2} & y = \dfrac{1}{(x-2)^2} & \xrightarrow{y \mapsto \frac{1}{4}y} & y = \dfrac{4}{(x-2)^2} \\ & & & & \big\downarrow y \mapsto -y \\ & & y = 1 + \dfrac{-4}{(x-2)^2} & \xleftarrow{y \mapsto y-1} & y = \dfrac{-4}{(x-2)^2} \end{array}$$

1. Translate the curve 2 units in the positive x-direction.
2. Stretch the curve with a scale factor of 4 parallel to the y-axis.
3. Reflect the curve about the x-axis.
4. Translate the curve 1 unit in the positive y-direction.

* * * * *

Problem 6. The transformations A, B, C and D are given as follows:

- A: A reflection about the y-axis.
- B: A translation of 2 units in the positive x-direction.
- C: A scaling parallel to the y-axis by a factor of 3.

- D: A translation of 1 unit in the positive y-direction.

A curve undergoes the transformations A, B, C and D in succession, and the equation of the resulting curve is $y = 3\sqrt{2-x}+1$. Determine the equation of the curve before the transformations were effected.

Solution.

$$\begin{aligned}
A\colon x \mapsto -x \quad &\Longrightarrow \quad A^{-1}\colon x \mapsto -x \\
B\colon x \mapsto x-2 \quad &\Longrightarrow \quad B^{-1}\colon x \mapsto x+2 \\
C\colon y \mapsto \frac{1}{3}y \quad &\Longrightarrow \quad C^{-1}\colon y \mapsto 3y \\
D\colon y \mapsto y-1 \quad &\Longrightarrow \quad D^{-1}\colon y \mapsto y+1
\end{aligned}$$

$$y = 3\sqrt{2-x}+1$$

$$\Big\downarrow D^{-1}$$

$$y+1 = 3\sqrt{2-x}+1$$

$$\Big\downarrow C^{-1}$$

$$3y+1 = 3\sqrt{2-x}+1$$

$$\Big\downarrow B^{-1}$$

$$3y+1 = 3\sqrt{2-(x+2)}+1$$

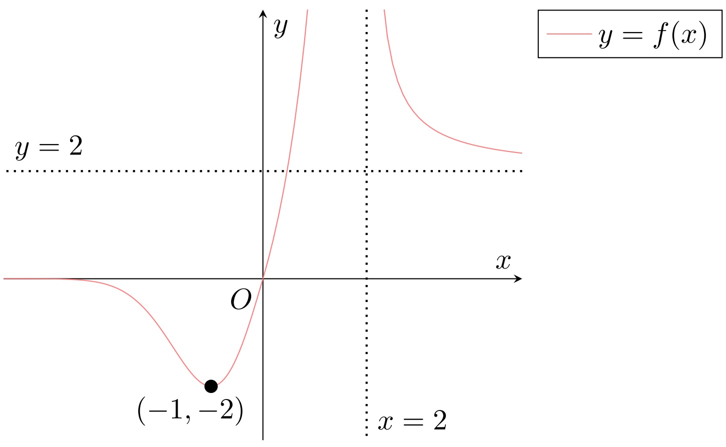
$$\Big\downarrow A^{-1}$$

$$3y+1 = 3\sqrt{2-(-x+2)}+1$$

Thus, the original curve has equation $y = \sqrt{x}$.

* * * * *

Problem 7.



The diagram shows the graph of $y = f(x)$. The curve passes through the origin and has minimum point $(-1,-2)$. The asymptotes are $x = 2$, $y = 0$ and $y = 2$.

Sketch, on separate diagrams, the graphs of

(a) $y = f(x-1)$

(b) $y = f(|x|)$

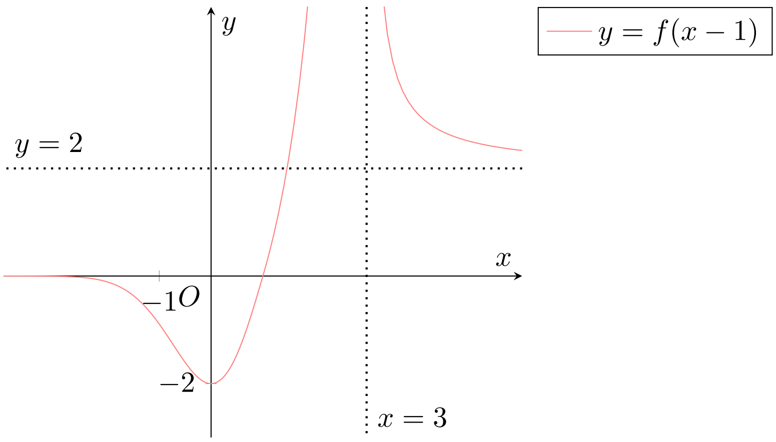
(c) $y = f(|x-1|)$

(d) $y = |f(x)|$

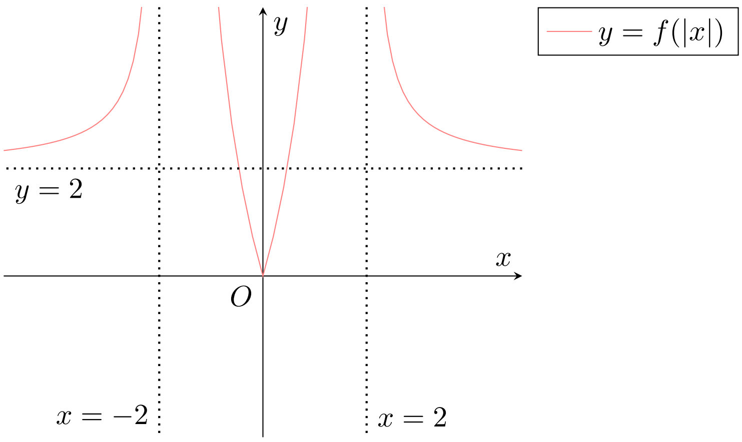
(e) $y = \frac{1}{f(x)}$

Solution.

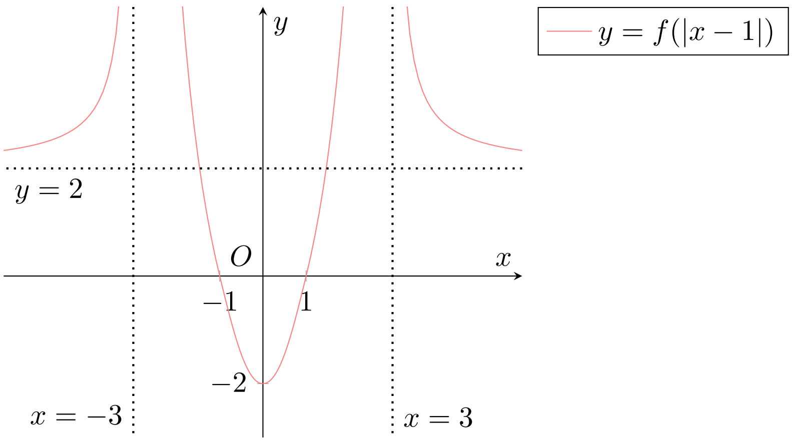
Part (a).



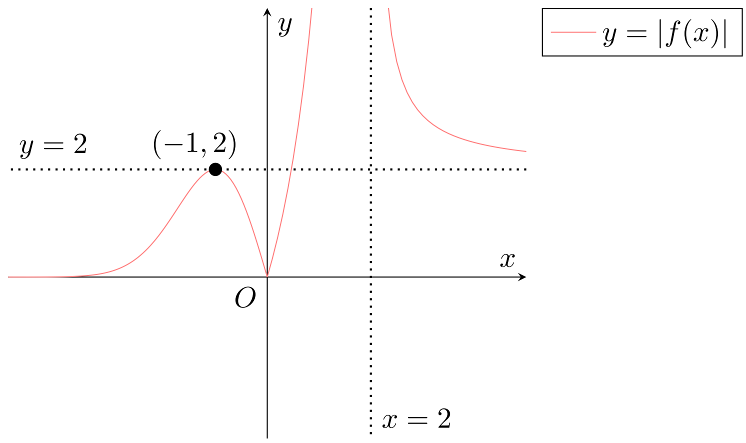
Part (b).



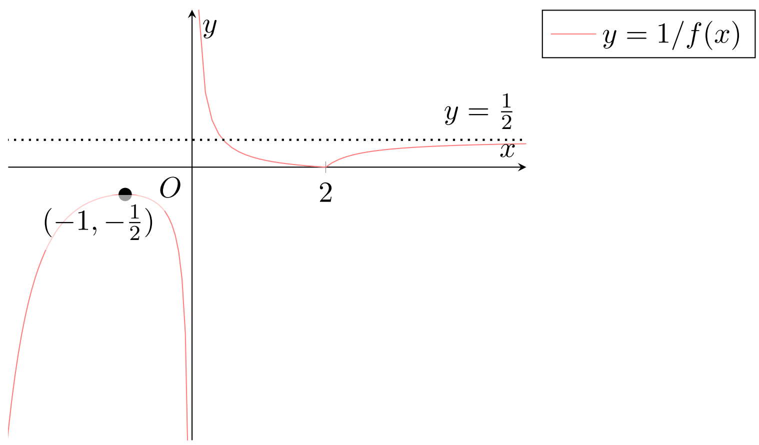
Part (c).



Part (d).



Part (e).



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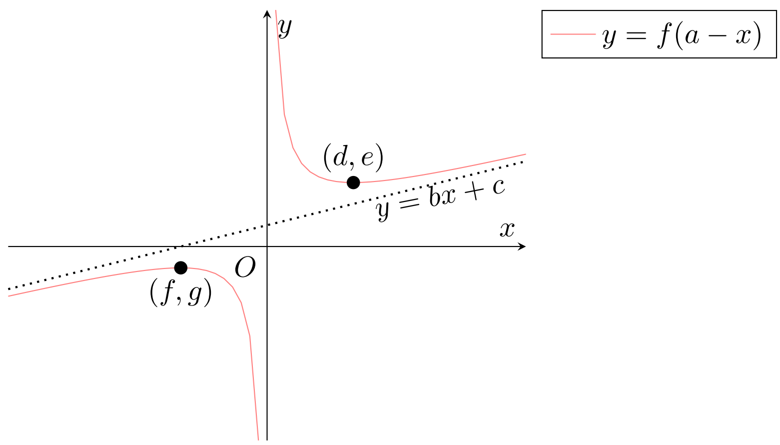
Problem 8.



The graph of $y = f(a - x)$ is shown in the figure, where $a > 0$. The curve has asymptotes $x = 0$, $y = bx + c$, a minimum point at (d, e) and a maximum point at (f, g).

Given $a > d$, sketch separately, the graphs of

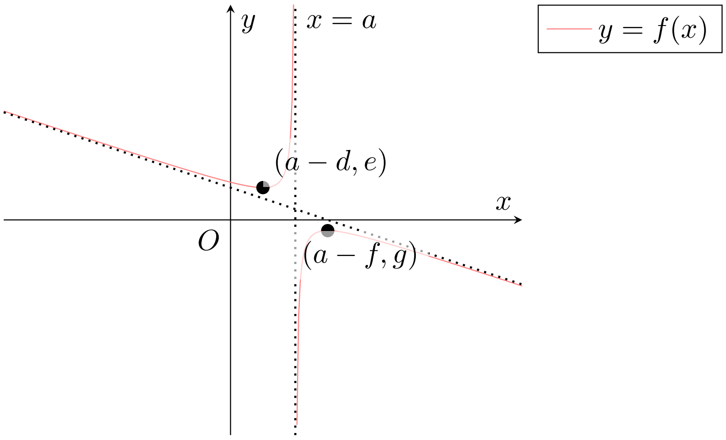
(a) $y = f(x)$

(b) $y = f(|x|)$

(c) $y = \frac{1}{f(x)}$

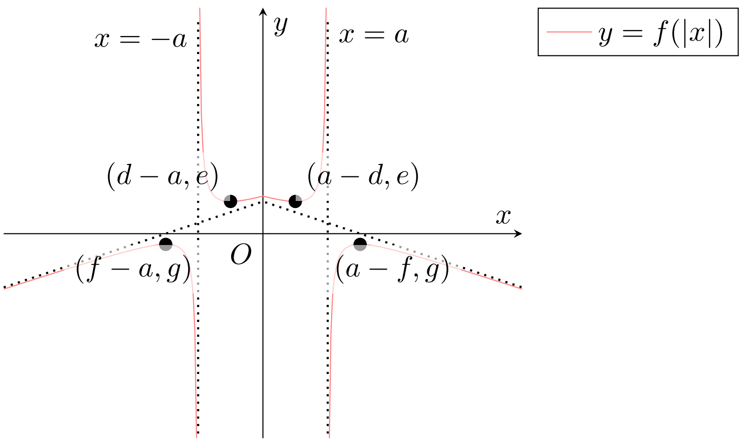
Solution.

Part (a).



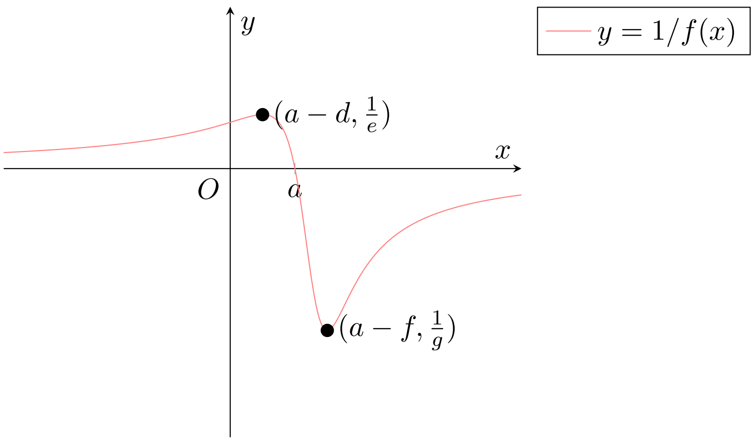
Equation of asymptote: $y = b(a - x) + c$

Part (b).



Equation of asymptotes: $y = b(a + x) + c$, $y = b(a - x) + c$

Part (c).



Problem 9. A curve C_1 is defined by the parametric equations

$$x = t(t+2),\ y = 2(t+1).$$

(a) Find the axial intercepts of the curve.

(b) Sketch C_1.

(c) A curve C_2 is defined by the parametric equations $x = t(t+2)$, $y = t+1$. Describe a geometrical transformation which maps C_1 to C_2. Hence, sketch the curve C_2 in the same diagram as C_1.

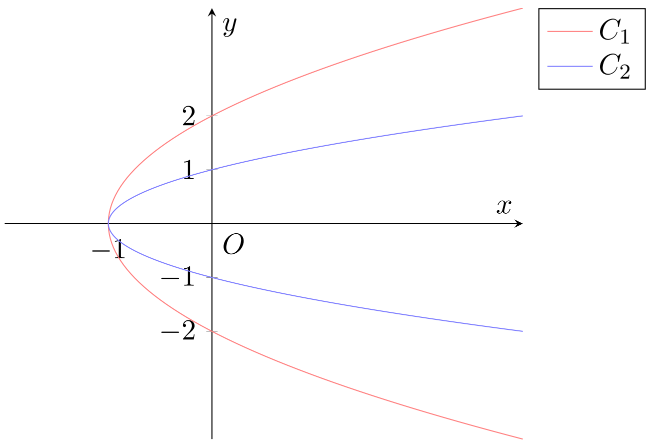
(d) Show that the Cartesian equation of the curve C_1 is given by $y^2 = 4(x+1)$.

Solution.

Part (a). Consider $x = 0$. Then $t(t+2) = 0$, whence $t = 0$ or $t = -2$. When $t = 0$, $y = 2$. When $t = -2$, $y = -2$. Hence, the curve intercepts the y-axis at $(0, 2)$ and $(0, -2)$.

Consider $y = 0$. Then $t = -1$, whence $x = -1$. Hence, the curve intercepts the x-axis at $(-1, 0)$.

Part (b).



Part (c). Scale by a factor of $\frac{1}{2}$ parallel to the y-axis.

Part (d).

$$y^2 = (2(t+1))^2 = 4(t^2 + 2t + 1) = 4(t(t+1) + 1) = 4(x+1).$$

Self-Practice B2

Problem 1. Show that the equation $y = \frac{2x+7}{x+2}$ can be written as $y = A + \frac{B}{x+2}$, where A and B are constants to be found. Hence, state a sequence of transformations which transform the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{2x+7}{x+2}$.

Sketch the graph of $y = \frac{2x+7}{x+2}$, giving the equations of any asymptotes and the coordinates of any points of intersection with the x- and y-axes.

* * * * *

Problem 2. The diagram shows the curve with equation $y = f(x)$. The curve passes through the origin, and has asymptotes $x = a$ and $y = b$, where a and b are positive constants.

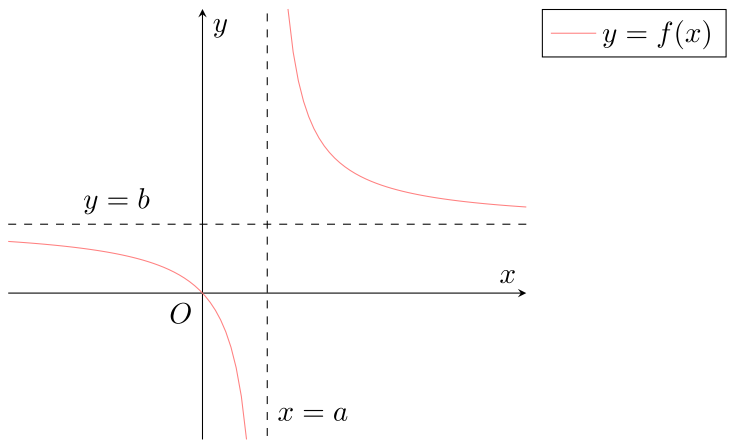


Figure 23.1

On separate diagrams, draw sketches of the graphs of

(a) $y = f(x + a) - b$,

(b) $y = 1/f(x)$,

showing clearly the axial intercepts and asymptotes (if any).

* * * * *

Problem 3. The curves C_1 and C_2 are given by the equations $x^2 + y^2 = 1$ and $x^2 - 2x + 9y^2 = a$ respectively, where a is a real constant. The curve C_2 cuts the x-axis at the origin O and is symmetrical about the line $x = b$, as shown in the diagram below.

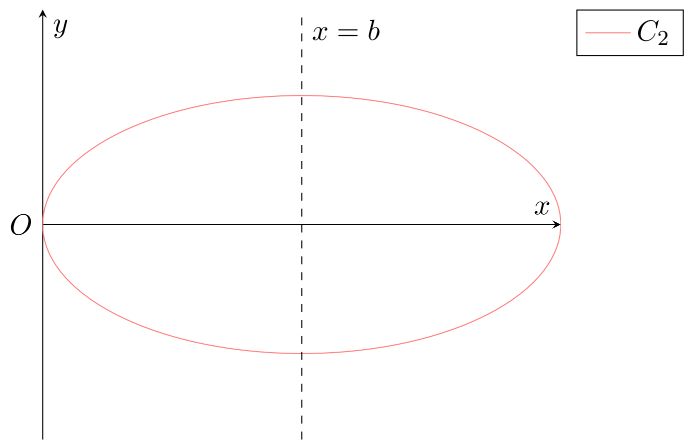


Figure 23.2

(a) Determine the values of a and b.

(b) Describe clearly a sequence of transformations that maps C_1 onto C_2.

* * * * *

Problem 4. It is given that the curve $y = f(x)$, where $f(x) = \frac{ax+b}{2x+c}$, where a, b, c are constants, has an asymptote $x = \frac{1}{2}$. The point A with coordinates $(2, \frac{5}{3})$ lies on the curve. The tangent to the curve at A has gradient $\frac{2}{9}$.

(a) Write down the value of c.

(b) Show that $a = 4$ and $b = -3$.

(c) Sketch the graph of $y = f(x)$, showing clearly all the asymptotes and the exact coordinates of the intersection with the axes.

(d) Describe a sequence of three transformations which transforms the graph $y = 2 + \frac{1}{x}$ to $y = f(x)$.

* * * * *

Problem 5. The curve whose equation is $\frac{(x-3)^2}{2^2} + \frac{y^2}{3^2} = 1$ undergoes, in succession, the following transformations:

A : A translation of magnitude 1 unit in the direction of the x-axis.

B : A reflection in the y-axis.

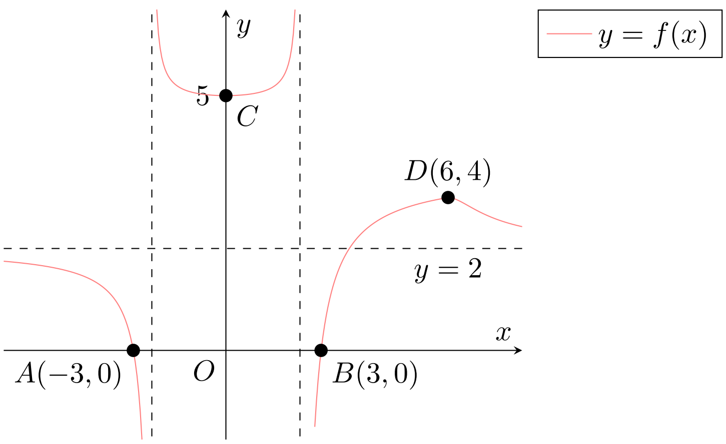
C : A scaling parallel to the y-axis by a scale factor of k.

(a) Find the equation of the resulting curve.

(b) State the value of k for which the resulting curve takes on the shape of a circle.

* * * * *

Problem 6. The diagram shows the graph of $y = f(x)$.



On separate diagrams, sketch the graphs of

(a) $y = f(4x + 3)$,

(b) $y = 1/f(x)$.

In each case, state the equations of any asymptotes and the coordinates of the points corresponding to A, B, C and D where appropriate.

* * * * *

Problem 7. A curve C_1 is defined parametrically by

$$x = \frac{2}{t-1}, \quad y = \frac{4}{t+1}, \quad t \neq \pm 1.$$

Sketch a clearly labelled diagram of C_1.

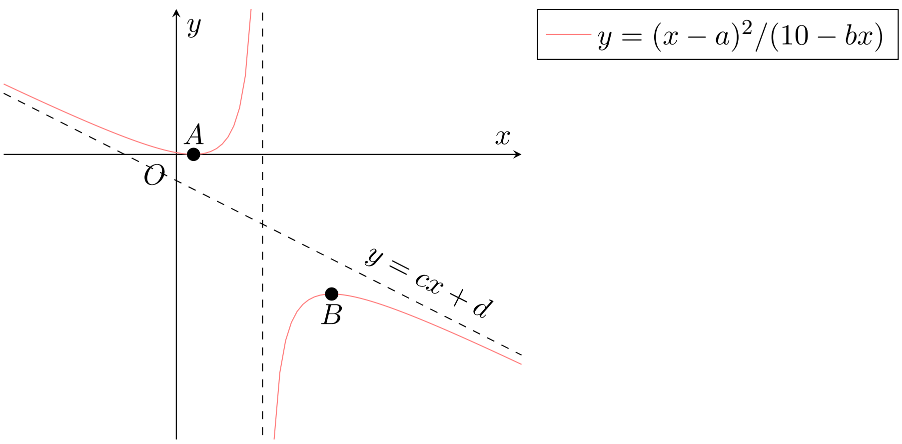
Describe a sequence of geometrical transformations which maps C_1 to C_2 defined by

$$x = \frac{1}{1-t}, \quad t = \frac{4}{t+1}, \quad t \neq \pm 1.$$

Sketch C_3, which is the reciprocal function of C_1, stating the equations of any asymptotes and any points of intersection with the axes.

* * * * *

Problem 8 (🌶). The curve of $y = \frac{x^2}{4-x}$ undergoes two transformations. The resulting curve whose equation is $y = \frac{(x-a)^2}{10-bx}$ has stationary points $A(1, 0)$ and $B(9, -8)$, and asymptotes $x = 5$ and $y = cx + d$, where a, b, c and d are constants.

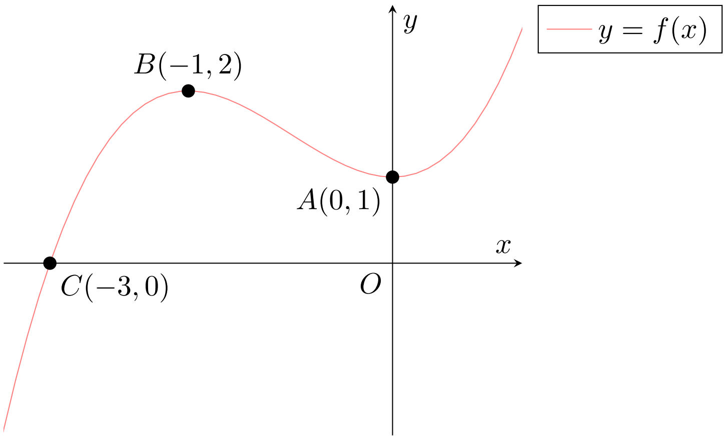


(a) Show that $a = 1$, and find the values of b, c and d.

(b) Describe the sequence of transformations undergone by the graph of $y = \frac{x^2}{4-x}$ to attain that of $y = \frac{(x-a)^2}{10-bx}$, where a and b are the values found in (a).

Assignment B2

Problem 1.



The diagram shows the graph of $y = f(x)$. The points A, B and C have coordinates $(0, 1)$, $(-1, 2)$ and $(-3, 0)$ respectively. Sketch, separately, the graphs of

(a) $y = f(-x)$

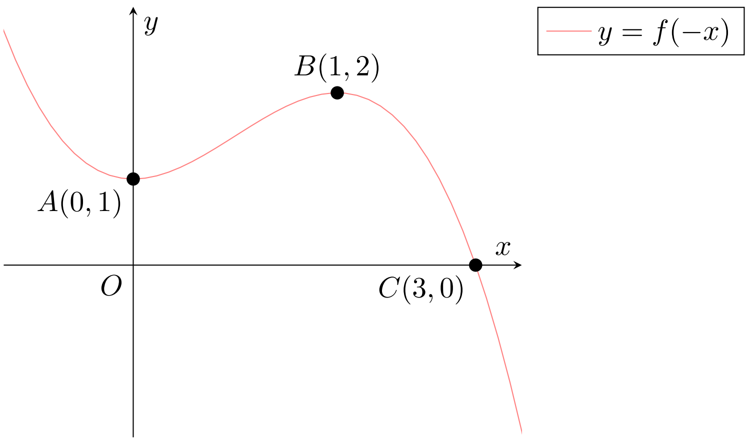
(b) $y = 3 - 2f(x)$

(c) $y = 3f\left(\frac{x}{2} + 1\right)$

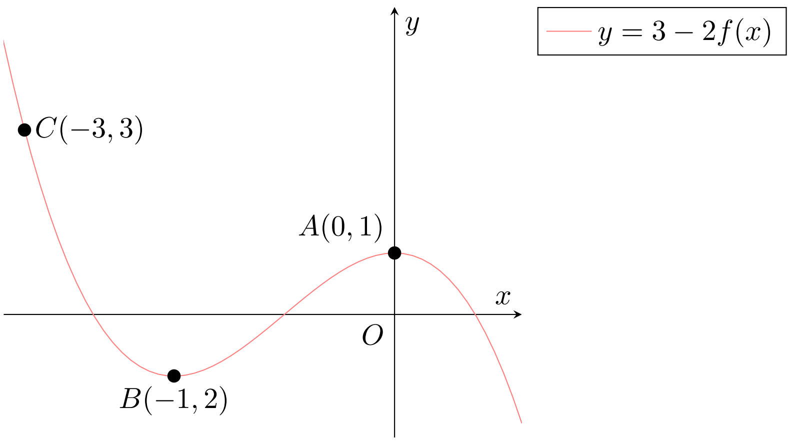
showing in each case the coordinates of the points corresponding to A, B and C.

Solution.

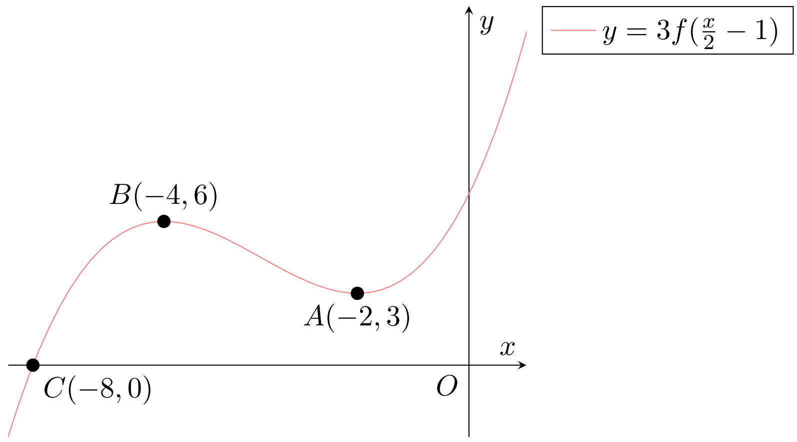
Part (a).



Part (b).



Part (c).



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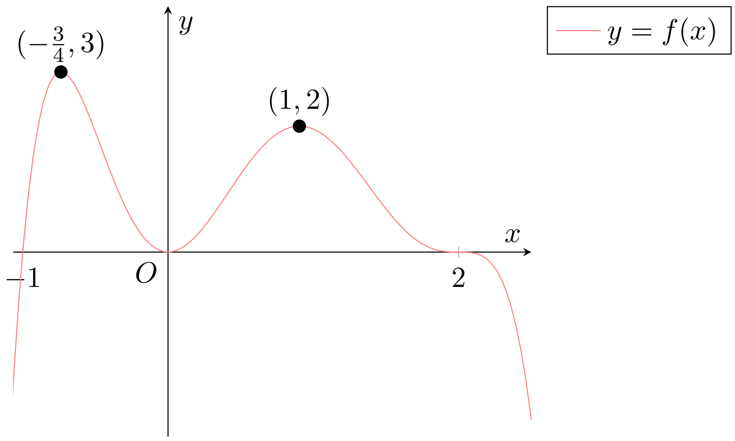
Problem 2.



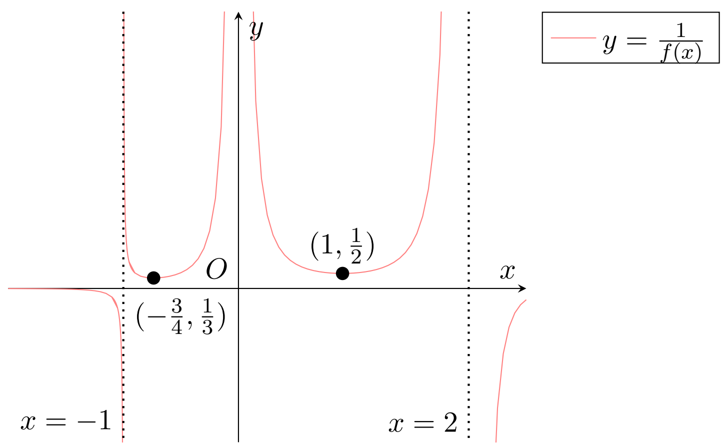
The curve shown is the graph of $y = f(x)$. The x-axis is a tangent at the origin and at $(2, 0)$. The curve has two maximum points at $\left(-\frac{3}{4}, 3\right)$ and $(1, 2)$. On two separate diagrams, sketch the graphs of the following equations. Show clearly the shapes of the graphs where they meet the x-axis and any asymptotes.

(a) $y = \frac{1}{f(x)}$, $x \neq -1, 0, 2$

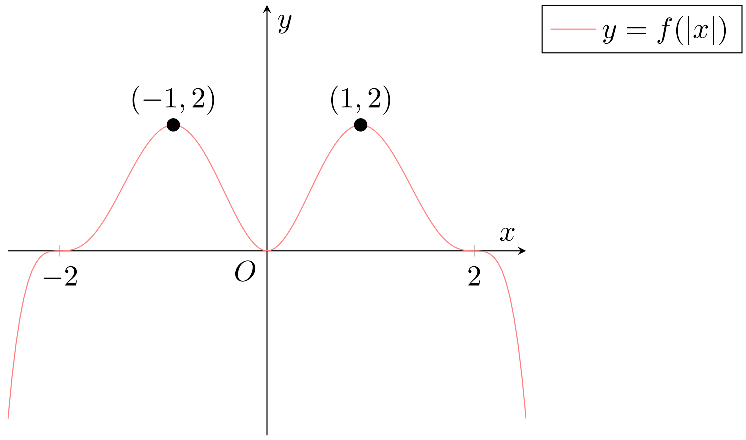
(b) $y = f(|x|)$

Solution.

Part (a).



Part (b).



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Problem 3. A graph with equation $y = f(x)$ undergoes transformation A followed by transformation B where A and B are described as follows:

- A: a translation of 1 unit in the positive direction of the x-axis
- B: a scaling parallel to the x-axis by a factor $\frac{1}{2}$

The resulting equation is $y = 4x^2 - 4x + 1$. Find the equation $y = f(x)$.

Solution. Note that

$$\begin{aligned} A\colon x &\mapsto x - 1 \implies A^{-1}\colon x \mapsto x + 1 \\ B\colon x &\mapsto 2x \quad\;\; \implies B^{-1}\colon x \mapsto \frac{1}{2}x. \end{aligned}$$

Hence,

$$y = 4x^2 - 4x + 1$$

$$\Big\downarrow B^{-1}$$

$$y = 4\left(\frac{1}{2}x\right)^2 - 4\left(\frac{1}{2}x\right) + 1$$

$$\Big\downarrow A^{-1}$$

$$y = 4\left[\frac{1}{2}(x+1)\right]^2 - 4\left[\frac{1}{2}(x+1)\right] + 1$$

Observe that y simplifies to

$$y = 4\left[\frac{1}{2}(x+1)\right]^2 - 4\left[\frac{1}{2}(x+1)\right] + 1 = (x+1)^2 - 2(x+1) + 1 = x^2 + 2x + 1 - 2x - 2 + 1 = x^2.$$

B3 Functions

Tutorial B3

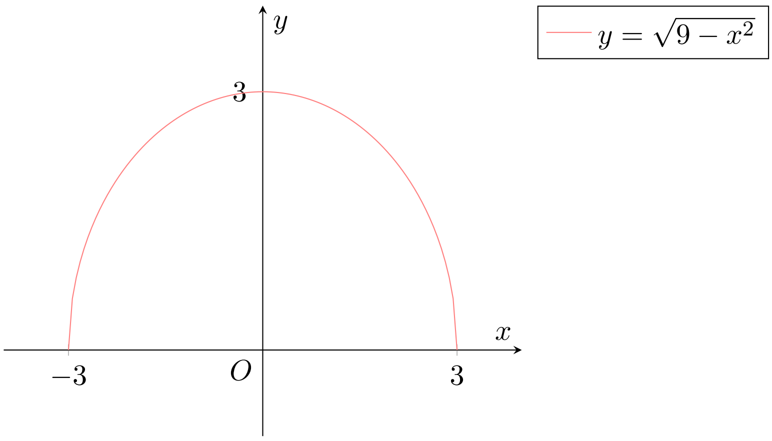
Problem 1. Sketch the following graphs and determine whether each graph represents a function for the given domain.

(a) $y = \sqrt{9 - x^2}$, $-3 \leq x \leq 3$

(b) $x = (y - 4)^2$, $y \in \mathbb{R}$

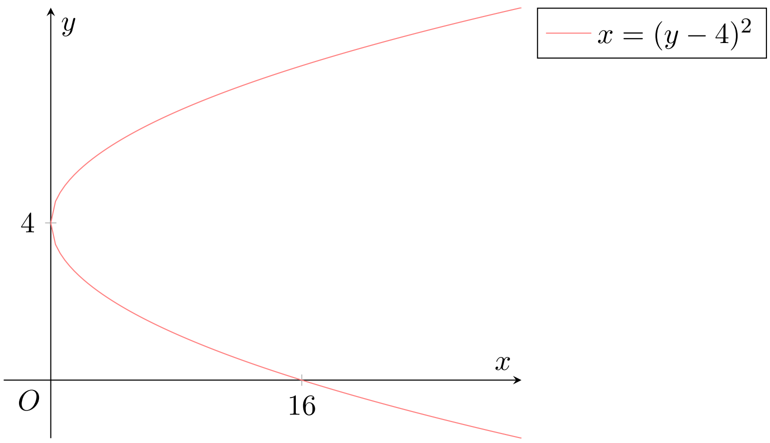
Solution.

Part (a).



$y = \sqrt{9 - x^2}$ passes the vertical line test for $-3 \leq x \leq 3$ and is hence a function.

Part (b).



$x = (y - 4)^2$ does not pass the vertical line test for $y \in \mathbb{R}$ and is hence not a function.

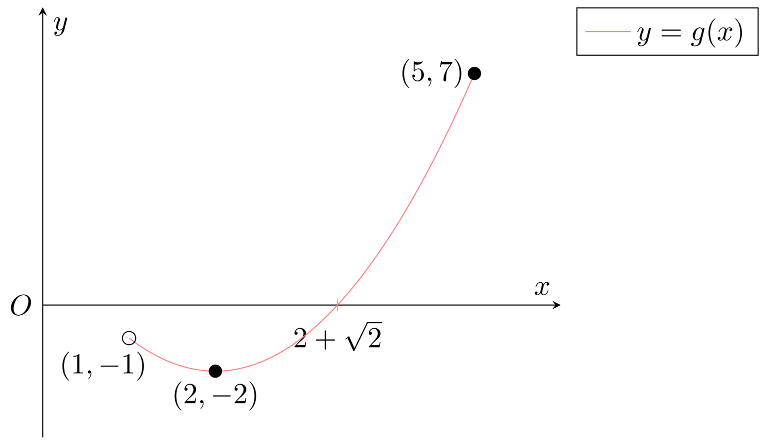
Problem 2. Sketch the graph and find the range for each the following functions.

(a) $g\colon x \mapsto x^2 - 4x + 2,\ 1 < x \le 5$

(b) $h\colon x \mapsto |2x-3|\,,\ x < 3$

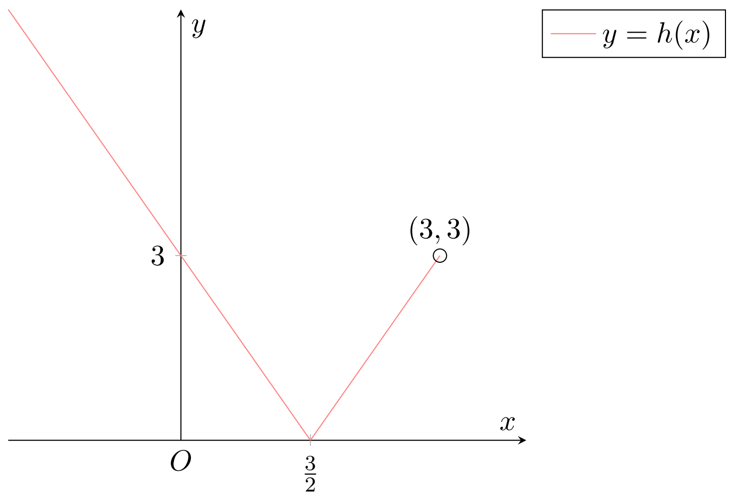
Solution.

Part (a).



From the graph, $R_g = [-2, 7)$.

Part (b).



From the graph, $R_h = [0, \infty)$.

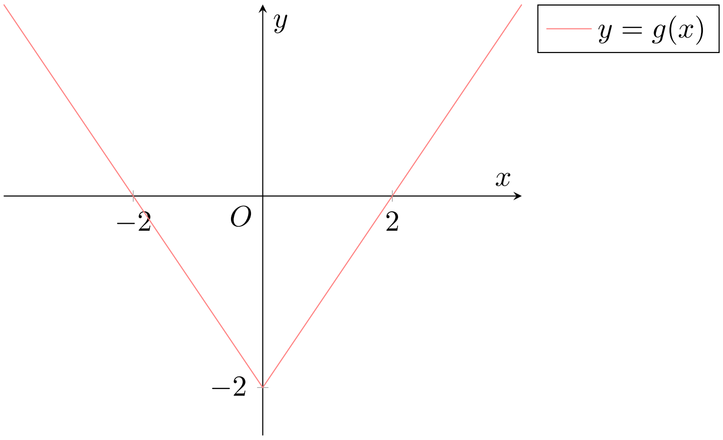
Problem 3. For each of the following functions, sketch its graph and determine if the function is one-one. If it is, find its inverse in a similar form.

(a) $g\colon x \mapsto |x| - 2,\ x \in \mathbb{R}$

(b) $h\colon x \mapsto x^2 + 2x + 5,\ x \leq -2$

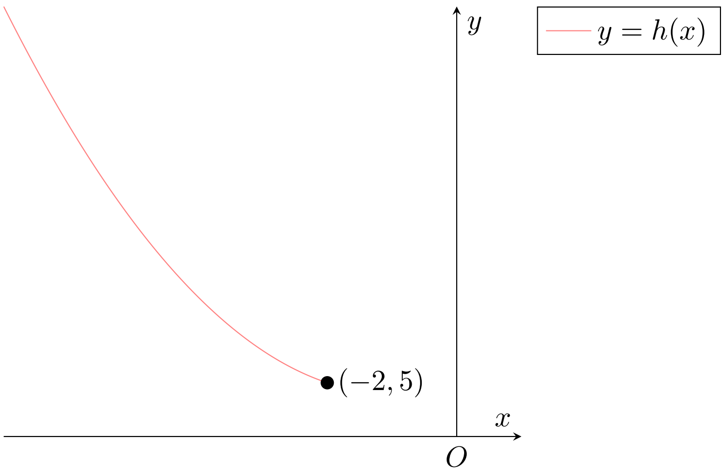
Solution.

Part (a).



$y = g(x)$ does not pass the horizontal line test. Hence, g is not one-one.

Part (b).



$y = h(x)$ passes the horizontal line test. Hence, h is one-one.
Note that $y = h(x) \implies x = h^{-1}(y)$. Now consider $y = h(x)$.

$$y = h(x) = x^2 + 2x + 5 = (x+1)^2 + 4 \implies x = -1 \pm \sqrt{y-4}.$$

Since $x \leq -2$, we reject $x = -1 + \sqrt{y-4}$. Note that $D_{h^{-1}} = R_h = [5, \infty)$. Hence,

$$h^{-1}\colon x \mapsto -1 - \sqrt{x-4},\ x \geq 5.$$

Problem 4. The function f is defined by

$$f\colon x \mapsto x + \frac{1}{x},\ x \neq 0.$$

(a) Sketch the graph of f and explain why f^{-1} does not exist.

(b) The function h is defined by $h\colon x \mapsto f(x)$, $x \in \mathbb{R}$, $x \geq \alpha$, where $\alpha \in \mathbb{R}^+$. Find the smallest value of α such that the inverse function of h exists.

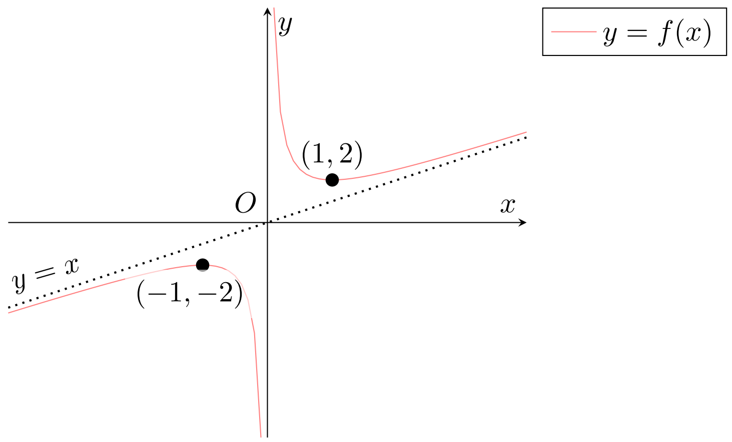
Using this value of α,

(c) State the range of h.

(d) Express h^{-1} in a similar form and sketch on a single diagram, the graphs of h and h^{-1}, showing clearly their geometrical relationship.

Solution.

Part (a).



$y = f(x)$ does not pass the horizontal line test. Hence, f is not one-one. Hence, f^{-1} does not exist.

Part (b). Consider $f'(x) = 0$ for $x > 0$.

$$f'(x) = 1 - \frac{1}{x^2} = 0 \implies x^2 = 1 \implies x = 1.$$

Note that we reject $x = -1$ since $x > 0$.

Looking at the graph of $y = f(x)$, we see that $f(x)$ achieves a minimum at $x = 1$. Hence, f is increasing for all $x \geq 1$. Thus, the smallest value of α is 1.

Part (c). Note $f(1) = 2$. Hence, from the graph, $R_h = [2, \infty)$.

Part (d). Note that $y = h(x) \implies x = h^{-1}(y)$. Now consider $y = h(x)$.

$$y = x + \frac{1}{x} \implies xy = x^2 + 1 \implies x^2 - yx + 1 = 0 \implies x = \frac{1}{2}\left(y \pm \sqrt{y^2 - 4}\right).$$

Note that $f(2) = \frac{5}{2}$. Since $2 = \frac{1}{2}\left(\frac{5}{2} + \sqrt{\left(\frac{5}{2}\right)^2 - 4}\right)$ and $2 \neq \frac{1}{2}\left(\frac{5}{2} - \sqrt{\left(\frac{5}{2}\right)^2 - 4}\right)$, we reject $x = \frac{1}{2}(y - \sqrt{y^2 - 4})$. Since $D_{f^{-1}} = R_f = [2, \infty)$, we thus have

$$h^{-1}\colon x \mapsto \frac{1}{2}\left(x + \sqrt{x^2 - 4}\right),\ x \geq 2.$$

Problem 5. The function f is defined as follows:

$$f: x \mapsto x^3 + x - 7,\ x \in \mathbb{R}.$$

(a) By using a graphical method or otherwise, show that the inverse of f exists.

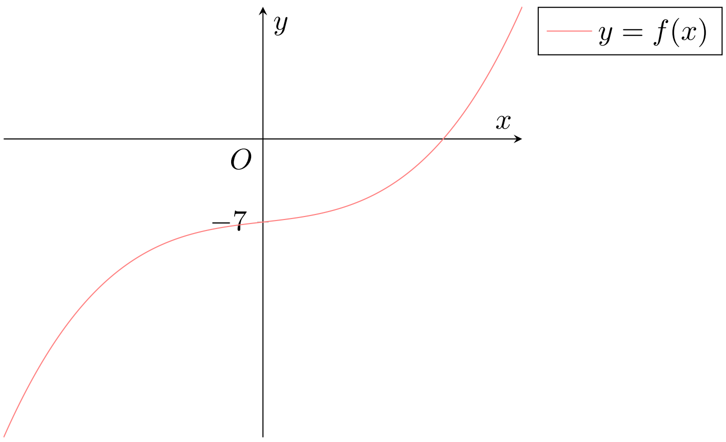
(b) Solve exactly the equation $f^{-1}(x) = 0$. Sketch the graph of f^{-1} together with the graph of f on the same diagram.

(c) Find, in exact form, the coordinates of the intersection point(s) of the graphs of f and f^{-1}.

(d) Given that the gradient of the tangent to the curve with equation $y = f^{-1}(x)$ is $\frac{1}{4}$ at the point with $x = p$, find the possible values of p.

Solution.

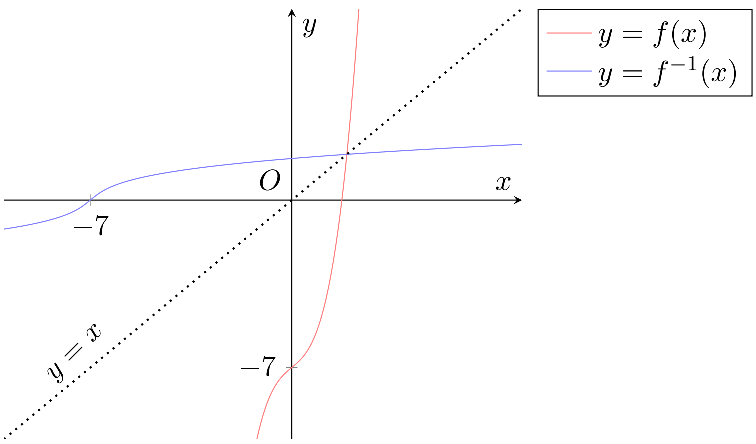
Part (a).



$y = f(x)$ passes the horizontal line test. Hence, f is one-one. Thus, f^{-1} exists.

Part (b). We have

$$f^{-1}(x) = 0 \implies x = f(0) = -7.$$



Part (c). Let (α, β) be the coordinates of the intersection between $f(x)$ and f^{-1}. From the graph, we see that $\alpha = \beta$, hence $f(\alpha) = \alpha$. Hence,

$$f(\alpha) = \alpha^3 + \alpha - 7 = \alpha \implies \alpha^3 = 7 \implies \alpha = \sqrt[3]{7}.$$

The coordinates are thus $\left(\sqrt[3]{7}, \sqrt[3]{7}\right)$.

Part (d). Note that

$$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}.$$

Evaluating at $x = p$, we obtain

$$\frac{1}{4} = \left.\frac{1}{f'(f^{-1}(x))}\right|_{x=p} \implies f'(f^{-1}(x))\big|_{x=p} = 4.$$

Since $f'(x) = 3x^2 + 1$,

$$3f^{-1}(p)^2 + 1 = 4 \implies f^{-1}(p)^2 = 1 \implies f^{-1}(p) = \pm 1.$$

Case 1: $f^{-1}(p) = 1$. Then $p = f(1) = -5$.
Case 2: $f^{-1}(p) = -1$. Then $p = f(-1) = -9$.
Hence, $p = -5$ or $p = -9$.

* * * * *

Problem 6. The functions g and h are defined as follows:

$$g\colon x \mapsto \ln(x+2), \qquad x \in (-1, 1)$$
$$h\colon x \mapsto x^2 - 2x - 1, \qquad x \in \mathbb{R}^+$$

(a) Sketch, on separate diagrams, the graphs of g and h.

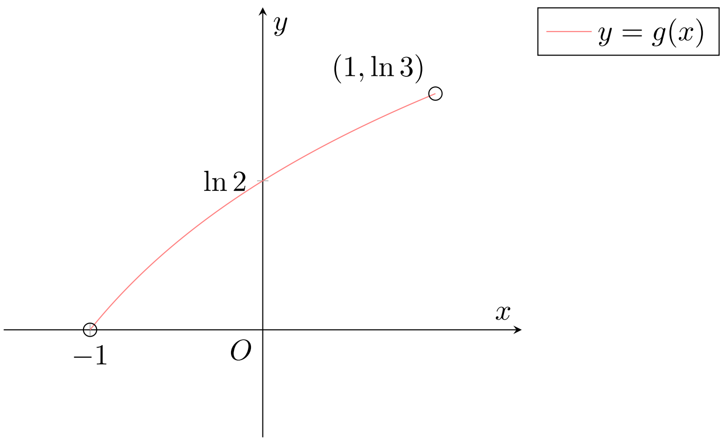
(b) Determine whether the composite function gh exists.

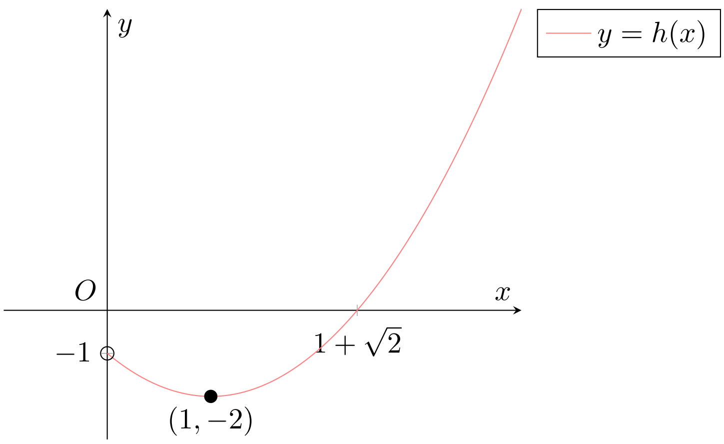
(c) Give the rule and domain of the composite function hg and find its range.

(d) The image of a under the composite function hg is -1.5. Find the value of a.

Solution.

Part (a).





Part (b). Observe that $R_h = [-2, \infty)$ and $D_g = (-1, 1)$. Hence, $R_h \nsubseteq D_g$. Thus, gh does not exist.

Part (c).

$$hg(x) = h(\ln(x+2)) = \ln(x+2)^2 - 2\ln(x+2) - 1.$$

Also note that $D_{hg} = D_g = (-1, 1)$. Hence,

$$hg\colon x \mapsto \ln(x+2)^2 - 2\ln(x+2) - 1,\ x \in (-1, 1).$$

Observe that h is decreasing on the interval $(0, 1]$ and increasing on the interval $[1, \infty)$. Note that $R_g = (0, \ln 3)$. Hence,

$$R_{hg} = [-2, \max\{h(0), h(\ln 3)\}) = [-2, -1).$$

Part (d). Note that $h(x) = (x-1)^2 - 2$. Hence, $h^{-1}(x) = 1 + \sqrt{x+2}$ (we reject $h^{-1}(x) = 1 - \sqrt{x+2}$ since $R_{h^{-1}} = D_h = \mathbb{R}^+$). Also note that $g^{-1} = \mathrm{e}^x - 2$. Thus,

$$\begin{aligned} hg(a) = -1.5 &\implies g(a) = h^{-1}(-1.5) = 1 + \sqrt{-1.5+2} = 1 + \frac{1}{\sqrt{2}} \\ &\implies a = g^{-1}\left(1 + \frac{1}{\sqrt{2}}\right) = \mathrm{e}^{1+\frac{1}{\sqrt{2}}} - 2. \end{aligned}$$

* * * * *

Problem 7. The functions f and g are defined as follows:

$$\begin{aligned} f\colon x &\mapsto 3 - x, && x \in \mathbb{R} \\ g\colon x &\mapsto \frac{4}{x}, && x \in \mathbb{R},\ x \neq 0 \end{aligned}$$

(a) Show that the composite function fg exists and express the definition of fg in a similar form. Find the range of fg.

(b) Find, in similar form, g^2 and g^3, and deduce g^{2017}.

(c) Find the set of values of x for which $g(x) = g^{-1}(x)$.

Solution.

Part (a). Note that $R_g = \mathbb{R} \setminus \{0\}$ and $D_g = \mathbb{R}$. Hence, $R_g \subseteq D_g$. Thus, fg exists.

$$fg(x) = f\left(\frac{4}{x}\right) = 3 - \frac{4}{x}.$$

Observe that $D_{fg} = D_g = \mathbb{R} \setminus \{0\}$. Thus,

$$fg\colon x \mapsto 3 - \frac{4}{x},\ x \in \mathbb{R} \setminus \{0\}.$$

Since $\frac{4}{x}$ can take on any value except 0, then $fg(x) = 3 - \frac{4}{x}$ can take on any value except 3. Thus,

$$R_{fg} = \mathbb{R} \setminus \{3\}.$$

Part (b). We have

$$g^2(x) = g\left(\frac{4}{x}\right) = \frac{4}{4/x} = x.$$

Hence,

$$g^2 \colon x \mapsto x,\, x \in \mathbb{R} \setminus \{0\}\,.$$

We have

$$g^3(x) = g(g^2(x)) = g(x) = \frac{4}{x}.$$

Hence,

$$g^3 \colon x \mapsto \frac{4}{x},\, x \in \mathbb{R} \setminus \{0\}\,.$$

Thus,

$$g^{2017} = g^{2016}(g(x)) = \left(g^2\right)^{1008} \circ g(x) = g(x) = \frac{4}{x}.$$

Hence,

$$g^{2017} \colon x \mapsto \frac{4}{x},\, x \in \mathbb{R} \setminus \{0\}\,.$$

Part (c). Note that $g(x) = g^{-1}(x) \implies g^2(x) = x$. From the definition of $g^2(x)$, we know that $g^2(x) = x$ for all x in D_{g^2}. Hence, the solution set is $\mathbb{R} \setminus \{0\}$.

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Problem 8. The function f is defined by

$$f(x) = \begin{cases} 2x+1, & 0 \leq x < 2 \\ (x-4)^2+1, & 2 \leq x < 4. \end{cases}$$

It is further given that $f(x) = f(x+4)$ for all real values of x.

(a) Find the values of $f(1)$ and $f(5)$ and hence explain why f is not one-one.

(b) Sketch the graph of $y = f(x)$ for $-4 \leq x < 8$.

(c) Find the range of f for $-4 \leq x < 8$.

Solution.

Part (a). We have

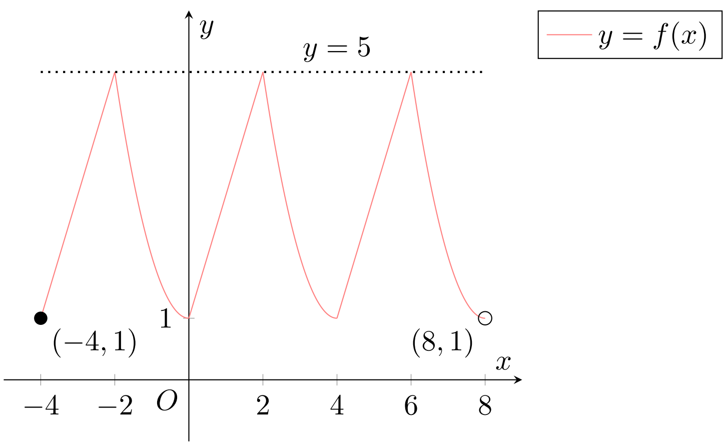
$$f(1) = 2(1) + 1 = 3$$

and

$$f(5) = f(1+4) = f(1) = 3.$$

Since $f(1) = f(5)$, but $1 \neq 5$, f is not one-one.

Part (b).



Part (c). From the graph, $R_f = [1, 5]$.

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Problem 9.

(a) The function f is given by $f\colon x \mapsto 1 + \sqrt{x}$ for $x \in \mathbb{R}^+$.

(i) Find $f^{-1}(x)$ and state the domain of f^{-1}.

(ii) Find $f^2(x)$ and the range of f^2.

(iii) Show that if $f^2(x) = x$ then $x^3 - 4x^2 + 4x - 1 = 0$. Hence, find the value of x for which $f^2(x) = x$. Explain why this value of x satisfies the equation $f(x) = f^{-1}(x)$.

(b) The function g, with domain the set of non-negative integers, is given by

$$g(n) = \begin{cases} 1, & n = 0 \\ 2 + g\left(\frac{1}{2}n\right), & n \text{ even} \\ 1 + g(n-1), & n \text{ odd} \end{cases}$$

(i) Find $g(4)$, $g(7)$ and $g(12)$.

(ii) Does g have an inverse? Justify your answer.

Solution.

Part (a).

Part (a)(i). Let $y = f(x)$. Then $x = f^{-1}(y)$.

$$y = f(x) = 1 + \sqrt{x} \implies \sqrt{x} = y - 1 \implies x = (y-1)^2.$$

Hence, $f^{-1}(x) = (x-1)^2$.

Observe that $D_{f^{-1}} = R_f = (1, \infty)$. Thus, $D_{f^{-1}} = (1, \infty)$.

Part (a)(ii). We have

$$f^2(x) = f(1 + \sqrt{x}) = 1 + \sqrt{1 + \sqrt{x}}.$$

Observe that $\sqrt{1 + \sqrt{x}} > 1$. Hence, $1 + \sqrt{1 + \sqrt{x}} > 1 + 1 = 2$, whence $R_{f^2} = (2, \infty)$.

Part (a)(iii). Note that $f^2(x) = x \implies 1 + \sqrt{1 + \sqrt{x}} = x$, whence x satisfies the recursion $1 + \sqrt{x} = x$. Hence,

$$1 + \sqrt{x} = x \implies \sqrt{x} = x - 1 \implies x = x^2 - 2x + 1 \implies x^2 - 3x + 1 = 0.$$

We can manipulate this to form the desired cubic equation:

$$0 = x\left(x^2 - 3x + 1\right) - \left(x^2 - 3x + 1\right) = x^3 - 4x^2 + 4x - 1.$$

Solving the initial quadratic equation yields $x = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$. Observe that $\frac{3-\sqrt{5}}{2} < 2$ and $\frac{3+\sqrt{5}}{2} > 2$. Thus, the sole solution is $x = \frac{3+\sqrt{5}}{2}$.

Consider $f(x) = f^{-1}(x)$. Applying f on both sides of the equation, we have $f^2(x) = f(x)$. Since $x = \frac{3+\sqrt{5}}{2}$ satisfies $f^2(x) = f(x)$, it also satisfies $f(x) = f^{-1}(x)$.

Part (b).

Part (b)(i). We have

$$g(4) = 2 + g(2) = 2 + 2 + g(1) = 4 + 1 + g(0) = 5 + 1 = 6,$$

$$g(7) = 1 + g(6) = 1 + 2 + g(3) = 3 + 1 + g(2) = 4 + (g(4) - 2) = 2 + 6 = 8,$$

and

$$g(12) = 2 + g(6) = 2 + (g(7) - 1) = 1 + 8 = 9.$$

Part (b)(ii). Consider $g(5)$ and $g(6)$.

$$g(5) = 1 + g(4) = 1 + 6 = 7, \quad g(6) = g(7) - 1 = 8 - 1 = 7.$$

Since $g(5) = g(6)$, but $5 \neq 6$, g is not one-one. Hence, g^{-1} does not exist.

Self-Practice B3

Problem 1. Functions f and g are defined by

$$f : x \mapsto \frac{3x-2}{x+1}, \quad x \in \mathbb{R}, \quad x \neq -1,$$
$$g : x \mapsto 3x+4, \quad x \in \mathbb{R}.$$

(a) Find $f^{-1}(x)$ and state the domain and range of f^{-1}.

(b) Express in similar form, the functions g^2 and gf.

(c) Find the value of x for which $gf^{-1}(x) = 0$.

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Problem 2. The function f is defined by

$$f : x \mapsto 2 - (x-1)^2, \quad x \leq 1.$$

(a) Sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = f^{-1}f(x)$ on a single diagram.

(b) If $f(\beta) = f^{-1}(\beta)$, find the values of the constant p and q such that

$$\beta^2 - p\beta + q = 0.$$

(c) Define f^{-1} in a similar form.

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Problem 3. The function f is defined by $f : x \mapsto \cos \frac{\pi x}{2}$, $x \in \mathbb{R}$, $-2 < x \leq 0$.

(a) Sketch the graph of $y = f(x)$ indicating clearly the coordinates of all axial intercepts and end points.

(b) Show that f^{-1} exists, and find its rule and domain.

The function g is defined by $g : x \mapsto (2x+1)^{2/3}$, $x \in \mathbb{R}$, $-2 < x \leq 2$.

(c) Find the set of values of x such that $g(x) \geq f(x)$

(d) Explain clearly why gf exists. Hence, find the range of gf.

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Problem 4. Functions f and$g$ are defined by:

$$f : x \mapsto x^2 + c, \quad x \leq 2,$$
$$g : x \mapsto 5 + \frac{3}{x}, \quad x \geq k,$$

where c, k are positive constants and $c > k$.

(a) Show that g^{-1} exists.

(b) Find g^{-1} in similar form, expressing its domain in terms of k.

(c) Determine whether each of the two functions, fg and gf, exists. Where it exists, express the composite function in similar form and state its range.

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Problem 5 (🌶). Functions f and g are defined such that

$$f : x \mapsto \arccos\left(x^2\right), \quad -1 \leq x \leq 1,$$
$$g : x \mapsto x^3 + 1, \quad x \in \mathbb{R}.$$

(a) Explain why the composite function fg does not exist.

The function h is defined such that $h(x) = g(x)$ and the domain of h is $a \leq x \leq 0$. It is given that $a = -5/4$.

(b) Find the range of fh in exact form.

(c) Determine all the possible value(s) of x that satisfies $g^{-1}(x^2)$. Hence, explain why $h^{-1}(x^2) = 2$ has no solution.

Assignment B3

Problem 1. Functions f and g are defined as follows:

$$f\colon x \mapsto (x-3)^2 + 6, \qquad x \in \mathbb{R},\ x \le 2$$
$$g\colon x \mapsto \ln(x-2), \qquad x \in \mathbb{R},\ x > 3$$

(a) Show that f^{-1} exists and define f^{-1} in a similar form.

(b) Sketch, on the same diagram, the graphs of f, f^{-1} and ff^{-1}.

(c) Find fg and gf if they exist, and find their ranges (where applicable).

Solution.

Part (a). Note that $f' = 2(x-3) < 0$ for all $x \le 2$. Thus, f is strictly decreasing. Since f is also continuous, f is one-one. Thus, f^{-1} exists.

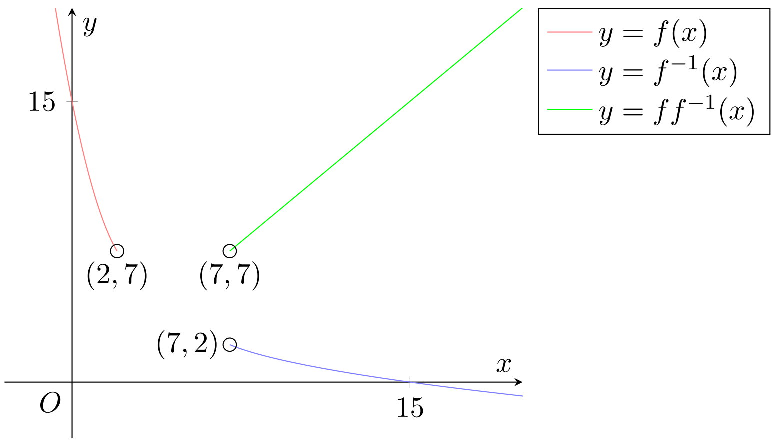
Let $y = f(x) \implies x = f^{-1}(y)$.

$$y = f(x) = (x-3)^2 + 6 \implies x = 3 \pm \sqrt{y-6}.$$

Since $x < 3$, we reject $x = 3 + \sqrt{y-6}$. Lastly, observe that $D_{f^{-1}} = R_f = [f(2), \infty) = [7, \infty)$. Thus,

$$f^{-1}\colon x \mapsto 3 - \sqrt{x-6},\ x \in \mathbb{R},\ x \ge 7.$$

Part (b).



Part (c). Note that $R_g = (0, \infty)$ and $D_f = (-\infty, 2]$. Hence, $R_g \not\subseteq D_f$. Thus, fg does not exist. Note that $R_f = [7, \infty)$ and $D_g = (3, \infty)$. Hence, $R_f \subseteq D_g$. Thus, gf exists.

Since $\ln x$ is a strictly increasing function, we have that g is also strictly increasing. Hence, $R_{gf} = [\ln(7-2), \infty) = [\ln 5, \infty)$.

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Problem 2. The function f is defined as follows:

$$f\colon x \mapsto \frac{1}{x^2-1}, \qquad x \in \mathbb{R},\ x \neq -1,\ x \neq 1.$$

(a) Sketch the graph of $y = f(x)$.

(b) If the domain of f is further restricted to $x \ge k$, state with a reason the least value of k for which the function f^{-1} exists.

In the rest of the question, the domain of f is $x \in \mathbb{R}$, $x \neq -1$, $x \neq 1$, as originally defined.

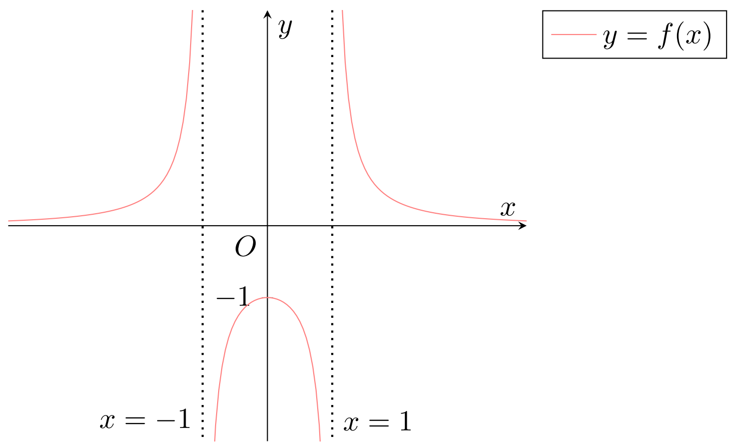
The function g is defined as follows:

$$g\colon x \mapsto \frac{1}{x-3}, \qquad x \in \mathbb{R},\, x \neq 2,\, x \neq 3, x \neq 4.$$

(c) Find the range of fg.

Solution.

Part (a).



Part (b). If the domain of f is further restricted to $x \geq 0$, f would pass the horizontal line test, whence f^{-1} would exist. Hence, $\min k = 0$.

Part (c). Observe that $R_g = \mathbb{R} \setminus \{g(2), g(4)\} = \mathbb{R} \setminus \{-1, 1\}$. Hence, $R_{fg} = R_f = \mathbb{R} \setminus (-1, 0]$.

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Problem 3. The function f is defined by

$$f\colon x \mapsto \frac{x}{x^2-1}, \qquad x \in \mathbb{R},\, x \neq -1,\, x \neq 1.$$

(a) Explain why f does not have an inverse.

(b) The function f has an inverse if the domain is restricted to $x \leq k$. State the largest value of k.

The function g is defined by

$$g\colon x \mapsto \ln x - 1, \qquad x \in \mathbb{R},\, 0 < x < 1.$$

(c) Find an expression for $h(x)$ for each of the following cases:

(i) $gh(x) = x$

(ii) $hg(x) = x^2 + 1$

Solution.

Part (a). Observe that $f(1/2) = -2/3$ and $f(-2) = -2/3$. Hence, $f(1/2) = f(-2)$. Since $1/2 \neq -2$, f is not one-one. Thus, f does not have an inverse.

Part (b). Clearly, $\max k = 0$.

Part (c).

Part (c)(i). Note that $gh(x) = x \implies h(x) = g^{-1}(x)$. Hence, consider $y = g(x) \implies x = h(y)$.

$$y = g(x) = \ln x - 1 \implies \ln x = y + 1 \implies x = \mathrm{e}^{y+1}.$$

Hence, $h(x) = \mathrm{e}^{x+1}$.

Part (c)(ii). Let $h = h_2 \circ h_1$ such that $h_1 g(x) = x \implies h_1(x) = g^{-1}(x) \implies h_1(x) = \mathrm{e}^{x+1}$. Then

$$hg(x) = x^2 + 1 \implies h_2 h_1 g(x) = x^2 + 1 \implies h_2(x) = x^2 + 1.$$

Hence, $h(x) = h_2 h_1(x) = h_2(\mathrm{e}^{x+1}) = \left(\mathrm{e}^{x+1}\right)^2 + 1 = \mathrm{e}^{2x+2} + 1$.

B4 Differentiation

Tutorial B4

Problem 1. Evaluate the following limits.

(a) $\lim_{x \to 5} (6x + 7)$

(b) $\lim_{x \to 1} \frac{x^3-1}{1-x}$

(c) $\lim_{x \to \infty} \frac{3x}{2x^2-5}$

Solution.

Part (a).

$$\lim_{x \to 5} (6x + 7) = 6(5) + 7 = 37.$$

Part (b).

$$\lim_{x \to 1} \frac{x^3 - 1}{1 - x} = \lim_{x \to 1} \frac{(x-1)\left(x^2 + x + 1\right)}{1 - x} = \lim_{x \to 1} -\left(x^2 + x + 1\right) = -(1^2 + 1 + 1) = -3.$$

Part (c).

$$\lim_{x \to \infty} \frac{3x}{2x^2 - 5} = \lim_{x \to \infty} \frac{3}{2x - 5/x}.$$

Note that as $x \to \infty$, $2x - \frac{5}{x} \to \infty$. Hence, $\lim_{x \to \infty} \frac{1}{2x - 5/x} = 0$.

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Problem 2. Differentiate the following with respect to x from first principles.

(a) $3x + 4$

(b) x^3

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}x}(3x + 4) = \lim_{h \to 0} \frac{[3(x+h) + 4] - [3x + 4]}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3.$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x} x^3 = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{3hx^2 + 3h^2x + h^3}{h} = \lim_{h \to 0} \left(3x^2 + 3hx + h^2\right) = 3x^2.$$

Problem 3. Differentiate each of the following with respect to x, simplifying your answer.

(a) $\left(x^2+4\right)^2\left(2x^3-1\right)$

(b) $\frac{x^2}{\sqrt{4-x^2}}$

(c) $\sqrt{1+\sqrt{x}}$

(d) $\left(\frac{x^3-1}{2x^3+1}\right)^4$

Solution.

Part (a).

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+4\right)^2\left(2x^3-1\right)=\left(2x^3-1\right)\left[4x\left(x^2+4\right)\right]+\left(x^2+4\right)^2\left(6x^2\right)\\&=2x\left(x^2+4\right)\left[2\left(2x^3-1\right)+3x\left(x^2+4\right)\right]=2x\left(x^2+4\right)\left(7x^3+12x-2\right).\end{aligned}$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{\sqrt{4-x^2}}=\frac{\sqrt{4-x^2}\,(2x)-x^2\left(\frac{-2x}{2\sqrt{4-x^2}}\right)}{4-x^2}=\frac{2x\left(4-x^2\right)+x^3}{(4-x^2)^{3/2}}=\frac{x\left(8-x^2\right)}{(4-x^2)^{3/2}}.$$

Part (c).

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+\sqrt{x}}=\frac{1}{2\sqrt{1+\sqrt{x}}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{4\sqrt{x(1+\sqrt{x})}}.$$

Part (d). Note that

$$\frac{x^3-1}{2x^3+1}=\frac{1}{2}\left(\frac{2x^3-2}{2x^3+1}\right)=\frac{1}{2}\left(1-\frac{3}{2x^3+1}\right)=\frac{1}{2}-\frac{3}{2}\left(\frac{1}{2x^3+1}\right).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3-1}{2x^3+1}=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}-\frac{3}{2}\left(\frac{1}{2x^3+1}\right)\right]=-\frac{3}{2}\left[\frac{-6x^2}{\left(2x^3+1\right)^2}\right]=\frac{9x^2}{\left(2x^3+1\right)^2}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-1}{2x^3+1}\right)^4=4\left(\frac{x^3-1}{2x^3+1}\right)^3\frac{9x^2}{\left(2x^3+1\right)^2}=\frac{36x^2\left(x^3-1\right)^3}{\left(2x^3+1\right)^5}.$$

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Problem 4. Using a graphing calculator, find the derivative of $\frac{e^{2x}}{x^2+1}$ when $x=1.5$.

Solution.

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^{2x}}{x^2+1}\right)\right|_{x=1.5}=6.66.$$

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Problem 5. Find the derivative with respect to x of

(a) $\cos x^\circ$

(b) $\cot\left(1-2x^2\right)$

(c) $\tan^3(5x)$

(d) $\frac{\sec x}{1+\tan x}$

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x^\circ = \frac{\mathrm{d}}{\mathrm{d}x}\cos\Big(\frac{\pi}{180}x\Big) = -\frac{\pi}{180}\sin\Big(\frac{\pi}{180}x\Big).$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot\big(1-2x^2\big) = 4x\csc\big(1-2x^2\big).$$

Part (c).

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^3(5x) = 15\tan^2(5x)\sec^2(5x).$$

Part (d). Note that

$$\frac{\sec x}{1+\tan x} = \frac{1}{\sin x+\cos x} = \frac{1}{\sqrt{2}\sin(x+\pi/4)} = \frac{1}{\sqrt{2}}\csc\Big(x+\frac{\pi}{4}\Big).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sec x}{1+\tan x} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{2}}\csc\Big(x-\frac{\pi}{4}\Big) = -\frac{1}{\sqrt{2}}\csc\Big(x+\frac{\pi}{4}\Big)\cot\Big(x+\frac{\pi}{4}\Big).$$

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Problem 6. Find the derivative with respect to x of

(a) $y = \mathrm{e}^{1+\sin 3x}$

(b) $y = x^2\mathrm{e}^{\frac{1}{x}}$

(c) $y = \ln\Big(\frac{1-x}{\sqrt{1+x^2}}\Big)$

(d) $y = \frac{\ln(2x)}{x}$

(e) $y = \log_2(3x^4-\mathrm{e}^x)$

(f) $y = 3^{\ln\sin x}$

(g) $y = a^{2\log_a x}$

(h) $y = \sqrt[3]{\frac{\mathrm{e}^x(x+1)}{x^2+1}}$, $x>0$

Solution.

Part (a).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{1+\sin 3x} = 3\mathrm{e}^{1+\sin 3x}\cos(3x).$$

Part (b).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}x^2\mathrm{e}^{1/x} = -x^2\left(-\frac{\mathrm{e}^{1/x}}{x^2}\right) + \mathrm{e}^{1/x}(2x) = \mathrm{e}^{1/x}(2x-1).$$

Part (c). Note that

$$y = \ln\left(\frac{1-x}{\sqrt{1+x^2}}\right) = \ln(1-x) - \frac{1}{2}\ln\left(1+x^2\right).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(1-x) - \frac{1}{2}\ln\left(1+x^2\right)\right] = -\frac{1}{1-x} - \frac{x}{1+x^2} = -\frac{1+x}{(1-x)\left(1+x^2\right)}.$$

Part (d).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln(2x)}{x} = \frac{x\left(\frac{1}{x}\right) - \ln(2x)(1)}{x^2} = \frac{1-\ln(2x)}{x^2}.$$

Part (e). Note that

$$y = \log_2\left(3x^4 - \mathrm{e}^x\right) \implies 2^y = 3x^4 - \mathrm{e}^x.$$

Implicitly differentiating with respect to x,

$$2^y \ln 2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 12x^3 - \mathrm{e}^x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12x^3 - \mathrm{e}^x}{2^y \ln 2} = \frac{12x^3 - \mathrm{e}^x}{\left(3x^4 - \mathrm{e}^x\right)\ln 2}.$$

Part (f). Note that

$$y = 3^{\ln\sin x} \implies \log_3 y = \frac{\ln y}{\ln 3} = \ln\sin x \implies \ln y = \ln 3 \ln\sin x.$$

Implicitly differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = \ln 3\left(\frac{\cos x}{\sin x}\right) = \ln 3 \cdot \cot x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \ln 3 \cdot y\cot x = \ln 3 \cdot \cot(x)\cdot 3^{\ln\sin x}.$$

Part (g). Observe that

$$y = a^{2\log_a x} = a^{\log_a x^2} = x^2 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x.$$

Part (h). Note that

$$y = \sqrt[3]{\frac{\mathrm{e}^x(x+1)}{x^2+1}} \implies \left(x^2+1\right)y^3 = \mathrm{e}^x(x+1).$$

Implicitly differentiating with respect to x,

$$\left(x^2+1\right)\left(3y^2\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right) + y^3(2x) = \mathrm{e}^x + (x+1)\mathrm{e}^x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x(x+2) - 2xy^3}{3\left(x^2+1\right)y^2}.$$

Now observe that

$$\frac{\mathrm{e}^x(x+2)}{3\left(x^2+1\right)y^2} = \frac{\mathrm{e}^x(x+1)(x+2)}{3\left(x^2+1\right)(x+1)y^2} = \frac{y^3(x+2)}{3(x+1)y^2} = y\left(\frac{x+2}{3(x+1)}\right).$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\left(\frac{x+2}{3(x+1)}\right) - y\left(\frac{2x}{3\left(x^2+1\right)}\right) = \frac{1}{3}\sqrt[3]{\frac{\mathrm{e}^x(x+1)}{x^2+1}}\left(\frac{x+2}{x+1} - \frac{2x}{x^2+1}\right).$$

Problem 7. Find the derivative with respect to x of

(a) $\arccos\left(\frac{x}{10}\right)$

(b) $\arctan\left(\frac{1}{1-x}\right)$

(c) $\arcsin(\tan x)$

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos\frac{x}{10} = -\frac{1}{10\sqrt{1-\left(\frac{x}{10}\right)^2}} = -\frac{1}{\sqrt{100-x^2}}.$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan\left(\frac{1}{1-x}\right) = \frac{1}{1+\left(\frac{1}{1-x}\right)^2}\left(\frac{1}{(1-x)^2}\right) = \frac{1}{(1-x)^2+1}.$$

Part (c).

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(\tan x) = \frac{\sec^2 x}{1-\tan^2 x}.$$

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Problem 8. Find an expression for $\mathrm{d}y/\mathrm{d}x$ in terms of x and y.

(a) $(y-x)^2 = 2a(y+x)$, where a is a constant

(b) $y^2 = \mathrm{e}^{2x}y + x\mathrm{e}^x$

(c) $y = x^y$

(d) $\sin x\cos y = \frac{1}{2}$

Solution.

Part (a). Implicitly differentiating with respect to x,

$$(y-x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}-1\right) = a\left(\frac{\mathrm{d}y}{\mathrm{d}x}+1\right) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a+y-x}{y-x-a}.$$

Part (b). Implicitly differential with respect to x,

$$2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\mathrm{e}^{2x}\frac{\mathrm{d}y}{\mathrm{d}x}+2ye^{2x}\right) + (x\mathrm{e}^x+\mathrm{e}^x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x\left(2y\mathrm{e}^x+x+1\right)}{2y-\mathrm{e}^{2x}}.$$

Part (c). Note that

$$y = x^y \implies \ln y = y\ln x.$$

Implicitly differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}+\ln x\cdot\frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y/x}{1/y-\ln x} = \frac{y^2}{x-xy\ln x}.$$

Part (d). Note that

$$\sin x\cos y = \frac{1}{2} \implies \cos y = \frac{1}{2}\csc x.$$

Implicitly differentiating with respect to x,

$$-\sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\csc x \cot x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\csc x \cot x}{2\sin y}.$$

* * * * *

Problem 9. It is given that x and y satisfy the equation

$$\arctan x + \arctan y + \arctan(xy) = \frac{7}{12}\pi.$$

(a) Find the exact value of y when $x = 1$.

(b) Express $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(xy)$ in terms of x, y and y'.

(c) Show that, when $x = 1$, $y' = -\frac{1}{3} - \frac{1}{2\sqrt{3}}$.

Solution.

Part (a). Substituting $x = 1$ into the given equation,

$$\frac{1}{4}\pi + 2\arctan y = \frac{7\pi}{12} \implies \arctan y = \frac{\pi}{6} \implies y = \frac{1}{\sqrt{3}}.$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(xy) = \frac{xy' + y}{1 + (xy)^2}.$$

Part (c). Differentiating the given equation with respect to x,

$$\frac{1}{1+x^2} + \frac{y'}{1+y^2} + \frac{xy'+y}{1+(xy)^2} = 0.$$

Substituting $x = 1$,

$$\frac{1}{2} + \frac{3y'}{4} + \frac{3}{4}\left(y' + \frac{1}{\sqrt{3}}\right) = 0 \implies y' = \frac{2}{3}\left(-\frac{3}{4\sqrt{3}} - \frac{1}{2}\right) = -\frac{1}{2\sqrt{3}} - \frac{1}{3}.$$

* * * * *

Problem 10. Find $\mathrm{d}y/\mathrm{d}x$ for

(a) $x = \frac{1}{1+t^2}$, $y = \frac{t}{1+t^2}$

(b) $x = \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t})$, $y = \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t})$

(c) $x = a\sec\theta$, $y = a\tan\theta$

(d) $x = \mathrm{e}^{3\theta}\cos(3\theta)$, $y = \mathrm{e}^{3\theta}\sin(3\theta)$

Solution.

Part (a). Observe that $y = xt$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right) + t = x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} + t = \frac{1}{1+t^2}\left(-\frac{2t}{(1+t^2)^2}\right)^{-1} + t = \frac{t^2-1}{2t}.$$

Part (b).

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\frac{\frac{1}{2}\left(\mathrm{e}^t-\mathrm{e}^{-t}\right)}{\frac{1}{2}\left(\mathrm{e}^t+\mathrm{e}^{-t}\right)}=\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{\mathrm{e}^t+\mathrm{e}^{-t}}.$$

Part (c). Recall that $\tan^2\theta+1=\sec^2\theta$. Hence, $y^2+a^2=x^2$. Implicitly differentiating with respect to x, we have

$$2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=2x \implies \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{y}=\frac{a\sec\theta}{a\tan\theta}=\csc\theta.$$

Part (d).

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}(3\theta)}{\mathrm{d}x/\mathrm{d}(3\theta)}=\frac{\mathrm{e}^{3\theta}\cos 3\theta+\mathrm{e}^{3\theta}\sin 3\theta}{-\mathrm{e}^{3\theta}\sin 3\theta+\mathrm{e}^{3\theta}\cos 3\theta}=\frac{\cos 3\theta+\sin 3\theta}{\cos 3\theta-\sin 3\theta}.$$

* * * * *

Problem 11. A curve is defined by the parametric equation

$$x=120t-4t^2,\ y=60t-6t^2.$$

Find the value of $\mathrm{d}y/\mathrm{d}x$ at each of the points where the curve cross the x-axis.

Solution. The curve crosses the x-axis when $y=0$:

$$y=60t-6t^2=6t(10-t)=0.$$

Hence, $t=0$ or $t=10$. Now, consider the derivative with respect to x of the curve.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\frac{60-12t}{120-8t}.$$

Case 1: $t=0$.

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=0}=\frac{60-12(0)}{120-8(0)}=\frac{1}{2}.$$

Case 2: $t=10$.

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=10}=\frac{60-12(10)}{120-8(10)}=-\frac{3}{2}.$$

* * * * *

Problem 12. A curve has parametric equations $x=2t-\ln(2t)$, $y=t^2-\ln t^2$, where $t>0$. Find the value of t at the point on the curve at which the gradient is 2.

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\frac{2t-2/t}{2-1/t}=\frac{2t^2-2}{2t-1}.$$

Consider $\mathrm{d}y/\mathrm{d}x=2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2t^2-2}{2t-1}=2 \implies \frac{t^2-1}{2t-1}=1 \implies t^2-1=2t-1 \implies t^2-2t=t(t-2)=0.$$

Hence, $t=0$ or $t=2$. Since $t>0$, we reject $t=0$. Thus, $t=2$.

Problem 13. If $y = \ln\left(\sin^3 2x\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and prove that $3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 36 = 0$.

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6\sin^2 2x \cos 2x}{\sin^3 2x} = 6\cot 2x.$$

Hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -12\csc^2 2x = -12\left(1+\cot^2 2x\right) = -12 - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2.$$

Thus, we clearly have

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 36 = 0.$$

* * * * *

Problem 14. Given that $y = \mathrm{e}^{\arcsin(2x)}$, show that $\left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 4y$. Differentiate this result further to obtain a differential equation for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

Solution. Note that

$$y = \mathrm{e}^{\arcsin(2x)} \implies \ln y = \arcsin(2x).$$

Implicitly differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\sqrt{1-4x^2}} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{\sqrt{1-4x^2}}.$$

Implicitly differentiating with respect to x once again,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\sqrt{1-4x^2}\left(2\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 2y\left(\frac{-4x}{\sqrt{1-4x^2}}\right)}{1-4x^2}.$$

Now observe that

$$2\sqrt{1-4x^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\left(\frac{2y}{\sqrt{1-4x^2}}\right) = 4y + 4x\cdot\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence,

$$\left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4y + 4x\cdot\frac{\mathrm{d}y}{\mathrm{d}x} \implies \left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 4y.$$

Implicitly differentiating with respect to x once again,

$$\left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 8x\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\left(x\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 4\cdot\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Rearranging,

$$\left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 12x\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 8\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Self-Practice B4

Problem 1. Differentiate each of the following with respect to x, simplifying your answer.

(a) $\frac{x^2-2x}{(x+2)^2}$,

(b) $x\left(x^3+1\right)^{1/3}$,

(c) $\cot x \csc x$,

(d) $\left(\sin^3 x\right)(\sin 3x)$,

(e) $\arctan\sqrt{x}$,

(f) $\arcsin\sqrt{1-x^2}$,

(g) $y=\frac{\mathrm{e}^{2x}}{1+\mathrm{e}^x}$,

(h) $y=\ln\frac{4}{x^2}$,

(i) $y=3^x$.

* * * * *

Problem 2. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

(a) $x^3+y^3+3xy-1=0$,

(b) $y^x=x$.

* * * * *

Problem 3. It is given that, at any point on the graph of $y=f(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{1+y^3}$.

(a) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{3}{2}y^2$.

(b) Find the expressions for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$ in terms of y and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

* * * * *

Problem 4. Given that $y=\mathrm{e}^{\sqrt{1+x}}$, show that

$$4(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=y.$$

* * * * *

Problem 5.

(a) A curve has parametric equations $x=\frac{t}{1+t}$, $y=\ln\cos t$, where $t\neq -1, \frac{\pi}{2}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

(b) A curve has equation $\arcsin y + x\mathrm{e}^y = 3x$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

* * * * *

Problem 6.

(a) Differentiate $\frac{x-2x^3}{\ln x}$ with respect to x.

(b) Given that $0 < x < \frac{\pi}{2}$, show that $\frac{\mathrm{d}}{\mathrm{d}x} \arcsin(cosx) = k$, where k is a real constant to be determined.

(c) Given that $\mathrm{e}^{xy} = \left(1 + y^2\right)^2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y, simplifying your answer.

* * * * *

Problem 7. It is given that $y = \ln \sin\left(\frac{\pi}{4} + x\right)$. Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1 = 0.$$

* * * * *

Problem 8.

(a) Differentiate the following with respect to x, giving your answers as single fractions.

(i) $\ln \frac{x}{\sqrt{1-2x}}$,

(ii) $\frac{1}{\arccos(x^2)}$.

(b) The variables x and y are related by

$$\mathrm{e}^{xy^2} = y\left(x^2 + 2\mathrm{e}^x\right).$$

Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0$ and $y = \frac{1}{2}$.

Assignment B4

Problem 1. Differentiate the following with respect to x.

(a) $\ln \frac{x^3}{\sqrt{1+x^2}}$

(b) $\arctan\left(\frac{x^2}{2}\right)$

(c) $e^{2x} \sec x$

Solution.

Part (a).

$$\ln \frac{x^3}{\sqrt{1+x^2}} = 3 \ln x - \frac{1}{2} \ln\left(1+x^2\right) \implies \frac{d}{dx}\left(\ln \frac{x^3}{\sqrt{1+x^2}}\right) = \frac{3}{x} - \frac{x}{1+x^2}.$$

Part (b).

$$\frac{d}{dx} \arctan\left(\frac{x^2}{2}\right) = \frac{x}{1+x^4/4} = \frac{4x}{4+x^4}.$$

Part (c).

$$\frac{d}{dx} e^{2x} \sec x = e^{2x} \left(\sec x \tan x\right) + \sec x \left(2e^{2x}\right) = e^{2x} \sec x \left(\tan x + 2\right).$$

* * * * *

Problem 2. Find the gradient of the curve $x^3 + xy^2 = 5y$ at the point where $x = 1$ and $0 < y < 1$, leaving your answer to 3 significant figures.

Solution. Substituting $x = 1$ into the given equation,

$$y^2 - 5y + 1 = 0 \implies y = \frac{5 \pm \sqrt{21}}{2}.$$

Since $0 < y < 1$, we reject $y = \frac{1}{2}\left(5 + \sqrt{21}\right)$ and take $y = \frac{1}{2}\left(5 - \sqrt{21}\right) = 0.20871$ (5 s.f.).

Implicitly differentiating the given equation,

$$3x^2 + 2xy \cdot y' + y^2 = 5y' \implies y' = \frac{3x^2 - y^2}{5 - 2xy}.$$

Substituting $x = 1$ and $y = 0.20871$ into the above equation,

$$y' = \frac{3(1)^2 - (0.20871)^2}{2(1)(0.20871) - 5} = 0.664 \text{ (3 s.f.)}.$$

Hence, the gradient of the curve is 0.664.

* * * * *

Problem 3. A curve C has parametric equations

$$x = \sin^3 \theta,\ y = 3 \sin^2 \theta \cos \theta, \qquad 0 \leq \theta \leq \frac{\pi}{2}.$$

Show that $dy/dx = a \cot \theta + b \tan \theta$, where a and b are values to be determined.

Solution. Note that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sin^2\theta\cos\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3\left(2\sin\theta\cos^2\theta - \sin^3\theta\right).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{3\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)}{3\sin^2\theta\cos\theta} = \frac{2\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = 2\cot\theta - \tan\theta.$$

Thus, $a = 2$ and $b = -1$.

B5 Applications of Differentiation

Tutorial B5A

Problem 1. The equation of a curve is $y = 2x^3 + 3x^2 + 6x + 4$. Find $\mathrm{d}y/\mathrm{d}x$ and hence show that y is increasing for all real values of x.

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 6x + 6 = 6\left(x + \frac{1}{2}\right)^2 + \frac{18}{4}.$$

For all $x \in \mathbb{R}$, we have $\left(x + \frac{1}{2}\right)^2 \geq 0$. Hence, $\mathrm{d}y/\mathrm{d}x > 0$. Thus, y is increasing for all real values of x.

* * * * *

Problem 2. Find, by differentiation, the x-coordinates of all the stationary points on the curve $y = \frac{x^3}{(x+1)^2}$ stating, with reasons, the nature of each point.

Solution.

$$y = \frac{x^3}{(x+1)^2} \implies (x+1)^2 y = x^3 \implies y'(x+1)^2 + 2y(x+1) = 3x^2.$$

For stationary points, $y' = 0$. Thus,

$$2y(x+1) = \frac{2x^3}{x+1} = 3x^2 \implies 2x^3 - 3x^2(x+1) = x^2(-x-3) = 0.$$

Hence, $x = 0$ or $x = -3$.

x	0^-	0	0^+	$(-3)^-$	-3	$(-3)^+$
$\mathrm{d}y/\mathrm{d}x$	+ve	0	+ve	+ve	0	-ve

By the first derivative test, there is a stationary point of inflexion at $x = 0$ and a maximum point at $x = -3$.

* * * * *

Problem 3. Differentiate $f(x) = 8\sin(x/2) - \sin x - 4x$ with respect to x and deduce that $f(x) < 0$ for $x > 0$.

Solution.

$$f'(x) = 4\cos\frac{x}{2} - \cos x - 4 = 4\cos\frac{x}{2} - \left(2\cos^2\frac{x}{2} - 1\right) - 4 = -2\left(\cos\frac{x}{2} - 1\right)^2 - 1.$$

Observe that for all $x \in \mathbb{R}$, $\left(\cos\frac{x}{2} - 1\right)^2 \geq 0$. Hence, $f'(x) < 0$ for all real values of x. Thus, $f(x)$ is strictly decreasing on $\mathbb{R}$.

Note that $f(0) = 0$. Since $f(x)$ is strictly decreasing, for all $x > 0$, $f(x) < f(0) = 0$.

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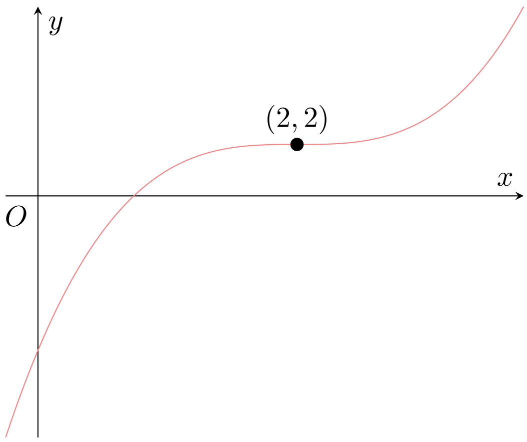
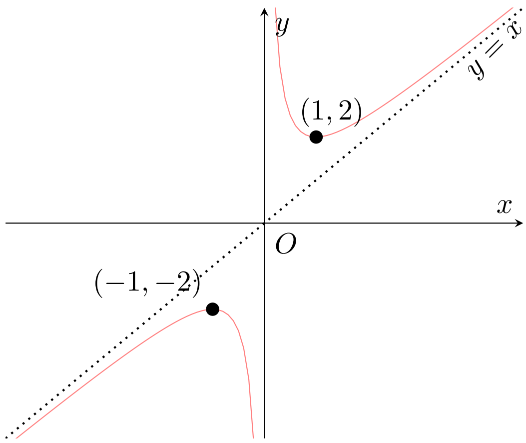
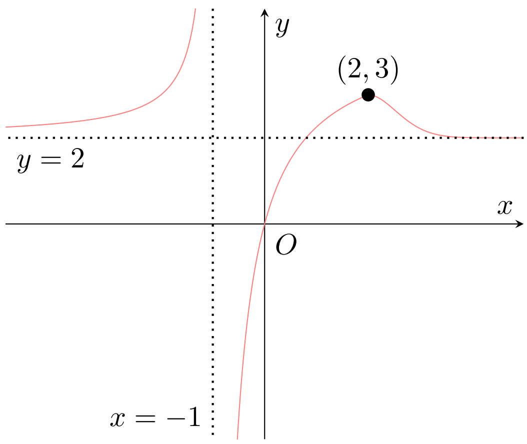
Problem 4. Sketch the graphs of the derivative functions for each of the graphs of the following functions below. In each graph, the point(s) labelled in coordinate form are stationary points.



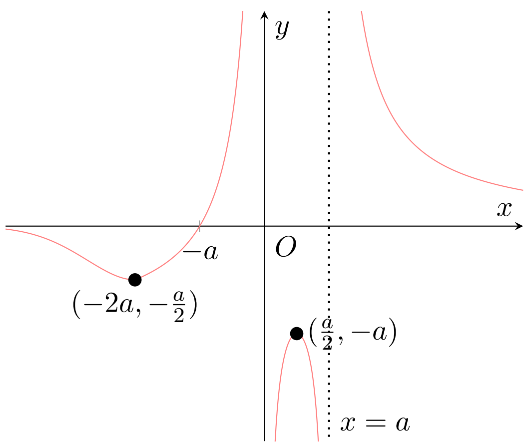
(a)



(b)



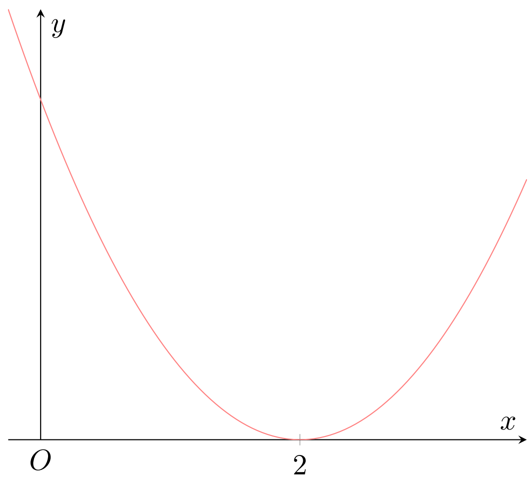
(c)



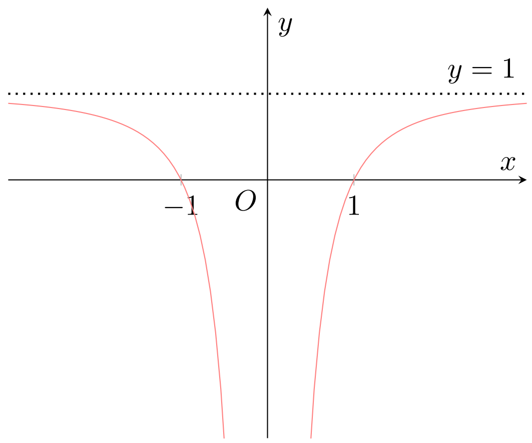
(d)

Solution.

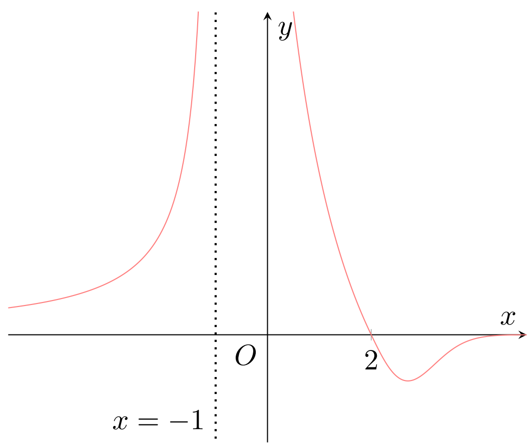
Part (a).



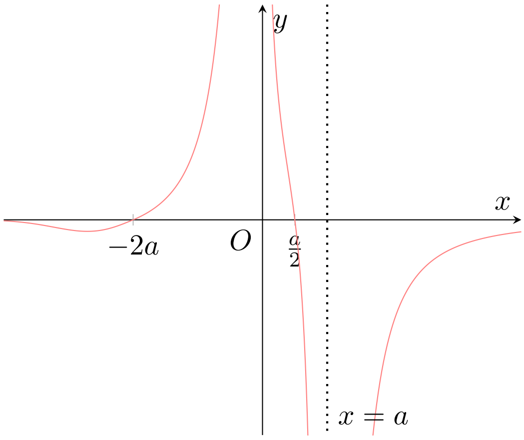
Part (b).



Part (c).



Part (d).



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Problem 5.

(a) Given that $y = ax\sqrt{x+2}$ where $a > 0$, find $\mathrm{d}y/\mathrm{d}x$, expressing your answer as a single algebraic fraction. Hence, show that the curve $y = ax\sqrt{x+2}$ has only one turning point, and state its coordinates in exact form.

(b) Sketch the graph of $y = f'(x)$, where $f(x) = ax\sqrt{x+2}$, where $a > 0$.

Solution.

Part (a).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\left(\frac{x}{2\sqrt{x+2}} + \frac{2(x+2)}{2\sqrt{x+2}}\right) = \frac{a(3x+4)}{2\sqrt{x+2}}.$$

Consider the stationary points of $y = ax\sqrt{x+2}$. For stationary points, $\mathrm{d}y/\mathrm{d}x = 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a(3x+4)}{2\sqrt{x+2}} = 0 \implies a(3x+4) = 0.$$

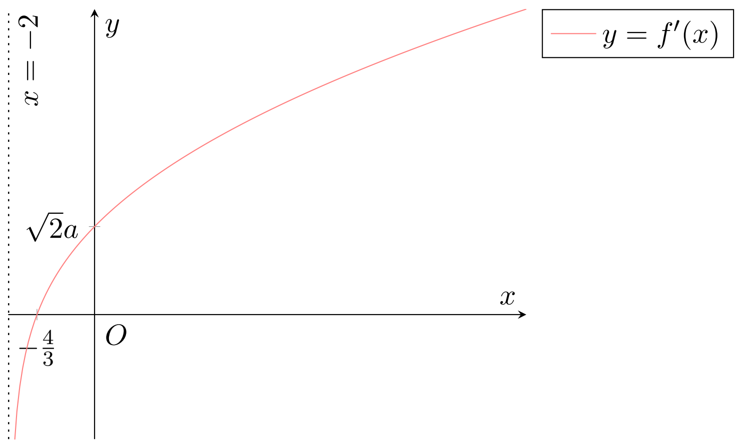
Since $a > 0$, we have $3x + 4 = 0$, whence $x = -4/3$.

x	$(-4/3)^-$	$-4/3$	$(-4/3)^+$
$\mathrm{d}y/\mathrm{d}x$	-ve	0	+ve

Hence, by the first derivative test, there is a turning point (minimum point) at $x = -4/3$. Thus, $y = ax\sqrt{x+2}$ has only one turning point.

Substituting $x = -4/3$ into $y = ax\sqrt{x+2}$, we see that the coordinate of the turning point is $(-\frac{4}{3}, -\frac{4a}{3}\sqrt{\frac{2}{3}})$.

Part (b).



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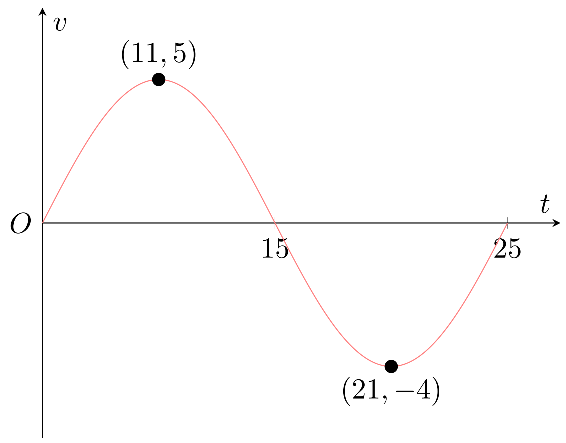
Problem 6. A particle P moves along the x-axis. Initially, P is at the origin O. At time t s, the velocity is v ms^{-1} and the acceleration is a ms^{-2}. Below is the velocity-time graph of the particle for $0 \leq t \leq 25$.



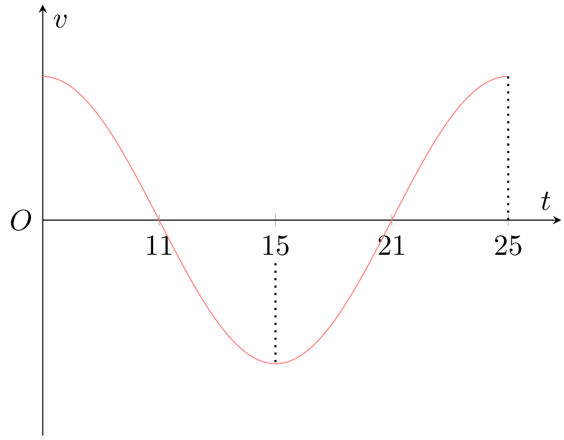
(a) Describe the motion of the particle for $0 \leq t \leq 25$.

(b) Sketch the acceleration-time graph of the particle P.

Solution.

Part (a). From $t = 0$ to $t = 11$, P speeds up and reaches a top speed of 5 ms^{-1}. From $t = 11$ to $t = 15$, P slows down. At $t = 15$, P is instantaneously at rest. From $t = 15$ to $t = 21$, P speeds up and moves in the opposite direction, reaching a top speed of 4 ms^{-1}. From $t = 21$ to $t = 25$, P slows down. At $t = 25$, P is instantaneously at rest.

Part (b).



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Problem 7. The function f defined by $f(x) = \ln x - 2(x - 1/2)$, where $x \in \mathbb{R}, x > 0$. Find $f'(x)$ and show that the function is decreasing for $x > 1/2$. Hence, show that for $x > 1/2$, $2(x - 1/2) - \ln x > \ln 2$.

Solution. Observe that $f'(x) = 1/x - 2 < 0$ for $x > 1/2$. Thus, $f(x)$ is decreasing for all $x > 1/2$. Since $f(1/2) = -\ln 2$, it follows that

$$\left(\forall x > \frac{1}{2}\right): \quad -\ln 2 = f(1/2) > f(x) = \ln x - 2\left(x - \frac{1}{2}\right) \implies 2\left(x - \frac{1}{2}\right) - \ln x > \ln 2.$$

Tutorial B5B

Problem 1. The equation of a closed curve is $(x+2y)^2+3(x-y)^2=27$.

(a) Show, by differentiation, that the gradient at the point (x, y) on the curve may be expressed in the form $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y-4x}{7y-x}$.

(b) Find the equations of the tangents to the curve that are parallel to

(i) the x-axis,

(ii) the y-axis.

Solution.

Part (a). Implicitly differentiating the given equation,

$$(x+2y)\left(1+2y'\right)+3(x-y)(1-y')=(-x+7y)y'+4x-y=0 \implies y'=\frac{y-4x}{7y-x}.$$

Part (b).

Part (b)(i). When the tangent to the curve is parallel to the x-axis, $y'=0$, whence $y=4x$. Substituting $y=4x$ into the given equation,

$$(9x)^2+3(-3x)^2=27 \implies 108x^2=27 \implies x^2=\frac{1}{4} \implies x=\pm\frac{1}{2} \implies y=\pm 2.$$

The equations of the tangents are hence $y=\pm 2$.

Part (b)(ii). When the tangent to the curve is parallel to the y-axis, y' is undefined. Hence, $7y-x=0 \implies x=7y$. Substituting $x=7y$ into the given equation,

$$(9y)^2+3(6y)^2=27 \implies 189y^2=27 \implies y^2=\frac{1}{7} \implies y=\pm\frac{1}{\sqrt{7}} \implies x=\pm\sqrt{7}.$$

The equations of the tangents are hence $x=\pm\sqrt{7}$.

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Problem 2. A piece of wire of length 8 cm is cut into two pieces, one of length x cm, the other of length $(8-x)$ cm. The piece of length x cm is bent to form a circle with circumference x cm. The other piece is bent to form a square with perimeter $(8-x)$ cm. Show that, as x varies, the sum of the areas enclosed by these two pieces of wire is a minimum when the radius of the circle is $\frac{4}{4+\pi}$ cm.

Solution. Let the radius of the circle be r cm. Then we have $x=2\pi r \implies r=x/2\pi$. Let the side length of the square be s cm. Then we have $8-x=4s \implies s=2-x/4$. Let the total area enclosed by the circle and the square be $A(x)$.

$$A(x)=\pi r^2+s^2=\pi\left(\frac{x}{2\pi}\right)^2+\left(2-\frac{x}{4}\right)^2=\left(\frac{1}{4\pi}+\frac{1}{16}\right)x^2-x+4.$$

Consider the stationary points of $A(x)$. For stationary points, $A'(x)=0$.

$$A'(x)=\left(\frac{1}{2\pi}+\frac{1}{8}\right)x-1=0 \implies x=\frac{1}{\frac{1}{2\pi}+\frac{1}{8}}=\frac{8\pi}{4+\pi}.$$

x	$\left(\frac{8\pi}{4+\pi}\right)^-$	$\frac{8\pi}{4+\pi}$	$\left(\frac{8\pi}{4+\pi}\right)^+$
$\mathrm{d}A/\mathrm{d}x$	$-$ve	0	$+$ve

Hence, by the first derivative test, the minimum value of $A(x)$ is achieved when $x = \frac{8\pi}{4+\pi}$, whence

$$r = \frac{1}{2\pi} \cdot \frac{8\pi}{4+\pi} = \frac{4}{4+\pi} \text{ cm.}$$

* * * * *

Problem 3. A spherical balloon is being inflated in such a way that its volume is increasing at a constant rate of 150 $\mathrm{cm}^3\mathrm{s}^{-1}$. At time t seconds, the radius of the balloon is r cm.

(a) Find $\mathrm{d}r/\mathrm{d}t$ when $r = 50$.

(b) Find the rate of increase of the surface area of the balloon when its radius is 50 cm.

Solution. Let the volume of the balloon be $V(r) = \frac{4}{3}\pi r^3$ cm^3.
Part (a). Note that $\frac{\mathrm{d}V}{\mathrm{d}t} = 150$ and $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r/\mathrm{d}V}{\mathrm{d}t/\mathrm{d}V} = \frac{\mathrm{d}V/\mathrm{d}t}{\mathrm{d}V/\mathrm{d}r} = \frac{150}{4\pi r^2} = \frac{75}{2\pi r^2}.$$

Evaluating $\frac{\mathrm{d}r}{\mathrm{d}t}$ at $r = 50$,

$$\left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{r=50} = \frac{75}{2\pi \cdot 50^2} = \frac{3}{200\pi}.$$

Part (b). Let the surface area of the balloon be $A(r) = 4\pi r^2$. Observe that $\frac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \implies \left.\frac{\mathrm{d}A}{\mathrm{d}t}\right|_{r=50} = (8\pi \cdot 50)\left(\frac{3}{200\pi}\right) = 6.$$

Thus, the rate of increase of the surface area of the balloon when its radius is 50 cm is 6 cm/s.

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Problem 4. A curve has parametric equations $x = 5\sec\theta$, $y = 3\tan\theta$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Find the exact coordinates of the point on the curve at which the normal is parallel to the line $y = x$.

Solution. Observe that $x^2 = 25\sec^2\theta$ and $\frac{25}{9}y^2 = 25\tan^2\theta$. Using the identity $\tan^2\theta + 1 = \sec^2\theta$, we get

$$\frac{25}{9}y^2 + 25 = x^2. \qquad (*)$$

Implicitly differentiating with respect to x, we get

$$\frac{25}{9}y \cdot y' = x.$$

Since the normal is parallel to $y = x$, the tangent is parallel to $y = -x$, whence $y' = -1$. Thus,

$$y = -\frac{9}{25}x.$$

Substituting $y = -\frac{9}{25}x$ into $(*)$,

$$\frac{25}{9}\left(-\frac{9}{25}x\right)^2 + 25 = x^2 \implies \frac{16}{25}x^2 = 25 \implies \frac{4}{5}x = \pm 5 \implies x = \pm\frac{25}{4}.$$

Observe that for $-\pi/2 < \theta < \pi/2$, $x = 5\sec\theta \geq 5$. We thus take $x = 25/4$, whence $y = -9/4$. The coordinate of the required point is thus $(25/4, -9/4)$.

Problem 5. The parametric equations of a curve are

$$x = t^2,\ y = \frac{2}{t}.$$

(a) Find the equation of the tangent to the curve at the point $(p^2, 2/p)$, simplifying your answer.

(b) Hence, find the coordinates of the points Q and R where this tangent meets the x- and y-axes respectively.

(c) The point F is the mid-point of QR. Find a Cartesian equation of the curve traced by F as p varies.

Solution.

Part (a). Observe that $\mathrm{d}x/\mathrm{d}t = 2t$ and $\mathrm{d}y/\mathrm{d}t = -2/t^2$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-2/t^2}{2t} = -\frac{1}{t^3}.$$

Using the point-slope formula, the tangent to the curve at $(p^2, 2/p)$ is given by the equation

$$y - \frac{2}{p} = -\frac{1}{p^3}\left(x - p^2\right) \implies y = \frac{3}{p} - \frac{1}{p^3}x.$$

Part (b). Consider the case where $y = 0$:

$$0 = \frac{3}{p} - \frac{1}{p^3}x \implies x = 3p^2 \implies Q\left(3p^2, 0\right).$$

Consider the case where $x = 0$:

$$y = \frac{3}{p} \implies R\left(0, \frac{3}{p}\right).$$

Part (c). Note that

$$F = \left(\frac{3}{2}p^2, \frac{3}{2p}\right).$$

As p varies, F traces a curve given by the parametric equations $x = 3p^2/2$, $y = 3/2p$. Hence,

$$p^2 = \frac{2}{3}x = \frac{9}{4y^2} \implies y^2 = \frac{27}{8x}.$$

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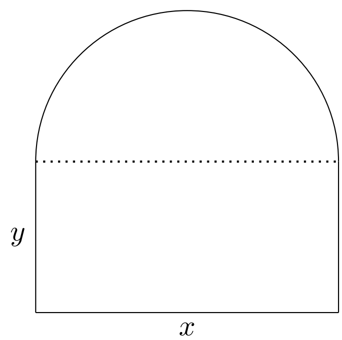
Problem 6.



A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle x m by y m together with a semicircle of diameter x m, as shown in the diagram. A low wall will be built around the flowerbed. The time needed to build the wall will be 3 hours per metre for the straight parts and 9 hours per metre for the semicircular part. Given that a total time of 180 hours is taken to build the wall, find, using differentiation, the values of x and y which give a flower-bed of maximum area.

Solution. Observe that the length of the straight parts is $(2y + x)$ m, while the length of the semicircular part is $\frac{1}{2}\pi x$ m. Since a total time of 180 hours is taken to build the wall,

$$3(2y + x) + 9\left(\frac{1}{2}\pi x\right) = 180 \implies 4y + 2x + 3\pi x = 120 \implies x = \frac{120 - 4y}{2 + 3\pi}.$$

Differentiating with respect to y, we get $x' = -4/(2 + 3\pi)$. Let $A(y)$ be the total area enclosed by the garden, in m^2. Observe that

$$A(y) = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = xy + \frac{\pi}{8}x^2.$$

Consider the stationary points of $A(y)$. For stationary points, $A'(y) = 0$.

$$A'(y) = \left(x'y + x\right) + \frac{\pi}{4}x \cdot x' = 0.$$

Substituting in our values of x and x', we get

$$\left[y\left(-\frac{4}{2 + 3\pi}\right) + \frac{120 - 4y}{2 + 3\pi}\right] + \left[\frac{\pi}{4}\left(\frac{120 - 4y}{2 + 3\pi}\right)\left(-\frac{4}{2 + 3\pi}\right)\right] = 0.$$

Using G.C., we get $y = 12.6$ (3 s.f.), whence $x = 6.09$ (3 s.f.).

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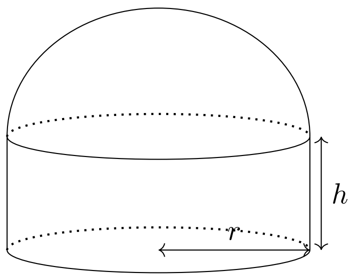
Problem 7.



A model of a concert hall is made up of three parts.

- The roof is modelled by the curved surface of a hemisphere of radius r cm.
- The walls are modelled by the curved surface of a cylinder of radius r cm and height h cm.
- The floor is modelled by a circular disc of radius r cm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

(a) It is given that the volume of the model is a fixed value k cm^3, and the external surface area is a minimum. Use differentiation to find the values of r and h in terms of k. Simplify your answers.

(b) It is given instead that the volume of the model is 200 cm^3 and its external surface area is 180 cm^2. Show that there are two possible values of r. Given also that $r < h$, find the value of r and the value of h.

Solution.

Part (a). Let the volume of the model be V cm^3. Then

$$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + \pi r^2 h = k \implies h = \frac{k}{\pi r^2} - \frac{2}{3}r. \tag{1}$$

Let the external surface area of the model be A cm^2. Then

$$A = \frac{4\pi r^2}{2} + 2\pi rh + \pi r^2 = 3\pi r^2 + 2\pi r\left(\frac{k}{\pi r^2} - \frac{2}{3}r\right) = \frac{5\pi}{3}r^2 + \frac{2k}{r}. \tag{2}$$

Consider the stationary points of A. For stationary points, $\mathrm{d}A/\mathrm{d}r = 0$.

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{10\pi}{3}r - \frac{2k}{r^2} = 0 \implies r^3 = \frac{3k}{5\pi} \implies r = \sqrt[3]{\frac{3k}{5\pi}}.$$

r	$\sqrt[3]{\frac{3k}{5\pi}}^-$	$\sqrt[3]{\frac{3k}{5\pi}}$	$\sqrt[3]{\frac{3k}{5\pi}}^+$
$\mathrm{d}A/\mathrm{d}r$	$-$ve	0	$+$ve

Hence, by the first derivative test, A is at a minimum when $r = \sqrt[3]{\frac{3k}{5\pi}}$.
Substituting $r^3 = \frac{3k}{5\pi}$ into (1),

$$\frac{2}{3}\pi\left(\frac{3k}{5\pi}\right) + \pi r^2 h = \frac{2}{5}k + \pi r^2 h = k \implies r^2 h = \frac{3k}{5\pi} = r^3 \implies h = r = \sqrt[3]{\frac{3k}{5\pi}}.$$

Part (b). From (2), we have

$$\frac{5\pi}{3}r^2 + \frac{2(200)}{r} = 180 \implies \pi r^3 - 108r + 240 = 0.$$

Let $f(r) = \pi r^3 - 108r + 240$. Consider the stationary points of $f(r)$. For stationary points, $f'(r) = 0$.

$$f'(r) = 3\pi r^2 - 108 = 0 \implies r^2 = \frac{36}{\pi} \implies r = \pm\frac{6}{\sqrt{\pi}}.$$

Since $f(r)$ is a cubic with two turning points, it follows that there is exactly one root in each of the following three intervals:

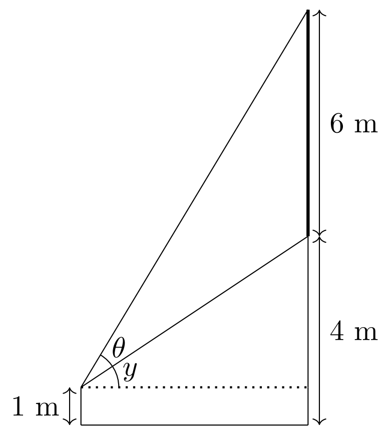
$$\left(-\infty, -\frac{6}{\sqrt{\pi}}\right), \quad \left(-\frac{6}{\sqrt{\pi}}, \frac{6}{\sqrt{\pi}}\right), \quad \left(\frac{6}{\sqrt{\pi}}, \infty\right).$$

We now show that the root in the interval $\left(-\frac{6}{\sqrt{\pi}}, \frac{6}{\sqrt{\pi}}\right)$ is positive. Since $f(r)$ has a positive leading coefficient, it must be decreasing in the interval $\left(-\frac{6}{\sqrt{\pi}}, \frac{6}{\sqrt{\pi}}\right)$. Since $f(0) = 240 > 0$, the root in said interval must be positive. Hence, $f(r) = 0$ has two positive roots. Using G.C., the roots are $r = 3.04$ and $r = 3.72$. From (1), we know that

$$h = \frac{200}{\pi r^2} - \frac{2}{3}r.$$

When $r = 3.04$, $h = 4.88 > r$. When $r = 3.72$, $h = 2.12 < r$. Thus, given that $r < h$, we have $r = 3.04$ and $h = 4.88$.

Problem 8.



A movie screen on a vertical wall is 6 m high and 4 m above the horizontal floor. A boy who is standing at x m away from the wall has eye level at 1 m above the floor as shown in the diagram.

The viewing angle of the boy at that position is θ and the angle of elevation of the bottom of the screen is y.

(a) Express y in terms of x.

(b) By expressing θ in terms of x or otherwise, find the stationary value of θ, giving your answers in exact form. Determine if the value is a maximum or minimum value, showing your working clearly.

Solution.

Part (a). Observe that $\tan y = 3/x$, whence $y = \arctan(3/x)$.

Part (b). Observe that $\tan(y+\theta) = 9/x$. Hence,

$$\tan(y+\theta) = \frac{\tan y + \tan\theta}{1 - \tan y \tan\theta} = \frac{3/x + \tan\theta}{1 - (3/x)\tan\theta} = \frac{3 + x\tan\theta}{x - 3\tan\theta} = \frac{9}{x} \implies \tan\theta = \frac{6x}{x^2+27}.$$

Hence,

$$\theta = \arctan\left(\frac{6x}{x^2+27}\right).$$

Differentiating with respect to x,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{1+\left(\frac{6x}{x^2+27}\right)^2}\left[\frac{6\left(x^2+27\right) - 6x(2x)}{\left(x^2+27\right)^2}\right] = \frac{-6x^2+162}{36x^2+\left(x^2+27\right)^2}.$$

For stationary points, $\mathrm{d}\theta/\mathrm{d}x = 0$. Hence,

$$-6x^2 + 162 = 0 \implies x^2 = 27 \implies x = \pm 3\sqrt{3}.$$

Since $x > 0$, we only take $x = 3\sqrt{3}$. Thus,

$$\theta = \arctan\left(\frac{6\left(3\sqrt{3}\right)}{27+27}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

x	$3\sqrt{3}^-$	$3\sqrt{3}$	$3\sqrt{3}^+$
$\mathrm{d}\theta/\mathrm{d}x$	+ve	0	−ve

Thus, by the first derivative test, $\theta = \frac{\pi}{6}$ is a maximum value.

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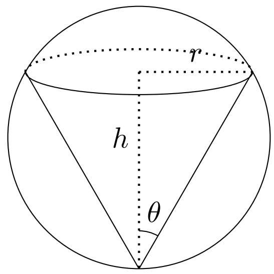
Problem 9.



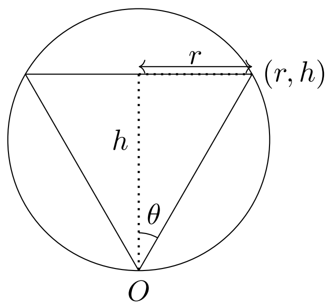
The diagram shows a right inverted cone of radius r, height h and semi-vertical angle θ, which is inscribed in a sphere of radius 1 unit.

Prove that $r^2 = 2h - h^2$.

(a) As r and h varies, determine the exact maximum volume of the cone.

(b) Show that $h = 2\cos^2\theta$. The volume of the cone is increasing at a rate of 6 unit^3/s when $h = \frac{3}{2}$. Determine the rate of change of θ at this instant, leaving your answer in an exact form.

Solution. Consider the following diagram of the cone and sphere.



Let the origin be the tip of the cone. Since the sphere has radius 1 unit, the circle is given by the equation $x^2 + (y-1)^2 = 1$. Since the point (r, h) lies on the circle,

$$r^2 + (h-1)^2 = 1 \implies r^2 = 2h - h^2. \qquad (*)$$

Part (a). Implicitly differentiating $(*)$ with respect to r,

$$2r = 2 \cdot \frac{\mathrm{d}h}{\mathrm{d}r} - 2h \cdot \frac{\mathrm{d}h}{\mathrm{d}r} \implies \frac{\mathrm{d}h}{\mathrm{d}r} = \frac{r}{1-h}.$$

Let the volume of the cone be $V(r)$ units^3. Then

$$V(r) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(2h - h^2\right)h = \frac{1}{3}\pi\left(2h^2 - h^3\right).$$

Differentiating $V(r)$ with respect to r,

$$V'(r) = \frac{1}{3}\pi\left(4h \cdot \frac{\mathrm{d}h}{\mathrm{d}r} - 3h^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}r}\right) = \frac{1}{3}\left(\frac{\pi r h}{1-h}\right)(4-3h).$$

Consider the stationary values of $V(r)$. For stationary values, $V'(r) = 0$, whence $h = 4/3$. Substituting this into $(*)$, we obtain

$$r^2 = 2\left(\frac{4}{3}\right) - \left(\frac{4}{3}\right)^2 = \frac{8}{9} \implies r = \sqrt{\frac{8}{9}}.$$

Note that we reject $r = -\sqrt{8/9}$ as $r > 0$.

r	$\sqrt{8/9}^-$	$\sqrt{8/9}$	$\sqrt{8/9}^+$
$V'(r)$	+ve	0	−ve

Hence, the maximum volume is achieved when $r = \sqrt{8/9}$. Note that

$$V\left(\sqrt{\frac{8}{9}}\right) = \frac{1}{3}\pi\left(\frac{8}{9}\right)\left(\frac{4}{3}\right) = \frac{32}{81}\pi.$$

The maximum volume of the cone is hence $32\pi/81$ units^3.

Part (b). From the diagram, we have

$$\cos\theta = \frac{h}{\sqrt{r^2+h^2}} \implies 2\cos^2\theta = \frac{2h^2}{r^2+h^2} = \frac{2h^2}{2h-h^2+h^2} = h.$$

Observe that

$$V = \frac{\pi}{3}\left(2h^2 - h^3\right) = \frac{\pi}{3}\left(8\cos^4\theta - 8\cos^6\theta\right) = \frac{8\pi}{3}\left(\cos^4\theta - \cos^6\theta\right).$$

Differentiating with respect to θ, we get

$$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{8\pi}{3}\left(-4\cos^3\theta\sin\theta + 6\cos^5\theta\sin\theta\right) = \frac{16\pi}{3}\cos^3\theta\sin\theta\left(-2+3\cos^2\theta\right).$$

Since $2\cos^2\theta = h = 3/2$, we clearly have $\theta = \pi/6$. Thus,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\theta}\right|_{h=3/2} = \frac{16\pi}{3}\cos^3\frac{\pi}{6}\sin\frac{\pi}{6}\left(-2+3\cos^2\frac{\pi}{6}\right) = \frac{\sqrt{3}\pi}{4}.$$

Hence,

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{h=3/2} = \left.\left(\frac{\mathrm{d}\theta}{\mathrm{d}V}\cdot\frac{\mathrm{d}V}{\mathrm{d}t}\right)\right|_{h=3/2} = \frac{6}{\sqrt{3}\pi/4} = \frac{8\sqrt{3}}{\pi}.$$

θ is thus increasing at a rate of $8\sqrt{3}/\pi$ radians per second when $h = \frac{3}{2}$.

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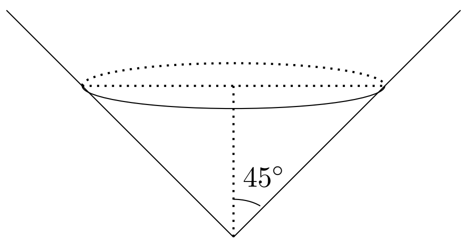
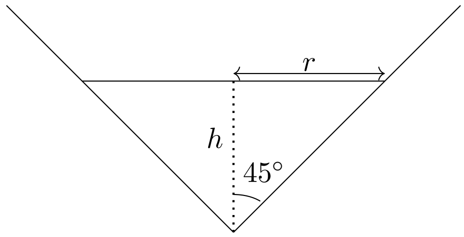
Problem 10.



A hollow cone of semi-vertical angle 45° is held with its axis vertical and vertex downwards. At the beginning of an experiment, it is filled with 390 cm^3 of liquid. The liquid runs out through a small hole at the vertex at a constant rate of 2 cm^3/s.

Find the rate at which the depth of the liquid is decreasing 3 minutes after the start of the experiment.

Solution. Consider the following diagram.



Let the volume of liquid be $V = \frac{1}{3}\pi r^2 h$ cm^3. From the diagram, we have $r = h$. Thus,

$$V = \frac{1}{3}\pi h^3.$$

Differentiating V with respect to h,

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi \cdot 3h^2 = \pi h^2.$$

Let t be the time since the start of the experiment in seconds. Consider $\mathrm{d}h/\mathrm{d}t$.

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = \left(\frac{\mathrm{d}h}{\mathrm{d}V}\right)^{-1} \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{-2}{\pi h^2}.$$

When $t = 180$, there is $390 - 180(2) = 30$ cm^3 of liquid left in the cone. Thus,

$$V = \frac{1}{3}\pi h^3 = 30 \implies h^3 = \frac{90}{\pi} \implies h = \sqrt[3]{\frac{90}{\pi}}.$$

Evaluating $\mathrm{d}h/\mathrm{d}t$ at $t = 180$,

$$\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{t=180} = \frac{-2}{\pi\left(\sqrt[3]{\frac{90}{\pi}}\right)^2} = -0.0680 \text{ (3 s.f.)}.$$

Thus, the depth of the liquid is decreasing at a rate of 0.0680 cm/s 3 minutes after the start of the experiment.

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Problem 11. A particle is projected from the origin O, and it moves freely under gravity in the x-y plane. At time t s after projection, the particle is at the point (x, y) where $x = 30t$ and $y = 20t - 5t^2$, with x and y measured in metres.

(a) Given that the particle passes through two points A and B which are at a distance 15 m above the x-axis, find the time taken for the particle to travel from A to B. Find also the distance AB.

(b) It is known that the particle always travels in a direction tangential to its path. Show that, when $x = 10$, the particle is travelling at an angle of $\arctan(5/9)$ above the horizontal.

The speed of the particle is given by $\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}$. Find the speed of the particle when $x = 10$.

(c) Show that the equation of trajectory is $y = \frac{2}{3}x - \frac{1}{180}x^2$.

Solution.

Part (a). Consider $y = 15$.

$$y = 20t - 5t^2 = 15 \implies t^2 - 4t + 3 = (t-1)(t-3) = 0.$$

Hence, $t = 1$ or $t = 3$. Thus, the particle takes $3 - 1 = 2$ seconds to travel from A to B.

Note that $x = 30$ when $t = 1$, and $x = 90$ when $t = 3$. Hence, $A(30, 15)$ and $B(90, 15)$, whence $AB = 60$ m.

Part (b). Note that $\mathrm{d}x/\mathrm{d}t = 30$ and $\mathrm{d}y/\mathrm{d}t = 20 - 10t$. Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{20 - 10t}{30} = \frac{2 - t}{3}.$$

When $x = 10$, $t = 1/3$. Evaluating $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $t = 1/3$,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\frac{1}{3}} = \frac{2 - 1/3}{3} = \frac{5}{9}.$$

Hence, the line tangent to the curve at $x = 10$ has gradient $5/9$. Thus, the particle is travelling at an angle of $\arctan(5/9)$ above the horizontal when $x = 10$.

Note that

$$\left.\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\right|_{t=\frac{1}{3}} = \sqrt{30^2 + \left(20 - \frac{10}{3}\right)^2} = 34.3 \text{ (3 s.f.)}.$$

Hence, the particle is travelling at a speed of 34.3 m/s when $x = 10$.

Part (c). Note that $t = x/30$. Hence,

$$y = 20t - 5t^2 = 20\left(\frac{x}{30}\right) - 5\left(\frac{x}{30}\right)^2 = \frac{2}{3}x - \frac{1}{180}x^2.$$

Self-Practice B5A

Problem 1. It is given that $f(x) = \frac{x^2-2x}{e^x}$.

Find the range of values of x for which the curve $y = f(x)$ is concave upward. Hence, sketch the graph of $y = f(x)$, indicating clearly the equations of any asymptotes and the coordinates of any stationary points and any intersections with the axes.

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Problem 2. It is given that x and y satisfy the equation

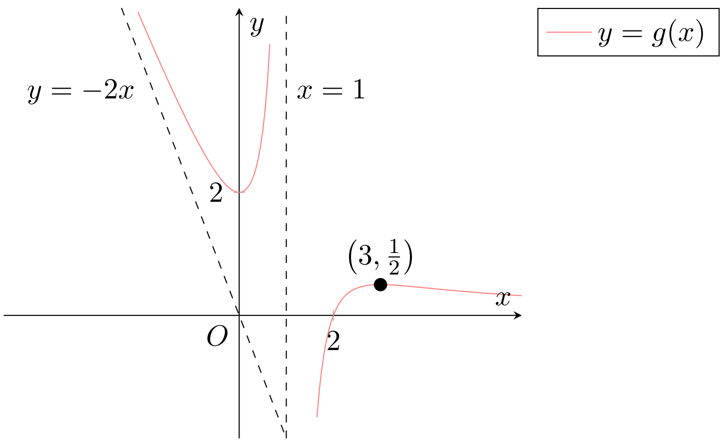
$$y^4 - \ln\frac{y^2}{4} = x^4 - 6x^2, \quad y > 0.$$

(a) Show that $\frac{dy}{dx} = \frac{2xy\left(x^2-3\right)}{2y^4-1}$.

(b) Hence, obtain the possible exact value(s) of $\frac{dy}{dx}$ when $y = 2$.

* * * * *

Problem 3. The diagram below shows the graph of $y = g(x)$. The graph has a minimum point at $(0, 2)$ and a maximum point at $(3, \frac{1}{2})$. The equations of the asymptotes are $x = 1$, $y = 0$ and $y = -2x$.



(a) State the interval(s) on which g is

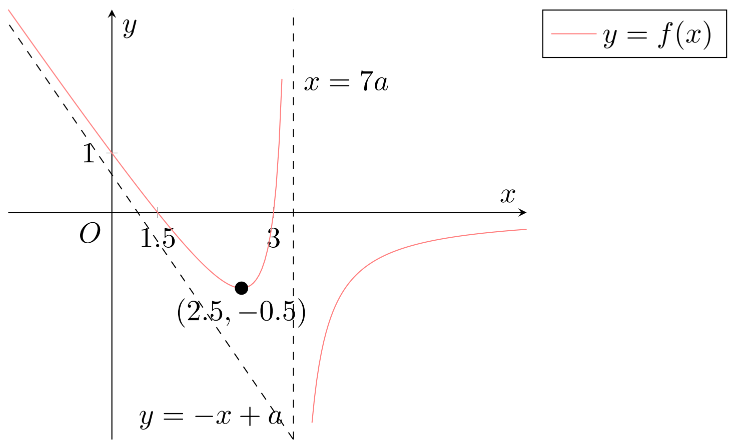
   (i) increasing;

   (ii) increasing and concave upward.

(b) Sketch $y = g'(x)$, showing clearly the equations of the asymptotes and the coordinates of the turning points and axial intercepts, where applicable.

* * * * *

Problem 4. The diagram below shows the graph of $y = f(x)$. It cuts the axes at the points $(0, 1)$, $(1.5, 0)$ and $(3, 0)$. It has a minimum point at $(2.5, -0.5)$. The horizontal, vertical and oblique asymptotes are $y = 0$, $x = 7a$ and $y = -x + a$ respectively, where a is a positive constant.



On separate diagrams, sketch the graphs of

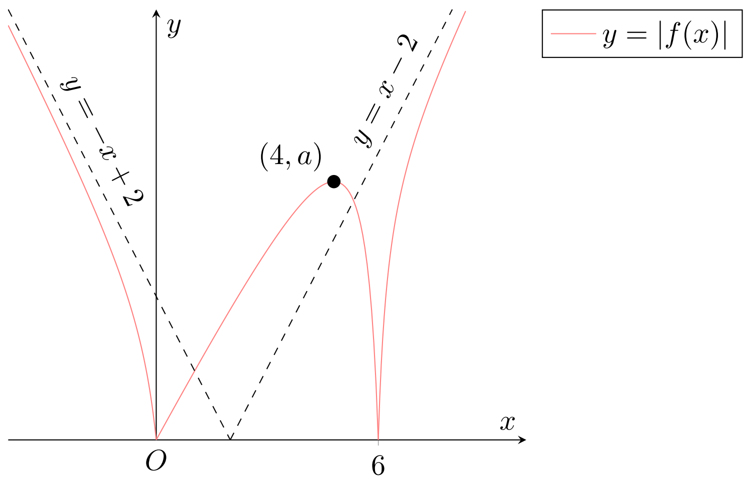
(a) $y = \frac{1}{f(x)}$,

(b) $y = f'(x)$,

showing clearly the axial intercepts, the stationary points and the equations of the asymptotes where applicable.

* * * * *

Problem 5 (🌶). The graph of $y = |f(x)|$ is shown in the diagram, with a maximum point $(4, a)$, and $x = 0$ and $x = 6$ are tangents to both graphs.



It is given that the graph of the continuous function f has **only** one oblique asymptote, and that $f'(1) > 0$ and $f'(7) < 0$.

Sketch the graph of $y = f'(x)$, showing clearly the stationary point(s), the asymptote(s) and the intercept(s), if any.

Self-Practice B5B

Problem 1. Find the coordinates of the points on the curve $3x^2 + xy + y^2 = 33$ at which the tangent is parallel to the x-axis.

* * * * *

Problem 2. Given the equation $x^{1/2} + y^{1/2} = k^{1/2}$, where k is a constant,

(a) show that the equation of the tangent at the point $(p,\, q)$ is given by

$$y = -\sqrt{\frac{q}{p}}x + q + \sqrt{pq}.$$

(b) Hence or otherwise, prove that the sum of the x and y-intercepts of any tangent line to the curve $x^{1/2} + y^{1/2} = k^{1/2}$ is constant and equal to k.

* * * * *

Problem 3. A curve C is defined by the parametric equations $x = t^2(t+1)$, $y = 4t - 5$, $t \geq 0$.

(a) Find the equation of the tangent to the curve C at the point where $y = -5$.

(b) Find the equation of the normal to the curve C when $t = 2$ and hence show that this normal does not intersect the curve C again.

* * * * *

Problem 4. A curve C has parametric equations

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}.$$

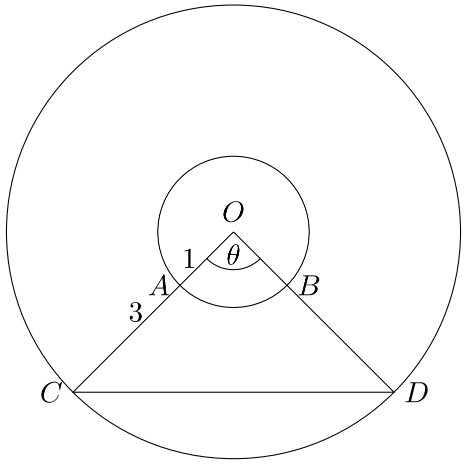
(a) The point P on the curve has parameter p. Show that the equation of the tangent at P is $\left(p^2+1\right)x - \left(p^2-1\right)y = 4p$.

(b) The tangent at P meets the line $y = x$ at the point A and the line $y = -x$ at the point B. Show that the area of triangle OAB is independent of p, where O is the origin.

(c) Find a Cartesian equation of C. Sketch C, giving the coordinates of any points where C crosses the x- and y-axes and the equations of any asymptotics.

* * * * *

Problem 5. The diagram shows two circles, of radii 1 and 3, each with centre O. The angle between the lines OAC and OBD is θ radians. The region R is bounded by the minor arc AB and the lines AC, CD and DB.



(a) Find the area of R.

(b) Find the value of θ for which the area of R is greatest.

(c) Find the greatest value of θ which ensures that the whole line segment CD lies between the two circles.

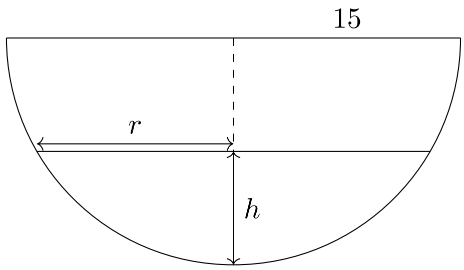
* * * * *

Problem 6. A company manufactures closed hollow cylindrical cans of cross-sectional radius r cm^2 and height h cm. A can is made of two different materials. Its top and base cost 0.09 cents per cm^2 and its curved surface costs 0.06 cents per cm^2 to manufacture.

Show that the radius of the cheapest can of volume 300 cm^3 is $\sqrt[3]{a/\pi}$, where a is a constant to be determined.

* * * * *

Problem 7. A hemispherical goldfish tank with radius 15 cm (as shown in the figure above) was initially filled with water. The tank has a defect and water is leaking at a constant rate of 20 cm^3 per min. The volume of water in the tank is given by $V = \frac{\pi}{3}\left(45h^2 - h^3\right)$ where h is the depth of water at the centre of the tank in cm. Show that r, the radius of the water surface in cm, is given by $r = \sqrt{30h - h^2}$.



Given that the minimum depth of water needed for the goldfish to survive is 5 cm, find, at this instant,

(a) the rate of change of the depth of water, and

(b) the rate of decrease of the radius of the water surface.

* * * * *

Problem 8. A circular cylinder is expanding in such a way that, at time t seconds, the height of the cylinder is y cm and the area of the cross-section is $\frac{1}{3}y^2$ cm^2. At the instant when $y = 3$ cm, the height is increasing at a rate of 0.5 cm/s. Find the rate of increase, at this instant, of:

(a) the area of the cross-section of the cylinder,

(b) the volume of the cylinder.

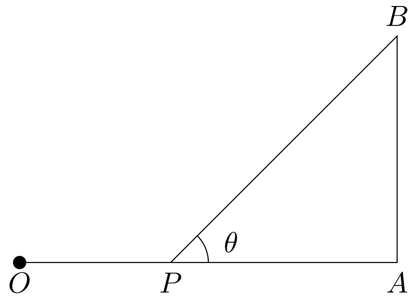
* * * * *

Problem 9. Two variables u and v are connected by the relation $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, where f is a constant.

Given that u and v both vary with time, t, find an equation connecting $\frac{\mathrm{d}u}{\mathrm{d}t}$, $\frac{\mathrm{d}v}{\mathrm{d}t}$, u and v. Given also that u is decreasing at a constant rate of 2 cm per second and that $f = 10$ cm, calculate the rate of increase of v when $u = 50$ cm.

* * * * *

Problem 10. In the diagram, O and A are fixed points 1000 m apart on horizontal ground. The point B is vertically above A, and represents a balloon which is ascending at a steady rate of 2 ms^{-1}. The balloon is being observed from a moving point P on the line OA.



At time $t = 0$, the balloon is at A and the observer is at O. The observation point P moves towards A with steady speed 6 ms^{-1}. At time t, the angle APB is θ radians.

Show that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{500}{t^2 + (500 - 3t)^2}.$$

* * * * *

Problem 11 (🌶). The normal to the rectangular hyperbola $xy = c^2$ at the point $P(cp, c/p)$, $p > 0$, meets the curve again at the point Q.

(a) Determine the coordinates of Q.

(b) Prove that $PQ^2 = 3OP^2 + OQ^2$.

Assignment B5A

Problem 1.

(a) Show, algebraically, that the derivative of the function

$$\ln(1+x) - \frac{2x}{x+2}$$

is never negative.

(b) Hence, show that $\ln(1+x) \geq \frac{2x}{x+2}$ when $x \geq 0$.

Solution. Let

$$f(x) = \ln(1+x) - \frac{2x}{x+2} = \ln(1+x) - 2 + \frac{4}{x+2}.$$

Part (a).

$$f'(x) = \frac{1}{1+x} - \frac{4}{(x+2)^2} = \frac{x^2}{(1+x)(x+2)^2}.$$

Given that $\ln(1+x)$ is defined, it must be that $1+x>0$. We also know that $x^2 \geq 0$ and $(x+2)^2 \geq 0$. Hence, $f'(x) \geq 0$ for all x in the domain of f and is thus never negative.

Part (b). Note that $f(0) = 0$. Since $f'(x) \geq 0$ for all $x \geq 0$,

$$\ln(1+x) - \frac{2x}{x+2} = f(x) \geq f(0) = 0 \implies \ln(1+x) \geq \frac{2x}{x+2}.$$

* * * * *

Problem 2. The equation of a curve is $y = ax^2 - 2bx + c$, where a, b and c are constants, with $a > 0$.

(a) Using differentiation, find the coordinates of the turning point on the curve, in terms of a, b and c. State whether it is a maximum point or a minimum point.

(b) Given that the turning point of the curve lies on the line $y = x$, find an expression for c in terms of a and b. Show that in this case, whatever the value of b, $c \geq -1/4a$.

(c) Find the numerical values of a, b and c when the curve passes through the point $(0,6)$ and has a turning point at $(2,2)$.

Solution.

Part (a). For stationary points, $\mathrm{d}y/\mathrm{d}x = 0$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax - 2b = 0 \implies x = \frac{b}{a} \implies y = a\left(\frac{b}{a}\right)^2 - 2b\left(\frac{b}{a}\right) + c = -\frac{b^2}{a} + c.$$

Since $a > 0$, the graph of y is concave upwards. Thus, there is a maximum point at $\left(\frac{b}{a}, -\frac{b^2}{a} + c\right)$.

Part (b). Since the turning point $\left(\frac{b}{a}, -\frac{b^2}{a} + c\right)$ lies on the line $y = x$,

$$\frac{b}{a} = -\frac{b^2}{a} + c \implies c = \frac{b+b^2}{a} = \frac{(b+1/2)^2 - 1/4}{a}.$$

Since $(b+1/2)^2 \geq 0$, it follows that $c \geq -1/4a$.

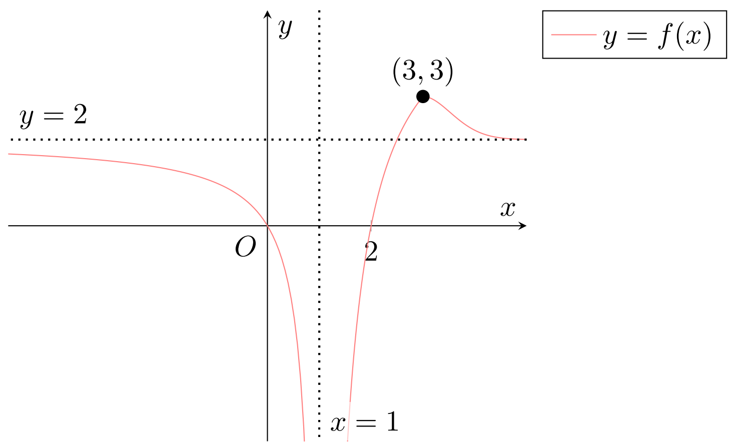
Part (c). Since the curve passes through $(0, 6)$, it is obvious to see that $c = 6$. Furthermore, since the curve has a turning point at $(2, 2)$, we know that $\frac{b}{a} = 2$ and $-\frac{b^2}{a} + c = 2$. Hence,

$$-\frac{b^2}{a} = 2 - c = -4 \implies b\left(\frac{b}{a}\right) = 4 \implies b = 2 \implies a = 1.$$

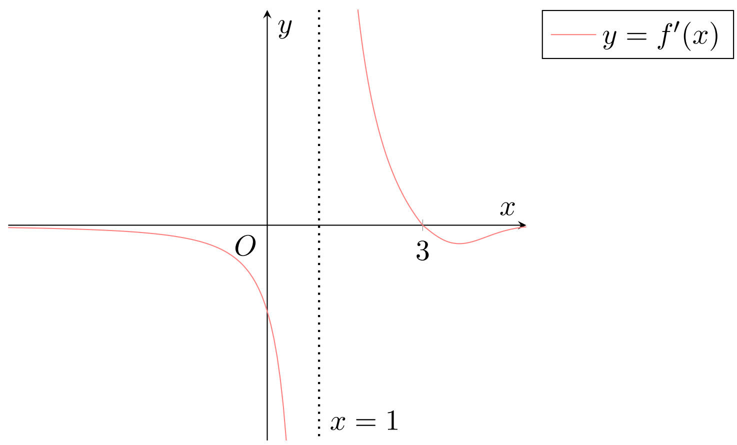
Thus, $a = 1$, $b = 2$, and $c = 6$.

* * * * *

Problem 3. The diagram below shows the graph of $y = f(x)$. Sketch the graph of $y = f'(x)$.



Solution.



Problem 4. The curve C has equation

$$x - y = (x+y)^2.$$

It is given that C has only one turning point.

(a) Show that $1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x+2y+1}$.

(b) Hence, or otherwise, show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.

(c) Hence, state, with a reason, whether the turning point is a maximum or a minimum.

Solution.

Part (a). Implicitly differentiating the given equation,

$$1 - \frac{\mathrm{d}y}{\mathrm{d}x} = 2(x+y)\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-(2x+2y)}{2x+2y+1} \implies 1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x+2y+1}.$$

Part (b). Implicitly differentiating the above equation,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\left(2 + 2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(2x+2y+1)^2} = -\left(\frac{2}{2x+2y+1}\right)^2\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) = -\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^3.$$

Part (c). For turning points, $\mathrm{d}y/\mathrm{d}x = 0$. Hence, $\mathrm{d}^2y/\mathrm{d}x^2 = -1 < 0$. Thus, the turning point is a maximum.

Assignment B5B

Problem 1. Sketch the curve with parametric equations

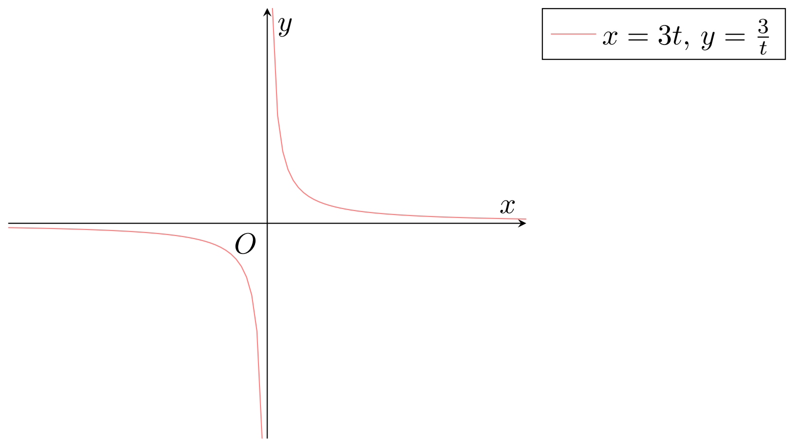
$$x = 3t,\ y = \frac{3}{t}.$$

The point P on the curve has parameter $t = 2$. The normal at P meets the curve again at the point Q.

(a) Show that the normal at P has equation $2y = 8x - 45$.

(b) Find the value of t at Q.

Solution.



Part (a). Consider $\mathrm{d}y/\mathrm{d}x$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-3/t^2}{3} = -\frac{1}{t^2}.$$

Hence, the tangent to the curve has gradient $-1/t^2$, whence the normal to the curve has gradient $\frac{-1}{-1/t^2} = t^2$. Thus, the normal to the curve at P has gradient $2^2 = 4$. Note that P has coordinates $(6, 3/2)$. Using the point-slope formula, the normal at P has equation

$$y - \frac{3}{2} = 4(x - 6) \implies 2y = 8(x - 6) + 3 = 8x - 45.$$

Part (b). Observe that

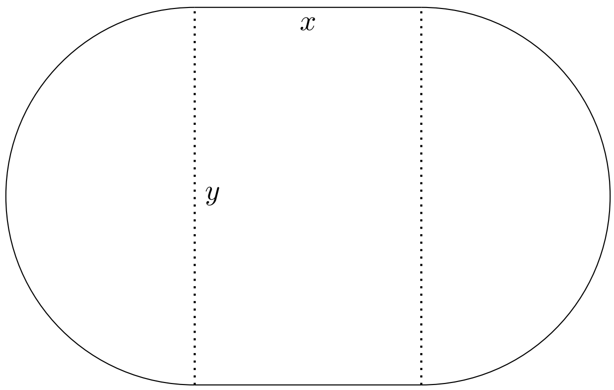
$$x = 3t \implies t = \frac{x}{3} \implies y = \frac{3}{x/3} = \frac{9}{x}.$$

Substituting $y = 9/x$ into the equation of the normal at P,

$$2\left(\frac{9}{x}\right) = 8x - 45 \implies 8x^2 - 45x - 18 = (x - 6)(8x + 3) = 0.$$

Hence, the x-coordinate of Q is $-3/8$ (note that we reject $x = 6$ since that corresponds to P). Thus, $t = -1/8$ at Q.

Problem 2.



A pond with a constant depth of 1 m is being designed for a park. The pond comprises a rectangle x m by y m and two semicircles of diameter y m, as shown in the diagram. The cost to build a boundary around the pond is \$30 per metre for straight parts and \$60 per metre for the semicircular parts. Given that the budget for the boundary is fixed at \$6000, using differentiation or otherwise, find in terms of π, the exact values of x and y which give the pond a maximum volume.

Solution. Observe that the total length of the straight parts is $2x$ m and the total length of the semicircular parts is πy m. Hence,

$$30(2x) + 60(\pi y) = 6000 \implies x + \pi y = 100 \implies x = 100 - \pi y.$$

Let $V(y)$ m^3 be the volume of the pond.

$$V(y) = \pi \left(\frac{y}{2}\right)^2 + xy = \frac{\pi}{4}y^2 + (100 - \pi y)\, y = -\frac{3\pi}{4}y^2 + 100y.$$

Consider the stationary points of $V(y)$. For stationary points, $V'(y) = 0$.

$$V'(y) = -\frac{3\pi}{2}y + 100 = 0 \implies y = \frac{200}{3\pi}.$$

y	$\left(\frac{200}{3\pi}\right)^-$	$\frac{200}{3\pi}$	$\left(\frac{200}{3\pi}\right)^+$
$V'(y)$	+ve	0	−ve

By the first derivative test, the maximum volume of the pond is achieved when $y = 200/3\pi$. Thus, $x = 100 - \pi y = 100/3$.

* * * * *

Problem 3. A circular cylinder is expanding in such a way that, at time t seconds, the length of the cylinder is $20x$ cm and the area of the cross-section is x cm^2. Given that, when $x = 5$, the area of the cross-section is increasing at a rate of 0.025 cm^{2}s^{-1}, find the rate of increase at this instant of

(a) the length of the cylinder,

(b) the volume of the cylinder,

(c) the radius of the cylinder.

Solution. Let $A = x$ cm^2 be the cross-sectional area of the cylinder. Then

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}$$

and

$$\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right|_{x=5} = 0.025.$$

Part (a). Let $L = 20x$ cm be the length of the cylinder. Then

$$\frac{\mathrm{d}L}{\mathrm{d}t} = 20 \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \implies \left.\frac{\mathrm{d}L}{\mathrm{d}t}\right|_{x=5} = 20\,(0.025) = 0.5.$$

Thus, the length of the cylinder is increasing at a rate of 0.5 cm/s.

Part (b). Let $V = AL = 20x^2$ cm^3 be the volume of the cylinder. Then

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 40x \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \implies \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{x=5} = 40(5)(0.025) = 5.$$

Thus, the volume of the cylinder is increasing at a rate of 5 cm^3/s.

Part (c). Let R cm be the radius of the cylinder. Observe that

$$\pi R^2 = A = x \implies R = \sqrt{\frac{x}{\pi}} = \frac{\sqrt{x}}{\sqrt{\pi}}.$$

Hence,

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{x}} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \implies \left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{x=5} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{5}}\right)(0.025) = 0.00315 \text{ (3 s.f.)}.$$

Thus, the radius of the cylinder is increasing at a rate of 0.00315 cm/s.

* * * * *

Problem 4. The curve C has equation $2^{-y} = x$. The point A on C has x-coordinate a where $a > 0$. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{a \ln 2}$ at A and find the equation of the tangent to C at A. Hence, find the equation of the tangent to C which passes through the origin.

The straight line $y = mx$ intersects C at 2 distinct points. Write down the range of values of m.

Solution. Observe that

$$2^{-y} = x \implies y = -\log_2 x = -\frac{\ln x}{\ln 2} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x \ln 2}.$$

At A, $x = a$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{a \ln 2}.$$

Also, we clearly have $A(a, -\ln a / \ln 2)$. Using the point-slope formula, the tangent to C at A has equation

$$y - \left(-\frac{\ln a}{\ln 2}\right) = -\frac{1}{a \ln 2}(x - a) \implies y = -\frac{x}{a \ln 2} + \frac{1 - \ln a}{\ln 2}.$$

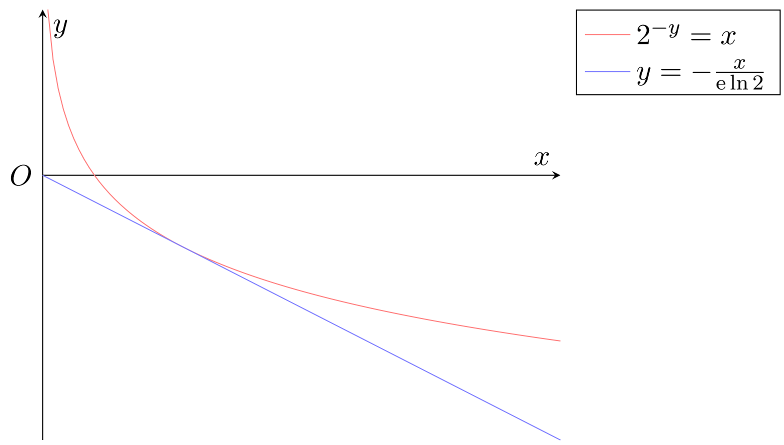
Consider the straight line $y = mx$ that is tangent to C and passes through the origin.

$$0 = -\frac{0}{a \ln 2} + \frac{1 - \ln a}{\ln 2} \implies 1 - \ln a = 0 \implies a = \mathrm{e}.$$

Hence, the equation of the tangent to C that passes through the origin is

$$y = -\frac{x}{\mathrm{e} \ln 2}.$$

Consider the graph of $2^{-y} = x$.



Hence, $m \in (-1/\mathrm{e}\ln 2, 0)$.

B6 MacLaurin Series

Tutorial B6

Problem 1.

(a) Given that $f(x) = \mathrm{e}^{\cos x}$, find $f(0)$, $f'(0)$ and $f''(0)$. Hence, write down the first two non-zero terms in the MacLaurin series for $f(x)$. Give the coefficients in terms of e.

(b) Given that $g(x) = \tan\left(2x + \frac{1}{4}\pi\right)$, find $g(0)$, $g'(0)$ and $g''(0)$. Hence, find the first three terms in the MacLaurin series of $g(x)$.

Solution.

Part (a). Note that

$$f'(x) = -\mathrm{e}^{\cos x} \sin x = -f(x) \sin x \implies f''(x) = -f(x) \cos x - f'(x) \sin x.$$

Evaluating $f(x)$, $f'(x)$ and $f''(x)$ at 0,

$$f(0) = \mathrm{e}, \quad f'(0) = 0, \quad f''(0) = -\mathrm{e}.$$

Hence,

$$f(x) = \frac{\mathrm{e}}{0!} + \frac{0}{1!}x + \frac{-\mathrm{e}}{2!}x^2 = \mathrm{e} - \frac{\mathrm{e}}{2}x^2 + \cdots .$$

Part (b). Note that

$$g'(x) = 2\sec^2\left(2x + \frac{\pi}{4}\right) = 2\left(1 + \tan^2\left(2x + \frac{\pi}{4}\right)\right) = 2 + 2g^2(x) \implies g''(x) = 4g(x)g'(x).$$

Evaluating $g(x)$, $g'(x)$ and $g''(x)$ at 0,

$$g(x) = 1, \quad g'(x) = 4, \quad g''(x) = 16.$$

Hence,

$$g(x) = \frac{1}{0!} + \frac{4}{1!}x + \frac{16}{2!}x^2 + \cdots = 1 + 4x + 8x^2 + \cdots .$$

* * * * *

Problem 2. Find the first three non-zero terms of the MacLaurin series for the following in ascending powers of x. In each case, find the range of values of x for which the series is valid.

(a) $\frac{(1+3x)^4}{\sqrt{1+2x}}$

(b) $\frac{\sin 2x}{2+3x}$

Solution.

Part (a). Observe that

$$(1+3x)^4 = 1 + 4(3x) + 6(3x)^2 + \cdots = 1 + 12x + 54x^2 + \cdots$$

and

$$(1+2x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(2x)^2 + \cdots = 1 - x + \frac{3}{2}x^2 + \cdots.$$

Thus,

$$y = \frac{(1+3x)^4}{\sqrt{1+2x}} = \left(1 + 12x + 54x^2 + \cdots\right)\left(1 - x + \frac{3}{2}x^2 + \cdots\right)$$
$$= \left(1 - x + \frac{3}{2}x^2\right) + \left(12x - 12x^2\right) + \left(54x^2\right) + \cdots = 1 + 11x + \frac{87}{2}x^2 + \cdots.$$

Note that the series is valid only when

$$|2x| < 1 \implies -\frac{1}{2} < x < \frac{1}{2}.$$

Part (b). Note that

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \cdots = 2x - \frac{4}{3}x^3 + \cdots$$

and

$$\frac{1}{2+3x} = \frac{1}{2}\left(1 + \frac{3x}{2}\right)^{-1} = \frac{1}{2}\left[1 - \frac{3x}{2} + \left(\frac{3x}{2}\right)^2 - \left(\frac{3x}{2}\right)^3 + \cdots\right]$$
$$= \frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \cdots.$$

Thus,

$$\frac{\sin 2x}{2+3x} = \left(2x - \frac{4}{3}x^3 + \cdots\right)\left(\frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \cdots\right)$$
$$= \left(x - \frac{3}{2}x^2 + \frac{9}{4}x^3\right) + \left(-\frac{2}{3}x^3\right) + \cdots = x - \frac{3}{2}x^2 + \frac{19}{12}x^3 + \cdots.$$

The series is only valid when

$$\left|\frac{3}{2}x\right| < 1 \implies -\frac{2}{3} < x < \frac{2}{3}.$$

* * * * *

Problem 3. Find the MacLaurin series of $\ln(1+\cos x)$, up to and including the term in x^4.

Solution. Let $y = \ln(1+\cos x)$. Then

$$y = \ln(1+\cos x) \implies \mathrm{e}^y = 1 + \cos x.$$

Implicitly differentiating repeatedly with respect to x,

$$e^y y' = -\sin x \implies e^y\left[\left(y'\right)^2 + y''\right] = -\cos x \implies e^y\left[\left(y'\right)^3 + 3y'y'' + y'''\right] = \sin x$$
$$\implies e^y\left[\left(y'\right)^4 + 3\left(y''\right)^2 + 6\left(y'\right)^2 y'' + 4y'y''' + y^{(4)}\right] = \cos x.$$

Evaluating the above at $x = 0$, we get

$$y(0) = \ln 2, \quad y'(0) = 0, \quad y''(0) = -\frac{1}{2}, \quad y'''(0) = 0, \quad y^{(4)}(0) = -\frac{1}{4}.$$

Thus,

$$\ln(1 + \cos x) = \ln 2 + \frac{-1/2}{2!}x^2 + \frac{-1/4}{4!}x^4 + \cdots = \ln 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4 + \cdots .$$

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Problem 4.

(a) Find the first three terms of the MacLaurin series for $e^x(1 + \sin 2x)$.

(b) It is given that the first two terms of this series are equal to the first two terms in the series expansion, in ascending powers of x, of $\left(1 + \frac{4}{3}x\right)^n$. Find n and show that the third terms in each of these series are equal.

Solution.

Part (a). Observe that

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

and

$$1 + \sin 2x = 1 + 2x + \cdots .$$

Hence,

$$e^x(1 + \sin 2x) = \left(1 + x + \frac{x^2}{2} + \cdots\right)(1 + 2x + \cdots)$$
$$= (1 + 2x) + \left(x + 2x^2\right) + \left(\frac{x^2}{2}\right) + \cdots = 1 + 3x + \frac{5}{2}x^2 + \cdots .$$

Part (b). Note that

$$\left(1 + \frac{4}{3}x\right)^n = 1 + n\left(\frac{4}{3}x\right) + \frac{n(n-1)}{2}\left(\frac{4}{3}x\right)^2 + \cdots = 1 + \frac{4n}{3}x + \frac{8n(n-1)}{9}x^2 \cdots .$$

Comparing the second terms of both series, we get

$$\frac{4n}{3} = 3 \implies n = \frac{9}{4}.$$

Thus, the third term of $(1 + \frac{4}{3}x)^n$ is

$$\frac{8(\frac{9}{4})(\frac{9}{4} - 1)}{9}x^2 = \frac{5}{2}x^2.$$

Hence, the third terms in each of these series are equal.

Problem 5.

(a) Show that the first three non-zero terms in the expansion of $\left(\frac{8}{x^3}-1\right)^{1/3}$ in ascending powers of x are in the form $\frac{a}{x}+bx^2+cx^5$, where a, b and c are constants to be determined.

(b) By putting $x=\frac{2}{3}$ in your result, obtain an approximation for $\sqrt[3]{26}$ in the form of a fraction in its lowest terms.

A student put $x=6$ into the expansion to obtain an approximation of $\sqrt[3]{26}$. Comment on the suitability of this choice of x for the approximation of $\sqrt[3]{26}$.

Solution.

Part (a). We have

$$\left(\frac{8}{x^3}-1\right)^{\frac{1}{3}}=\frac{2}{x}\left(1-\frac{x^3}{8}\right)^{\frac{1}{3}}=\frac{2}{x}\left[1+\frac{1}{3}\left(-\frac{x^3}{8}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2}\left(-\frac{x^3}{8}\right)^2+\cdots\right]$$
$$=\frac{2}{x}\left(1-\frac{x^3}{24}-\frac{x^6}{576}+\cdots\right)=\frac{2}{x}-\frac{x^2}{12}-\frac{x^5}{288}+\cdots.$$

Part (b). Evaluating the above equation at $x=2/3$,

$$\sqrt[3]{26}\approx\left(\frac{8}{(2/3)^3}-1\right)^{1/3}=\frac{2}{2/3}-\frac{(2/3)^2}{12}-\frac{(2/3)^5}{288}=\frac{6479}{2187}.$$

Observe that the validity range for the series is

$$\left|-\frac{x^3}{8}\right|<1\implies -2<x<2.$$

Since 6 is outside this range, it is not an appropriate choice.

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Problem 6. Let $f(x)=\mathrm{e}^x\sin x$.

(a) Sketch the graph of $y=f(x)$ for $-3\le x\le 3$.

(b) Find the series expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3.

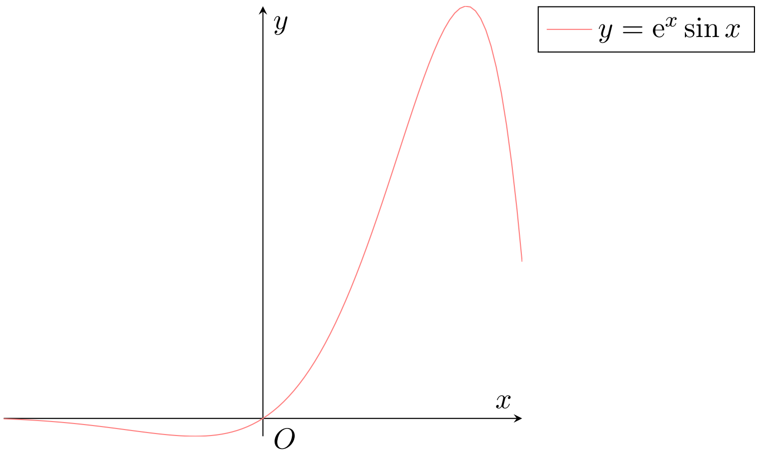
Denote the answer to part (b) by $g(x)$.

(c) On the same diagram, sketch the graph of $y=f(x)$ and $y=g(x)$. Label the two graphs clearly.

(d) Find, for $-3\le x\le 3$, the set of values of x for which the value of $g(x)$ is within ± 0.5 of the value of $f(x)$.

Solution.

Part (a).



Part (b). Observe that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

and

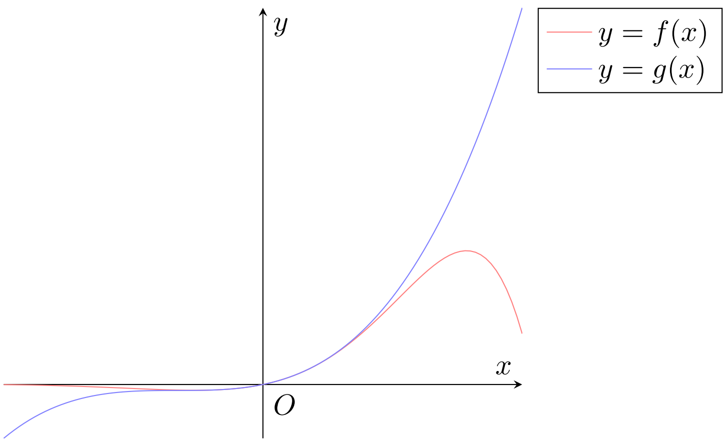
$$\sin x = x - \frac{x^3}{6} + \cdots .$$

Thus,

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{6} + \cdots\right)$$

$$= \left(x - \frac{x^3}{6}\right) + \left(x^2\right) + \left(\frac{x^3}{2}\right) + \cdots = x + x^2 + \frac{x^3}{3} + \cdots .$$

Part (c).



Part (d). Using G.C., $\{x \in \mathbb{R} : -1.96 \leq x \leq 1.56\}$.

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Problem 7. It is given that $y = 1/(1 + \sin 2x)$. Show that, when $x = 0$, $\mathrm{d}^2y/\mathrm{d}x^2 = 8$. Find the first three terms of the MacLaurin series for y.

(a) Use the series to obtain an approximate value for $\int_{-0.1}^{0.1} y \, \mathrm{d}x$, leaving your answer as a fraction in its lowest terms.

(b) Find the first two terms of the MacLaurin series for $\mathrm{d}y/\mathrm{d}x$.

(c) Write down the equation of the tangent at the point where $x = 0$ on the curve $y = 1/(1 + \sin 2x)$.

Solution. Differentiating with respect to x, we get

$$y' = -\frac{2\cos 2x}{(1+\sin 2x)^2} = -2y^2 \cos 2x.$$

Differentiating once more, we get

$$y'' = -2\left(-2y^2 \sin 2x + 2y \cdot y' \cos 2x\right).$$

Evaluating the above at $x = 0$, we obtain

$$y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 8.$$

Hence,

$$\frac{1}{1+\sin 2x} = \frac{1}{0!} + \frac{-2}{1!}x + \frac{8}{2!}x^2 + \cdots = 1 - 2x + 4x^2 + \cdots.$$

Part (a).

$$\int_{-0.1}^{0.1} y \,\mathrm{d}x \approx \int_{-0.1}^{0.1} \left(1 - 2x + 4x^2\right) \mathrm{d}x = \left[x - x^2 + \frac{4}{3}x^3\right]_{-0.1}^{0.1} = \frac{76}{275}.$$

Part (b).

$$y' = \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - 2x + 4x^2 + \cdots\right) = -2 + 8x + \cdots.$$

Part (c). Using the point-slope formula,

$$y - 1 = -2(x - 0) \implies y = -2x + 1.$$

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Problem 8. It is given that $y = \mathrm{e}^{\arcsin 2x}$.

(a) Show that $(1 - 4x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 4y$.

(b) By further differentiating this result, find the MacLaurin series for y in ascending powers of x, up to an including the term in x^3.

(c) Hence, find an approximation value of $\mathrm{e}^{\pi/2}$, by substituting a suitable value of x in the MacLaurin series for y.

(d) Suggest one way to improve the accuracy of the approximated value obtained.

Solution.

Part (a). Note that

$$y = \mathrm{e}^{\arcsin(2x)} \implies \ln y = \arcsin(2x).$$

Implicitly differentiating with respect to x,

$$\frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\sqrt{1-4x^2}} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{\sqrt{1-4x^2}}.$$

Implicitly differentiating with respect to x once again,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\sqrt{1-4x^2}\left(2\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 2y\left(\frac{-4x}{\sqrt{1-4x^2}}\right)}{1-4x^2}.$$

Now observe that

$$2\sqrt{1-4x^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\left(\frac{2y}{\sqrt{1-4x^2}}\right) = 4y + 4x\cdot\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence,

$$\left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4y + 4x\cdot\frac{\mathrm{d}y}{\mathrm{d}x} \implies \left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 4y.$$

Part (b). Implicitly differentiating with respect to x once again,

$$\left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 8x\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\left(x\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 4\cdot\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Rearranging,

$$\left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 12x\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 8\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Evaluating the above equations at $x = 0$, we get

$$y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 4, \quad y'''(0) = 16.$$

Hence,

$$y = \frac{1}{0!} + \frac{2}{1!}x + \frac{4}{2!}x^2 + \frac{16}{3!}x^3 + \cdots = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \cdots.$$

Part (c). Consider $y = \mathrm{e}^{\pi/2}$.

$$y = \arcsin 2x = \mathrm{e}^{\pi/2} \implies x = \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}.$$

Substituting $x = 1/2$ into the MacLaurin series for y,

$$\mathrm{e}^{\pi/2} \approx 1 + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2 + \frac{8}{3}\left(\frac{1}{2}\right)^3 = \frac{17}{6}.$$

Part (d). More terms of the MacLaurin series of y could be considered.

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Problem 9. The curve $y = f(x)$ passes through the point $(0, 1)$ and satisfies the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6-2y}{\cos 2x}$.

(a) Find the MacLaurin series of $f(x)$, up to and including the term in x^3.

(b) Using standard results given in the List of Formulae (MF27), express $\frac{1-\sin x}{\cos x}$ as a power series of x, up to and including the term in x^3.

(c) Using the two power series you have found, show to this degree of approximation, that $f(x)$ can be expressed as $a(\tan 2x - \sec 2x) + b$, where a and b are constants to be determined.

Solution.

Part (a). Note that

$$y' = \frac{6-2y}{\cos 2x} \implies y' \cos 2x = 6 - 2y.$$

Implicitly differentiating with respect to x,

$$-2y' \sin 2x + y'' \cos 2x = -2y'.$$

Implicitly differentiating once more,

$$-2\left(y'' \sin 2x + 2y' \cos 2x\right) + \left(y''' \cos 2x - 2y'' \sin 2x\right) = -2y''$$

Hence,

$$y(0) = 1, \quad y'(0) = 4, \quad y''(0) = -8, \quad y'''(0) = 32,$$

whence

$$f(x) = \frac{1}{0!}x + \frac{4}{1!}x + \frac{-8}{2!}x^2 + \frac{32}{3!}x^3 + \cdots = 1 + 4x - 4x^2 + \frac{16}{3}x^3 + \cdots.$$

Part (b). Note that

$$\frac{1}{\cos x} \approx \left(1 - \frac{x^2}{2}\right)^{-1} \approx 1 + \frac{x^2}{2}.$$

Hence,

$$\frac{1-\sin x}{\cos x} \approx \left(1 - x + \frac{x^3}{6}\right)\left(1 + \frac{x^2}{2}\right) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots.$$

Part (c). Note that

$$\frac{1-\sin x}{\cos x} = \sec x - \tan x.$$

Hence,

$$a(\tan 2x - \sec 2x) + b \approx -a\left[1 - 2x + \frac{(2x)^2}{2} - \frac{(2x)^3}{3}\right] + b$$

$$= a\left(-1 + 2x - 2x^2 + \frac{8}{3}x^3\right) + b = a\left(-\frac{3}{2} + \frac{f(x)}{2}\right) + b = -\frac{3}{2}a + b + \frac{a}{2}f(x).$$

Thus,

$$\frac{a}{2}f(x) - \frac{3}{2}a + b \approx a(\tan 2x - \sec 2x) + b.$$

In order to obtain an approximation for $f(x)$, we need $\frac{a}{2} = 1$ and $-\frac{3}{2}a + b = 0$, whence $a = 2$ and $b = 3$.

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Problem 10. Given that x is sufficiently small for x^3 and higher powers of x to be neglected, and that $13 - 59\sin x = 10(2 - \cos 2x)$, find a quadratic equation for x and hence solve for x.

Solution. Note that

$$13 - 59\sin x = 10\,(2 - \cos 2x) = 10\left[2 - \left(1 - 2\sin^2 x\right)\right] = 10 + 20\sin^2 x.$$

Thus,

$$20\sin^2 x + 59\sin x - 3 = (20\sin x - 1)(\sin x + 3) = 0,$$

whence $\sin x = 1/20$. Note that we reject $\sin x = -3$ since $|\sin x| \leq 1$. Since x is sufficiently small for x^3 and higher powers of x to be neglected, $\sin x \approx x$. Thus, $x \approx 1/20$.

Problem 11. In triangle ABC, angle $A = \pi/3$ radians, angle $B = (\pi/3 + x)$ radians and angle $C = (\pi/3 - x)$ radians, where x is small. The lengths of the sides BC, CA and AB are denoted by a, b and c respectively. Show that $b - c \approx 2ax/\sqrt{3}$.

Solution. By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Hence,

$$b = a\left(\frac{\sin B}{\sin A}\right) = \frac{2a}{\sqrt{3}}\sin B, \quad c = a\left(\frac{\sin C}{\sin A}\right) = \frac{2a}{\sqrt{3}}\sin C.$$

Thus,

$$\begin{aligned} b - c = \frac{2a}{\sqrt{3}}(\sin B - \sin C) &= \frac{2a}{\sqrt{3}}\left[\sin\left(\frac{\pi}{3} + x\right) - \sin\left(\frac{\pi}{3} - x\right)\right] \\ &= \frac{2a}{\sqrt{3}}\left[2\sin x \cos\frac{\pi}{3}\right] = \frac{2a}{\sqrt{3}}\sin x. \end{aligned}$$

Since x is small, $\sin x \approx x$. Hence,

$$b - c \approx \frac{2ax}{\sqrt{3}}.$$

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Problem 12. D'Alembert's ratio test states that a series of the form $\sum_{r=0}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, and diverges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$. When $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=0}^{\infty}\frac{x^r}{r!}$ converges for all real values of x and state the sum to infinity of this series, in terms of x.

Solution. Let $a_n = \frac{x^n}{n!}$ and consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{n}\right|$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\Big/\frac{x^n}{n!}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0.$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{n}\right| < 1$ for all $x \in \mathbb{R}$, it follows by D'Alembert's ratio test that $\sum_{r=0}^{\infty}\frac{x^r}{r!}$ converges for all real values of x. The sum to infinity of the series in question is e^x.

Self-Practice B6

Problem 1. Express $\frac{6x+4}{(1-2x)(1+3x^2)}$ in partial fractions. Hence, find the coefficients of x^5 and x^6 in the expansion, in ascending powers of x, of $\frac{6x+4}{(1-2x)(1+3x^2)}$.

Solution. Let

$$\frac{6x+4}{(1-2x)(1+3x^2)} = \frac{A}{1-2x} + \frac{Bx+C}{1+3x^2},$$

where A, B and C are constants to be determined. Using the cover-up rule, we immediately get

$$A = \frac{6(1/2)+4}{1+3(1/2)^2} = 4.$$

Clearing denominators, we get

$$6x+4 = 4\left(1+3x^2\right) + (Bx+C)(1-2x) = (12-2B)\,x^2\,(B-2C)\,x + (4+C)\,.$$

Comparing coefficients, we have $B = 6$ and $C = 0$. Hence,

$$\frac{6x+4}{(1-2x)(1+3x^2)} = \frac{4}{1-2x} + \frac{6x}{1+3x^2}.$$

Note that

$$\frac{4}{1-2x} = \cdots + (2x)^5 + (2x)^6 + \cdots = 4\left[\cdots + 128x^5 + 256x^6 + \ldots\right]$$

and

$$\frac{6x}{1+3x^2} = 6x\left[\cdots + \left(-3x^2\right)^2 + \ldots\right] = \cdots + 54x^5 + \ldots.$$

Hence,

$$\frac{6x+4}{(1-2x)(1+3x^2)} = \cdots + 182x^5 + 256x^6 + \ldots.$$

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Problem 2. If x is so small that terms in x^n, $n \geq 3$, can be neglected and $\frac{3+ax}{3+bx} = (1-x)^{1/3}$, find the values of a and b. Hence, find an approximation for $\sqrt[3]{0.96}$ in the form $\frac{p}{q}$, where p and q are integers.

Solution. Rearranging,

$$3+ax = (3+bx)(1-x)^{1/3} = (3+bx)\left(1 - \frac{x}{3} - \frac{x^2}{9}\right) = 3 + (b-1)x - \frac{b+1}{3}x^2.$$

Comparing coefficients, we have $a = -2$ and $b = -1$. Thus,

$$\frac{3-2x}{3-x} = (1-x)^{1/3}.$$

Substituting $x = 0.04$, we get

$$\sqrt[3]{0.96} = \frac{3-2(0.04)}{3-0.04} = \frac{73}{74}.$$

* * * * *

Problem 3. Given that $y = \tan\left(\frac{1}{2}\arctan x\right)$, show that

$$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(1+y^2\right).$$

By differentiating this result twice, show that, up to and including the term in x^3, the Maclaurin series for $\tan\left(\frac{1}{2}\arctan x\right)$ is $\frac{1}{2}x - \frac{1}{8}x^3$.

Solution. Note that $\arctan y = \frac{1}{2}\arctan x$. Differentiating with respect to x,

$$\frac{y'}{1+y^2} = \frac{1}{2}\cdot\frac{1}{1+x^2} \implies \left(1+x^2\right)y' = \frac{1}{2}\left(1+y^2\right).$$

Differentiating with respect to x,

$$\left(1+x^2\right)y'' + 2xy' = yy'.$$

Differentiating once more,

$$\left(1+x^2\right)y''' + 4xy'' + 2y' = y\cdot y'' + \left(y'\right)^2.$$

When $x = 0$, we get

$$y(0) = 0, \quad y'(0) = \frac{1}{2}, \quad y''(0) = 0, \quad y'''(0) = -\frac{3}{4}.$$

Thus,

$$y = \tan\left(\frac{1}{2}\arctan x\right) = \frac{1}{2}x + \frac{-3/4}{3!}x^3 = \frac{1}{2}x - \frac{1}{8}x^3.$$

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Problem 4. Given that $\cos y = \sqrt{1-\frac{1}{4}\mathrm{e}^x}$ and $0 < y < \frac{\pi}{2}$, show that $\sin(2y)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4}\mathrm{e}^x$. By further differentiation of this result, find the Maclaurin series for y, up to and including the term in x^2, leaving your answer in exact form. Deduce the equation of the tangent to the curve $y = \arccos\sqrt{1-\frac{1}{4}\mathrm{e}^x}$ at $x = 0$.

Solution. Rearranging, we get

$$\cos^2 y = 1 - \frac{1}{4}\mathrm{e}^x.$$

Differentiating with respect to x,

$$-2\cos y\sin y\cdot y' = -\frac{1}{4}\mathrm{e}^x \implies \sin(2y)y' = \frac{1}{4}\mathrm{e}^x.$$

Differentiating once more,

$$\sin(2y)y'' + 2\cos(2y)\left(y'\right)^2 = \frac{1}{4}\mathrm{e}^x.$$

When $x = 0$, we get

$$y(0) = \frac{\pi}{6}, \quad y'(0) = \frac{1}{2\sqrt{3}}, \quad y''(0) = \frac{1}{3\sqrt{3}}.$$

Thus,

$$y = \arccos\sqrt{1-\frac{1}{4}\mathrm{e}^x} = \frac{\pi}{6} + \left(\frac{1}{2\sqrt{3}}\right)x + \left(\frac{1}{3\sqrt{3}}\right)\left(\frac{x^2}{2}\right) + \cdots = \frac{\pi}{6} + \frac{x}{2\sqrt{3}} + \frac{x^2}{6\sqrt{3}} + \cdots.$$

The equation of the tangent at $x = 0$ is simply

$$y = \frac{\pi}{6} + \frac{x}{2\sqrt{3}}.$$

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Problem 5. By expressing $\sin\left(\frac{\pi}{3}+2x\right)$ in terms of $\sin 2x$ and $\cos 2x$, show that

$$\sin\left(\frac{\pi}{3}+2x\right) \approx \frac{\sqrt{3}}{2} + x - \sqrt{3}x^2$$

if x is sufficiently small. Hence, by using a suitable value of x, estimate the value of $\sin\frac{\pi}{9}$, giving your answer to 3 significant figures.

Solution. By the angle-sum formula,

$$\sin\left(\frac{\pi}{3}+2x\right)=\sin\frac{\pi}{3}\cos 2x+\cos\frac{\pi}{3}\sin 2x=\frac{\sqrt{3}}{2}\cos 2x+\frac{1}{2}\sin 2x.$$

For sufficiently small x, we have $\sin x \approx x$ and $\cos x = 1 - x^2/2$. Hence,

$$\sin\left(\frac{\pi}{3}+2x\right)\approx\frac{\sqrt{3}}{2}\left(1-\frac{(2x)^2}{2}\right)+\frac{1}{2}(2x)=\frac{\sqrt{3}}{2}+x-\sqrt{3}x^2.$$

Consider $\pi/3+2x=\pi/9$. Clearly $x=-\pi/9$. Substituting this into the above approximation, we get

$$\sin\frac{\pi}{9}\approx\frac{\sqrt{3}}{2}+\left(-\frac{\pi}{9}\right)-\sqrt{3}\left(-\frac{\pi}{9}\right)^2=0.306 \text{ (3 s.f.)}.$$

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Problem 6 (🌶). Consider the infinite series $\frac{1}{1!}+\frac{4}{2!}+\frac{7}{3!}+\frac{10}{4!}+\ldots$.

(a) If the series continues with the same pattern, find an expression for the nth term.

(b) By rewriting the infinite series in terms of sigma notation and using the standard series for e^x, show that the series evaluates to $\mathrm{e}+2$.

Solution.
Part (a). The nth term is given by $\frac{3n-2}{n!}$, where $n \geq 1$.
Part (b). The infinite series is given by

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3n-2}{n!}&=3\sum_{n=1}^{\infty}\frac{n}{n!}-2\sum_{n=1}^{\infty}\frac{1}{n!}\\&=3\sum_{n=1}^{\infty}\frac{1}{(n-1)!}-2\left(1+\sum_{n=1}^{\infty}\frac{1}{n!}\right)+2\\&=3\sum_{n=0}^{\infty}\frac{1}{n!}-2\sum_{n=0}^{\infty}\frac{1}{n!}+2\\&=3\mathrm{e}-2\mathrm{e}+2=\mathrm{e}+2.\end{aligned}$$

Problem 7 (🌶). Find the function represented by each of the following series by expressing it as a sum or difference of two standard series.

(a) $f(x) = 2 + x + \frac{x^3}{3!} + \frac{2x^4}{4!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{2x^8}{8!} + \dots$, $x \in \mathbb{R}$.

(b) $g(x) = (a+b)x - \frac{a^2+b^2}{2}x^2 + \frac{a^3+b^3}{3}x^3 - \frac{a^4+b^4}{4}x^4 + \dots$, where a and b are positive constants such that $-\frac{1}{a} < x \leq \frac{1}{a}$ and $-\frac{1}{b} < x \leq \frac{1}{b}$.

Solution.

Part (a). Observe that $f(x)$ is defined for all $x \in \mathbb{R}$. This suggests that $f(x)$ is composed of e^x, $\cos x$ and $\sin x$. Also observe that the powers of 2, 6, 10, ... are missing. This suggests that we are adding $\cos x$ to e^x:

$$\begin{aligned} f(x) &= 2 + x + \frac{x^3}{3!} + \frac{2x^4}{4!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{2x^8}{8!} + \dots \\ &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \dots\right) \\ &+ \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots\right) \\ &= \mathrm{e}^x + \cos x. \end{aligned}$$

Part (b). We can easily separate $g(x)$ as follows:

$$\begin{aligned} g(x) &= (a+b)x - \frac{a^2+b^2}{2}x^2 + \frac{a^3+b^3}{3}x^3 - \frac{a^4+b^4}{4}x^4 + \dots \\ &= \left(ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \frac{(ax)^4}{4} + \dots\right) + \left(bx - \frac{(bx)^2}{2} + \frac{(bx)^3}{3} - \frac{(bx)^4}{4} + \dots\right) \\ &= \ln(1+ax) + \ln(1+bx). \end{aligned}$$

Assignment B6

Problem 1. Expand $(1+2x)^{-\frac{1}{3}}$, where $|x|<\frac{1}{2}$, as a series of ascending powers of x, up to an including the term in x^2, simplifying the coefficients.

By choosing $x=\frac{1}{14}$, find an approximate value of $\sqrt[3]{7}$ in the form $\frac{p}{q}$, where p and q are to be determined.

Using your calculator, calculate the numerical value of $\sqrt[3]{7}$. Compare this value to the approximate value found, and with reference to the value of x chosen, comment on the accuracy of your approximation.

Solution.

$$(1+2x)^{-1/3}=1-\frac{1}{3}(2x)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2}(2x)^2+\cdots=1-\frac{2}{3}x+\frac{8}{9}x^2+\cdots.$$

Substituting $x=1/14$,

$$\left[1+2\left(\frac{1}{14}\right)\right]^{-1/3}=\frac{\sqrt[3]{7}}{2}\approx 1-\frac{2}{3}\left(\frac{1}{14}\right)+\frac{8}{9}\left(\frac{1}{14}\right)^2=\frac{422}{441}$$
$$\implies \sqrt[3]{7}\approx\frac{844}{441}=1.9138 \text{ (5 s.f.)}.$$

Since $\sqrt[3]{7}=1.9129$ (5 s.f.), the approximation is accurate.

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Problem 2. In the triangle ABC, $AB=1$, $BC=3$ and angle $ABC=\theta$ radians. Given that θ is a sufficiently small angle, show that

$$AC\approx(4+3\theta^2)^{\frac{1}{2}}\approx a+b\theta^2$$

for constants a and b to be determined.

Solution. By the cosine rule,

$$AC^2=AB^2+BC^2-2(AB)(BC)\cos ABC=1^2+3^2-2(1)(3)\cos\theta=10-6\cos\theta.$$

Since θ is sufficiently small, $\cos\theta\approx 1-\theta^2/2$. Hence,

$$AC^2\approx 10-6\left(1-\frac{\theta^2}{2}\right)=4+3\theta^2$$
$$\implies AC=\left(4+3\theta^2\right)^{1/2}=2\left(1+\frac{3\theta^2}{4}\right)^{1/2}\approx 2\left[1+\frac{1}{2}\left(\frac{3\theta^2}{4}\right)\right]=2+\frac{3\theta^2}{4}.$$

Hence, $a=2$ and $b=\frac{3}{4}$.

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Problem 3. Given that $y=\ln\sec x$, show that

(a) $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=2\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x}$

(b) the value of $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$ when $x=0$ is 2.

Write down the MacLaurin series for $\ln\sec x$ up to and including the term in x^4. By substituting $x=\frac{\pi}{4}$, show that $\ln 2\approx\frac{\pi^2}{16}+\frac{\pi^4}{1536}$.

Solution.

Part (a). Note that

$$y = \ln \sec x = -\ln \cos x \implies \mathrm{e}^{-y} = \cos x.$$

Implicitly differentiating with respect to x,

$$-y'\mathrm{e}^{-y} = -\sin x \implies y' = \tan x.$$

Differentiating repeatedly,

$$y'' = \sec^2 x \implies y''' = 2\sec^2 x \tan x.$$

Thus,

$$y''' = 2\sec^2 x \tan x = 2y'' \cdot y'.$$

Part (b). Implicitly differentiating the above differential equation,

$$y^{(4)} = 2\left[y''' \cdot y' + \left(y''\right)^2\right].$$

Evaluating the above equations at $x = 0$, we see that

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, \quad y^{(3)}(0) = 0, \quad y^{(4)}(0) = 2.$$

We have

$$\ln \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \cdots.$$

Substituting $x = \pi/4$,

$$\ln \sec \frac{\pi}{4} = \frac{1}{2}\ln 2 \approx \frac{1}{2}\left(\frac{\pi}{4}\right)^2 + \frac{1}{12}\left(\frac{\pi}{4}\right)^4 = \frac{\pi^2}{32} + \frac{\pi^4}{3072} \implies \ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536}.$$

B7 Integration Techniques

Tutorial B7

Problem 1. Find

(a) $\int \frac{1}{\sqrt{3-2x}}\,\mathrm{d}x$

(b) $\int \frac{1}{3-2x}\,\mathrm{d}x$

(c) $\int \frac{1}{3-2x^2}\,\mathrm{d}x$

(d) $\int \frac{1}{\sqrt{3-2x^2}}\,\mathrm{d}x$

(e) $\int \frac{x}{\sqrt{3-2x^2}}\,\mathrm{d}x$

(f) $\int \frac{1}{3+4x+2x^2}\,\mathrm{d}x$

Solution.

Part (a). Consider the substitution $u = 3 - 2x$.

$$\int \frac{1}{\sqrt{3-2x}}\,\mathrm{d}x = -\int \frac{1}{2\sqrt{u}}\,\mathrm{d}u = -\sqrt{u} + C = -\sqrt{3-2x} + C.$$

Part (b). Consider the substitution $u = 3 - 2x$.

$$\int \frac{1}{3-2x}\,\mathrm{d}x = -\frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|3-2x| + C.$$

Part (c).

$$\int \frac{1}{3-2x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{3/2-x^2}\,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{2\sqrt{3/2}}\right)\ln\left(\frac{\sqrt{3/2}+x}{\sqrt{3/2}-x}\right) + C$$
$$= \frac{1}{2\sqrt{6}}\ln\left(\frac{\sqrt{3}+\sqrt{2}x}{\sqrt{3}-\sqrt{2}x}\right) + C.$$

Part (d).

$$\int \frac{1}{\sqrt{3-2x^2}}\,\mathrm{d}x = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{3/2-x^2}}\,\mathrm{d}x = \frac{1}{\sqrt{2}}\arcsin\left(\frac{x}{\sqrt{3/2}}\right) + C$$
$$= \frac{\sqrt{2}}{2}\arcsin\left(\frac{\sqrt{6}x}{3}\right) + C.$$

Part (e). Consider the substitution $u = 3 - 2x^2$.

$$\int \frac{x}{\sqrt{3-2x^2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{1}{2\sqrt{u}}\,\mathrm{d}u = -\frac{\sqrt{u}}{2} + C = -\frac{\sqrt{3-2x^2}}{2} + C.$$

Part (f).

$$\int \frac{1}{3+4x+2x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{(x+1)^2+1/2}\,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{\sqrt{1/2}}\right)\arctan\left(\frac{x+1}{1/\sqrt{1/2}}\right)+C$$
$$= \frac{\arctan\left(\sqrt{2}(x+1)\right)}{\sqrt{2}}+C.$$

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Problem 2. Find

(a) $\int \frac{\sec^2 3x}{\tan 3x}\,\mathrm{d}x$

(b) $\int \cos(3x+\alpha)\,\mathrm{d}x$, where α is a constant

(c) $\int \cos^2 3x\,\mathrm{d}x$

(d) $\int \mathrm{e}^{1-2x}\,\mathrm{d}x$

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan 3x = 3\sec^2 3x \implies \int \frac{\sec^2 3x}{\tan 3x}\,\mathrm{d}x = \frac{1}{3}\int \frac{3\sec^2 3x}{\tan 3x}\,\mathrm{d}x = \frac{\ln\tan 3x}{3}+C.$$

Part (b).

$$\int \cos(3x+\alpha)\,\mathrm{d}x = \frac{\sin(3x+\alpha)}{3}+C$$

Part (c). Recall that

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \implies \cos^2 3x = \frac{1+\cos 6x}{2}.$$

Thus,

$$\int \cos^2 3x\,\mathrm{d}x = \frac{1}{2}\int (1+\cos 6x)\,\mathrm{d}x = \frac{1}{2}\left(x+\frac{\sin 6x}{6}\right)+C = \frac{x}{2}+\frac{\sin 6x}{12}+C.$$

Part (d).

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{1-2x} = -2\mathrm{e}^{1-2x} \implies \int \mathrm{e}^{1-2x}\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{1-2x}+C.$$

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Problem 3. Find

(a) $\int 2x\sqrt{3x^2-5}\,\mathrm{d}x$

(b) $\int \frac{x^2-1}{\sqrt{x^3-3x}}\,\mathrm{d}x$

(c) $\int \sin x\sqrt{\cos x}\,\mathrm{d}x$

(d) $\int \mathrm{e}^{2x}\left(1-\mathrm{e}^{2x}\right)^4\,\mathrm{d}x$

Solution.

Part (a). Consider the substitution $u = 3x^2 - 5$.

$$\int 2x\sqrt{3x^2-5}\,\mathrm{d}x = \frac{1}{3}\int \sqrt{u}\,\mathrm{d}u = \frac{1}{3}\left(\frac{2}{3}u^{3/2}\right) + C = \frac{2}{9}\left(3x^2-5\right)^{3/2} + C.$$

Part (b). Consider the substitution $u = x^3 - 3x$:

$$\int \frac{x^2-1}{\sqrt{x^3-3x}}\,\mathrm{d}x = \frac{2}{3}\int \frac{\mathrm{d}u}{2\sqrt{u}} = \frac{2}{3}\sqrt{u} + C = \frac{2}{3}\sqrt{x^3-3x} + C.$$

Part (c). Consider the substitution $u = \cos x$.

$$\int \sin x\sqrt{\cos x}\,\mathrm{d}x = -\int \sqrt{u}\,\mathrm{d}u = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}\cos^{3/2}x + C.$$

Part (d). Consider the substitution $u = 1 - \mathrm{e}^{2x}$.

$$\int \mathrm{e}^{2x}(1-\mathrm{e}^{2x})^4\,\mathrm{d}x = -\frac{1}{2}\int u^4\,\mathrm{d}u = -\frac{1}{2}\left(\frac{u^5}{5}\right) + C = -\frac{\left(1-\mathrm{e}^{2x}\right)^5}{10} + C.$$

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Problem 4. Find

(a) $\int \frac{1}{\sqrt{x}(1-\sqrt{x})}\,\mathrm{d}x$

(b) $\int \frac{3x}{x+3}\,\mathrm{d}x$

(c) $\int \frac{\sin x+\cos x}{\sin x-\cos x}\,\mathrm{d}x$

Solution.

Part (a). Consider the substitution $u = 1 - \sqrt{x}$.

$$\int \frac{1}{\sqrt{x}(1-\sqrt{x})}\,\mathrm{d}x = -2\int \frac{1}{u}\,\mathrm{d}u = -2\ln|u| + C = -2\ln\left|1-\sqrt{x}\right| + C.$$

Part (b).

$$\int \frac{3x}{x+3}\,\mathrm{d}x = \int \left(3 - \frac{9}{x+3}\right)\mathrm{d}x = 3x - 9\ln|x+3| + C.$$

Part (c). Consider the substitution $u = \sin x - \cos x$.

$$\int \frac{\sin x+\cos x}{\sin x-\cos x}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C = \ln|\sin x - \cos x| + C.$$

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Problem 5. Find

(a) $\int \frac{\mathrm{e}^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$

(b) $\int (\sin x)(\cos x)(\mathrm{e}^{\cos 2x})\,\mathrm{d}x$

Solution.

Part (a). Consider the substitution $u = -\sqrt{x}$.

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = -2\int e^u\,du = -2e^u + C = -2e^{-\sqrt{x}} + C.$$

Part (b). Consider the substitution $u = \cos 2x$.

$$\int (\sin x)(\cos x)(e^{\cos 2x})\,dx = \frac{1}{2}\int e^{\cos 2x}\sin 2x\,dx = -\frac{1}{4}\int e^u\,du$$
$$= -\frac{e^u}{4} + C = -\frac{e^{\cos 2x}}{4} + C.$$

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Problem 6. Find

(a) $\int \tan^2 2x\,dx$

(b) $\int \frac{1}{1+\cos 2t}\,dt$

(c) $\int \sin\left(\frac{5}{2}\theta\right)\cos\left(\frac{1}{2}\theta\right)d\theta$

(d) $\int \tan^4 x\,dx$

Solution.

Part (a).

$$\int \tan^2 2x\,dx = \int \left(\sec^2 2x - 1\right)\,dx = \frac{\tan 2x}{2} - x + C.$$

Part (b). Note that

$$\frac{1}{1+\cos 2t} = \frac{1}{1+(2\cos^2 t - 1)} = \frac{\sec^2 t}{2}.$$

Hence,

$$\int \frac{1}{1+\cos 2t}\,dt = \frac{1}{2}\int \sec^2 t\,dt = \frac{\tan t}{2} + C.$$

Part (c). By the product-to-sum identity,

$$\sin\left(\frac{5\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{\sin 3\theta + \sin 2\theta}{2}.$$

Hence,

$$\int \sin\left(\frac{5}{2}\theta\right)\cos\left(\frac{1}{2}\theta\right)d\theta = \frac{1}{2}\int (\sin 3\theta + \sin 2\theta)\,d\theta = -\frac{\cos 3\theta}{6} - \frac{\cos 2\theta}{4} + C$$

Part (d). Note that

$$\int \tan^2 x\,dx = \tan x - x + C$$

and

$$\int \tan^2 x \sec^2 x\,dx = \frac{\tan^3 x}{3} + C.$$

Hence,

$$\int \tan^4 x \,\mathrm{d}x = \int \tan^2 x \left(\sec^2 x - 1\right) \,\mathrm{d}x = \int \left(\tan^2 x \sec^2 x - \tan^2\right) \,\mathrm{d}x$$
$$= \frac{\tan^3 x}{3} - \tan x + x + C$$

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Problem 7. Find

(a) $\int \frac{1}{4x^2+2x+10} \,\mathrm{d}x$

(b) $\int \frac{x^2}{1-x^2} \,\mathrm{d}x$

(c) $\int \frac{1}{\sqrt{3+2x-x^2}} \,\mathrm{d}x$

Solution.

Part (a).

$$\int \frac{1}{4x^2+2x+10} \,\mathrm{d}x = 4\int \frac{1}{(4x+1)^2+39} \,\mathrm{d}x = 4\left(\frac{1}{4}\right)\left(\frac{1}{\sqrt{39}}\right) \arctan\left(\frac{4x+1}{\sqrt{39}}\right) + C$$
$$= \frac{1}{\sqrt{39}} \arctan\left(\frac{4x+1}{\sqrt{39}}\right) + C.$$

Part (b).

$$\int \frac{x^2}{1-x^2} \,\mathrm{d}x = \int \left(\frac{1}{1-x^2} - 1\right) \,\mathrm{d}x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) - x + C.$$

Part (c).

$$\int \frac{1}{\sqrt{3+2x-x^2}} \,\mathrm{d}x = \int \frac{1}{\sqrt{2^2-(x-1)^2}} \,\mathrm{d}x = \arcsin\left(\frac{x-1}{2}\right) + C.$$

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Problem 8. Evaluate the following without the use of graphic calculator:

(a) $\int_{\pi/3}^{2\pi/3} 4 \cot \frac{x}{2} \csc^2 \frac{x}{2} \,\mathrm{d}x$

(b) $\int_0^4 \frac{x+2}{\sqrt{2x+1}} \,\mathrm{d}x$

(c) $\int_0^1 \frac{2}{(1+x)(1+x^2)} \,\mathrm{d}x$

(d) $\int_{-4}^{-2} \frac{x^3+2}{x^2-1} \,\mathrm{d}x$

Solution.

Part (a).

$$\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} 4 \cot \frac{x}{2} \csc^2 \frac{x}{2} \,\mathrm{d}x = -4 \int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} \cot \frac{x}{2} \left(-\csc^2 \frac{x}{2}\right) \,\mathrm{d}x = -8 \left[\frac{\tan^2(x/2)}{2}\right]_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} = \frac{32}{3}.$$

Part (b). Consider the substitution $u = 2x + 1$.

$$\int_0^4 \frac{x+2}{\sqrt{2x+1}}\,\mathrm{d}x = \frac{1}{2}\int_0^4 \left(\sqrt{2x+1} + \frac{3}{\sqrt{2x+1}}\right)\mathrm{d}x$$
$$= \frac{1}{4}\int_1^9 \left(\sqrt{u} + \frac{3}{\sqrt{u}}\right)\mathrm{d}u = \frac{1}{4}\left[\frac{u^{3/2}}{3/2} + \frac{3u^{1/2}}{1/2}\right]_1^9 = \frac{22}{3}.$$

Part (c). Note that

$$\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\left[\ln|1+x^2|\right]_0^1 = \frac{\ln 2}{2}.$$

Thus,

$$\int_0^1 \frac{2}{(1+x)(1+x^2)}\,\mathrm{d}x = \int_0^1 \left(\frac{1}{1+x} + \frac{1}{1+x^2} - \frac{x}{1+x^2}\right)\mathrm{d}x$$
$$= [\ln|1+x|]_0^1 + [\arctan x]_0^1 - \frac{1}{2}\ln 2 = \ln 2 + \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{1}{2}\ln 2 + \frac{\pi}{4}.$$

Part (d).

$$\int_{-4}^{-2} \frac{x^3+2}{x^2-1}\,\mathrm{d}x = \int_{-4}^{-2}\left(x + \frac{3/2}{x-1} - \frac{1/2}{x+1}\right)\mathrm{d}x = \left[\frac{x^2}{2} + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right]_{-4}^{-2}$$
$$= -6 + 2\ln 3 - \frac{3}{2}\ln 5.$$

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Problem 9. Using the given substitution, find

(a) $\int \frac{x}{(2x+3)^3}\,\mathrm{d}x$ [$u = 2x + 3$]

(b) $\int \frac{1}{\mathrm{e}^x + 4\mathrm{e}^{-x}}\,\mathrm{d}x$ [$u = \mathrm{e}^x$]

(c) $\int_0^{\sqrt{2}} \sqrt{4 - y^2}\,\mathrm{d}y$ [$y = 2\sin\theta$]

(d) $\int_0^{\pi/2} \frac{1}{1+\sin\theta}\,\mathrm{d}\theta$ [$t = \tan\frac{\theta}{2}$]

Solution.

Part (a). Using the substitution $u = 2x + 3$,

$$\int \frac{x}{(2x+3)^3}\,\mathrm{d}x = \frac{1}{4}\int \frac{u-3}{u^3}\,\mathrm{d}x = \frac{1}{4}\int\left(\frac{1}{u^2} - \frac{3}{u^3}\right)\mathrm{d}u = \frac{1}{4}\left(\frac{u^{-1}}{-1} - \frac{3u^{-2}}{-2}\right) + C$$
$$= \frac{3}{8}(2x+3)^{-2} - \frac{1}{4}(2x+3)^{-1} + C.$$

Part (b). Using the substitution $u = \mathrm{e}^x$,

$$\int \frac{1}{\mathrm{e}^x + 4\mathrm{e}^{-x}}\,\mathrm{d}x = \int \frac{e^x}{e^{2x}+4}\,\mathrm{d}x = \int \frac{1}{u^2+4}\,\mathrm{d}u$$
$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C = \frac{1}{2}\arctan\left(\frac{\mathrm{e}^x}{2}\right) + C.$$

Part (c). Using the substitution $y = 2\sin\theta$,

$$\int_0^{\sqrt{2}} \sqrt{4-y^2}\,\mathrm{d}y = 2\int_0^{\pi/4} \cos\theta\sqrt{4-4\sin^2\theta}\,\mathrm{d}\theta = 4\int_0^{\pi/4} \cos\theta\sqrt{1-\sin^2\theta}\,\mathrm{d}\theta$$
$$= 4\int_0^{\pi/4} \cos^2\theta\,\mathrm{d}\theta = 4\int_0^{\pi/4} \frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta = 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = 1 + \frac{\pi}{2}.$$

Part (d). Consider the substitution $t = \tan\frac{\theta}{2}$. Then

$$\theta = 2\arctan t \implies \mathrm{d}\theta = \frac{2}{1+t^2}\,\mathrm{d}t$$

and

$$\sin\theta = \sin(2\arctan t) = 2\sin(\arctan t)\cos(\arctan t) = 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right) = \frac{2t}{1+t^2}.$$

Hence,

$$\int_0^{\pi/2} \frac{1}{1+\sin\theta}\,\mathrm{d}\theta = \int_0^1 \frac{2/(1+t^2)}{1+2t/(1+t^2)}\,\mathrm{d}u = \int_0^1 \frac{2}{(t+1)^2}\,\mathrm{d}t = 2\left[-\frac{1}{t+1}\right]_0^1 = 1.$$

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Problem 10. Find

(a) $\int \ln(2x+1)\,\mathrm{d}x$

(b) $\int x\arctan(x^2)\,\mathrm{d}x$

(c) $\int \mathrm{e}^{-2x}\cos 2x\,\mathrm{d}x$

(d) $\int_0^2 x^2 e^{-x}\,\mathrm{d}x$

Solution.

Part (a). Consider the substitution $u = 2x+1$.

$$\int \ln(2x+1)\,\mathrm{d}x = \frac{1}{2}\int \ln u\,\mathrm{d}u.$$

Integrating by parts,

	D	I
$+$	$\ln u$	1
$-$	$1/u$	u

Thus,

$$\int \ln(2x+1)\,\mathrm{d}x = \frac{1}{2}\left(u\ln u - \int u\left(\frac{1}{u}\right)\,\mathrm{d}u\right) = \frac{u\ln u - u}{2} + C$$
$$= \frac{(2x+1)\ln(2x+1) - (2x+1)}{2} + C = x\ln(2x+1) + \frac{\ln(2x+1)}{2} - x + C.$$

Part (b). Consider the substitution $u = x^2$.

$$\int x\arctan(x^2)\,\mathrm{d}x = \frac{1}{2}\int \arctan u\,\mathrm{d}u.$$

Integrating by parts,

	D	I
$+$	$\arctan u$	1
$-$	$1/(1+u^2)$	u

Thus,

$$\int x\arctan(x^2)\,\mathrm{d}x = \frac{1}{2}\left(u\arctan u - \int \frac{u}{1+u^2}\,\mathrm{d}u\right) = \frac{1}{2}\left[u\arctan u - \frac{\ln\left(1+u^2\right)}{2}\right] + C$$
$$= \frac{x^2\arctan x^2}{2} - \frac{\ln\left(1+x^4\right)}{4} + C.$$

Part (c). Let

$$I = \int \mathrm{e}^{-2x}\cos 2x\,\mathrm{d}x.$$

Integrating by parts, we have

	D	I
$+$	e^{-2x}	$\cos 2x$
$-$	$-2\mathrm{e}^{-2x}$	$\sin(2x)/2$
$+$	$4\mathrm{e}^{-2x}$	$-\cos(2x)/4$

Thus,

$$I = \frac{\mathrm{e}^{-2x}\sin 2x}{2} - \frac{\mathrm{e}^{-2x}\cos 2x}{2} - I \implies I = \frac{\mathrm{e}^{-2x}\left(\sin 2x - \cos 2x\right)}{4} + C.$$

Part (d). Integrating by parts, we get

	D	I
$+$	x^2	e^{-x}
$-$	$2x$	$-\mathrm{e}^{-x}$
$+$	2	e^{-x}
$-$	0	$-\mathrm{e}^{-x}$

Thus,

$$\int_0^2 x^2\mathrm{e}^{-x}\,\mathrm{d}x = \left[-x^2\mathrm{e}^{-x} - 2x\mathrm{e}^{-x} - 2\mathrm{e}^{-x}\right]_0^2 = 2 - 10\mathrm{e}^{-2}.$$

* * * * *

Problem 11.

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x + \tan x) = \sec x$.

(b) Find $\int x\sin x\,\mathrm{d}x$.

(c) Find the exact value of $\int_0^{\pi/4}(x\sin x)\ln(\sec x + \tan x)\,\mathrm{d}x$.

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x+\tan x)=\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}=\sec x\left(\frac{\tan x+\sec x}{\sec x+\tan x}\right)=\sec x.$$

Part (b). Integrating by parts,

	D	I
$+$	x	$\sin x$
$-$	1	$-\cos x$
$+$	0	$-\sin x$

Hence,

$$\int x\sin x\,\mathrm{d}x=-x\cos x+\sin x+C.$$

Part (c). Integrating by parts,

	D	I
$+$	$\ln(\sec x+\tan x)$	$x\sin x$
$-$	$\sec x$	$-x\cos x+\sin x$

Thus,

$$\begin{aligned}&\int_0^{\pi/4}(x\sin x)\ln(\sec x+\tan x)\,\mathrm{d}x\\&=\left[\ln(\sec x+\tan x)\left(-x\cos x+\sin x\right)\right]_0^{\pi/4}-\int_0^{\pi/4}\left(-x+\tan x\right)\,\mathrm{d}x\\&=\left[\ln(\sec x+\tan x)\left(-x\cos x+\sin x\right)-\frac{x^2}{2}-\ln\left|\cos x\right|\right]_0^{\pi/4}\\&=\frac{\sqrt{2}}{2}\left(1-\frac{\pi}{4}\right)\ln\left(\sqrt{2}+1\right)+\frac{\pi^2}{32}-\frac{\ln 2}{2}\end{aligned}$$

* * * * *

Problem 12.

(a) Use the fact that $7\cos x-4\sin x=\frac{3}{2}(\cos x+\sin x)+\frac{11}{2}(\cos x-\sin x)$ to find the exact value of $\int_0^{\pi/2}\frac{7\cos x-4\sin x}{\cos x+\sin x}\,\mathrm{d}x$.

(b) Use integration by parts to find the exact value of $\int_1^{\mathrm{e}}(\ln x)^2\,\mathrm{d}x$.

Solution.

Part (a). Note that

$$\frac{7\cos x-4\sin x}{\cos x+\sin x}=\frac{1}{2}\left(\frac{3(\cos x+\sin x)+11(\cos x-\sin x)}{\cos x+\sin x}\right)=\frac{3}{2}+\frac{11}{2}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right).$$

Thus,

$$\begin{aligned}\int_0^{\pi/2}\frac{7\cos x-4\sin x}{\cos x+\sin x}\,\mathrm{d}x&=\frac{1}{2}\int_0^{\pi/2}\left(3+11\cdot\frac{\cos x-\sin x}{\cos x+\sin x}\right)\,\mathrm{d}x\\&=\left[\frac{3x}{2}+\frac{11}{2}\ln\left|\cos x+\sin x\right|\right]_0^{\pi/2}=\frac{3\pi}{4}.\end{aligned}$$

Part (b). Integrating by parts,

	D	I
$+$	$(\ln x)^2$	1
$-$	$\frac{2\ln x}{x}$	x

Thus,

$$\int_1^{\mathrm{e}} (\ln x)^2 \,\mathrm{d}x = \left[x(\ln x)^2\right]_1^{\mathrm{e}} - 2\int_1^{\mathrm{e}} \ln x \,\mathrm{d}x = \left[x(\ln x)^2 - 2\left(x\ln x - x\right)\right]_1^{\mathrm{e}} = \mathrm{e} - 2.$$

* * * * *

Problem 13.

(a) Solve the inequality $x^2 + 2x - 3 < 0$.

(b) Without using the graphing calculator, evaluate

(i) $\int_{-4}^{4} \left|x^2 + 2x - 3\right| \,\mathrm{d}x$

(ii) $\int_0^2 x\left|x^2 + 2x - 3\right| \,\mathrm{d}x$

Solution.

Part (a).

$$x^2 + 2x - 3 = (x+1)^2 - 4 < 0 \implies (x+1)^2 < 4 \implies -2 < x + 1 < 2 \implies -3 < x < 1.$$

Part (b).

Part (b)(i). Let $F(x) = \int \left(x^2 + 2x - 3\right) \,\mathrm{d}x = \frac{1}{3}x^3 + x^2 - 3x + C$. Then,

$$\begin{aligned}
&\int_{-4}^{4} \left|x^2 + 2x - 3\right| \,\mathrm{d}x \\
&= \int_{-4}^{-3} \left|x^2 + 2x - 3\right| \,\mathrm{d}x + \int_{-3}^{1} \left|x^2 + 2x - 3\right| \,\mathrm{d}x + \int_{1}^{4} \left|x^2 + 2x - 3\right| \,\mathrm{d}x \\
&= \int_{-4}^{-3} x^2 + 2x - 3 \,\mathrm{d}x - \int_{-3}^{1} x^2 + 2x - 3 \,\mathrm{d}x + \int_{1}^{4} x^2 + 2x - 3 \,\mathrm{d}x \\
&= \Big[F(-3) - F(-4)\Big] - \Big[F(1) - F(-3)\Big] + \Big[F(4) - F(1)\Big] = 40.
\end{aligned}$$

Part (b)(ii). Let $F(x) = \int x\left(x^2 + 2x - 3\right) \,\mathrm{d}x = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + C$. Then,

$$\begin{aligned}
&\int_0^2 x\left|x^2 + 2x - 3\right| \,\mathrm{d}x \\
&= \int_0^1 x\left|x^2 + 2x - 3\right| \,\mathrm{d}x + \int_1^2 x\left|x^2 + 2x - 3\right| \,\mathrm{d}x \\
&= -\int_0^1 x(x^2 + 2x - 3) \,\mathrm{d}x + \int_1^2 x(x^2 + 2x - 3) \,\mathrm{d}x \\
&= -\Big[F(1) - F(0)\Big] + \Big[F(2) - F(1)\Big] = \frac{9}{2}.
\end{aligned}$$

Problem 14. The indefinite integral $\int \frac{P(x)}{x^3+1}\,\mathrm{d}x$, where $P(x)$ is a polynomial in x, is denoted by I.

(a) Find I when $P(x) = x^2$.

(b) By writing $x^3 + 1 = (x+1)\left(x^2 + Ax + B\right)$, where A and B are constants, find I when

  (i) $P(x) = x^2 - x + 1$

  (ii) $P(x) = x + 1$

(c) Using the results of parts (a) and (b), or otherwise, find I when $P(x) = 1$.

Solution.

Part (a).

$$\int \frac{x^2}{x^3+1}\,\mathrm{d}x = \frac{1}{3}\frac{3x^2}{x^3+1}\,\mathrm{d}x = \frac{\ln\left|x^3+1\right|}{3} + C.$$

Part (b).

$$x^3 + 1 = (x+1)\left(x^2 - x + 1\right).$$

Part (b)(i).

$$\int \frac{x^2-x+1}{x^3+1}\,\mathrm{d}x = \int \frac{x^2-x+1}{(x+1)\left(x^2-x+1\right)}\,\mathrm{d}x = \int \frac{1}{x+1}\,\mathrm{d}x = \ln|x+1| + C.$$

Part (b)(ii).

$$\begin{aligned}\int \frac{x+1}{x^3+1}\,\mathrm{d}x &= \int \frac{x+1}{(x+1)\left(x^2-x+1\right)}\,\mathrm{d}x = \int \frac{1}{x^2-x+1}\,\mathrm{d}x = \int \frac{1}{(x-1/2)^2+3/4}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{3/4}}\arctan\left(\frac{x-1/2}{\sqrt{3/4}}\right) + C = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.\end{aligned}$$

Part (c). Observe that $1 = \frac{1}{2}\left[\left(x^2-x+1\right) - x^2 + (x+1)\right]$. Hence,

$$\begin{aligned}\int \frac{1}{x^3+1}\,\mathrm{d}x &= \frac{1}{2}\left(\int \frac{x^2-x+1}{x^3+1}\,\mathrm{d}x - \int \frac{x^2}{x^3+1}\,\mathrm{d}x + \int \frac{x+1}{x^3+1}\,\mathrm{d}x\right) \\ &= \frac{1}{2}\left[\ln|x+1| - \frac{\ln\left|x^3+1\right|}{3} + \frac{2}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\right] + C \\ &= \frac{1}{2}\ln|x+1| - \frac{\ln\left|x^3+1\right|}{6} + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.\end{aligned}$$

Self-Practice B7

Problem 1. Find

(a) $\int \tan\left(\frac{\pi}{6} - 3x\right) \mathrm{d}x$,

(b) $\int \tan x \sec^4 x \,\mathrm{d}x$,

(c) $\int \frac{1}{x^2+3x+2} \,\mathrm{d}x$,

(d) $\int \cos \frac{3x}{2} \cos \frac{5x}{2} \,\mathrm{d}x$,

(e) $\int \frac{2}{x \ln x^2} \,\mathrm{d}x$,

(f) $\int x\mathrm{e}^{-x^2} \,\mathrm{d}x$.

Solution.
Part (a).

$$\int \tan\left(\frac{\pi}{6} - 3x\right) \mathrm{d}x = -\frac{1}{3} \int -3\tan\left(\frac{\pi}{6} - 3x\right) \mathrm{d}x = -\frac{1}{3} \ln\left|\sec\left(\frac{\pi}{6} - 3x\right)\right| + C.$$

Part (b).

$$\int \tan x \sec^4 x \,\mathrm{d}x = \int (\sec x \tan x) \sec^3 x \,\mathrm{d}x = \frac{\sec^4 x}{4} + C.$$

Part (c).

$$\int \frac{1}{x^2+3x+2} \,\mathrm{d}x = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) \mathrm{d}x = \ln|x+1| - \ln|x+2| + C = \ln\left|\frac{x+1}{x+2}\right| + C.$$

Part (d).

$$\int \cos \frac{3x}{2} \cos \frac{5x}{2} \,\mathrm{d}x = \int (\cos 4x + \cos x)\, \mathrm{d}x = \frac{\sin 4x}{8} + \frac{\sin x}{2} + C.$$

Part (e).

$$\int \frac{2}{x \ln x^2} \,\mathrm{d}x = \int \frac{1/x}{\ln x} \,\mathrm{d}x = \ln|\ln x| + C.$$

Part (f).

$$\int x\mathrm{e}^{-x^2} \,\mathrm{d}x = -\frac{1}{2} \int -2x\mathrm{e}^{-x^2} \,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x^2} + C.$$

* * * * *

Problem 2. Using the substitution $x = \tan\theta$, find the exact value of $\int_0^1 \frac{1-x^2}{(1+x^2)^2} \,\mathrm{d}x$.

Solution. Note that $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$. Hence,

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2} \,\mathrm{d}x = \int_0^{\pi/4} \frac{1-\tan^2\theta}{\sec^4\theta} \left(\sec^2\theta \,\mathrm{d}\theta\right) = \int_0^{\pi/4} \frac{1-\tan^2\theta}{\sec^2\theta} \,\mathrm{d}\theta.$$

Using trigonometric identities to simplify the integrand, we get

$$\int_0^{\pi/4} \frac{1-\tan^2\theta}{\sec^2\theta} \,\mathrm{d}\theta = \int_0^{\pi/4} \cos 2\theta \,\mathrm{d}t = \left[\frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}.$$

Problem 3. State the derivative of $\sin x^2$. Hence, find $\int x^3 \cos x^2 \, \mathrm{d}x$.

Solution. We have

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x^2 = 2x \cos x^2.$$

Consider the substitution $u = \sin x^2$. Using the above result, we have

$$\int x^2 \cos x^2 \, \mathrm{d}x = \frac{1}{2} \int \left(2x \cos x^2\right) x^2 \, \mathrm{d}x = \frac{1}{2} \int \arcsin u \, \mathrm{d}u.$$

Integrating by parts, we get

$$\frac{1}{2} \left(u \arcsin u - \int \frac{u}{\sqrt{1-u^2}} \, \mathrm{d}u \right).$$

The integral is fairly simple to evaluate:

$$\int \frac{u}{\sqrt{1-u^2}} = -\frac{1}{2} \int \frac{-2u}{\sqrt{1-u^2}} \, \mathrm{d}u = -\sqrt{1-u^2} + C.$$

Thus,

$$\int x^3 \cos x^2 \, \mathrm{d}x = \frac{1}{2} \left(x^2 \sin x^2 + \sqrt{1 - \sin^2 x^2} \right) + C = \frac{1}{2} \left(x^2 \sin x^2 + \cos x^2 \right) + C.$$

* * * * *

Problem 4. Find the exact value of p such that $\int_0^1 \frac{1}{4-x^2} \, \mathrm{d}x = \int_0^{1/2p} \frac{1}{\sqrt{1-p^2x^2}} \, \mathrm{d}x$.

Solution. Using standard integration results, the LHS evaluates to

$$\int_0^1 \frac{1}{4-x^2} \, \mathrm{d}x = \left[\frac{1}{4} \ln \frac{2+x}{2-x} \right]_0^1 = \frac{1}{4} \ln 3.$$

Meanwhile, under the substitution $u = px$, the RHS evaluates as

$$\int_0^{1/2p} \frac{1}{\sqrt{1-p^2x^2}} \, \mathrm{d}x = \frac{1}{p} \int_0^{1/2} \frac{1}{\sqrt{1-u^2}} \, \mathrm{d}u = \frac{1}{p} \left[\arcsin u \right]_0^{1/2} = \frac{\pi}{6p}.$$

Equating the two, we get

$$\frac{1}{4} \ln 3 = \frac{\pi}{6p} \implies p = \frac{2\pi}{3 \ln 3}.$$

Problem 5.

(a) Find $\int \frac{x+3}{\sqrt{4x-x^2}}\,\mathrm{d}x$.

(b) If $x = 4\cos^2\theta + 7\sin^2\theta$, show that $7 - x = 3\cos^2$, and find a similar expression for $x - 4$. By using the substitution $x = 4\cos^2\theta + 7\sin^2\theta$, evaluate $\int_4^7 \frac{1}{\sqrt{(x-4)(7-x)}}\,\mathrm{d}x$.

Solution.

Part (a). Note that

$$\int \frac{x+3}{\sqrt{4x-x^2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{-2x-6}{\sqrt{4x-x^2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{-2x-4}{\sqrt{4x-x^2}}\,\mathrm{d}x + 5\int \frac{1}{\sqrt{4x-x^2}}\,\mathrm{d}x.$$

Also note that $4x - x^2 = 4 - (x-2)^2$. Hence,

$$\int \frac{x+3}{\sqrt{4x-x^2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{-2x-4}{\sqrt{4x-x^2}}\,\mathrm{d}x + 5\int \frac{1}{\sqrt{4-(x-2)^2}}\,\mathrm{d}x,$$

which we can easily evaluate as

$$\int \frac{x+3}{\sqrt{4x-x^2}}\,\mathrm{d}x = -\sqrt{4x-x^2} + 5\arcsin\frac{x-2}{2} + C.$$

Part (b). Clearly,

$$x = 4\cos^2\theta + 7\sin^2\theta = 7\left(\cos^2\theta + \sin^2\theta\right) - 3\cos^2\theta = 7 - 3\cos^2,$$

whence $7 - x = 3\cos^2\theta$ as desired. Similarly,

$$x = 4\left(\cos^2\theta + \sin^2\theta\right) + 3\sin^2\theta = 4 + 3\sin^2\theta,$$

whence $x - 4 = 3\sin^2\theta$.

Under the substitution $u = 4\cos^2\theta + 7\sin^2\theta$, the integral transforms as

$$\int_4^7 \frac{1}{\sqrt{(x-4)(7-x)}}\,\mathrm{d}x = \int_0^{\pi/2} \frac{6\cos\theta\sin\theta}{\sqrt{\left(3\cos^2\theta\right)\left(3\sin^2\theta\right)}}\,\mathrm{d}\theta = 2\int_0^{\pi/2}\mathrm{d}\theta = \pi.$$

* * * * *

Problem 6. Express $\frac{x^2+x+28}{(1-x)(x^2+9)}$ in partial fractions. Hence, show that $\int_0^3 \frac{x^2+x+28}{(1-x)(x^2+9)}\,\mathrm{d}x = \frac{\pi}{12} - 2\ln 2$.

Solution. Let

$$\frac{x^2+x+28}{(1-x)(x^2+9)} = \frac{A}{1-x} + \frac{Bx+C}{x^2+9},$$

where A, B and C are constants to be determined. By the cover-up rule, we immediately get

$$A = \frac{1+1+28}{1^2+9} = 3.$$

Clearing denominators, we get

$$x^2 + x + 28 = 3\left(x^2+9\right) + (Bx+C)(1-x) = (3-B)x^2 + (B-C)x + (27+C).$$

Comparing coefficients, we get $B = 2$ and $C = 1$, whence

$$\frac{x^2+x+28}{(1-x)(x^2+9)} = \frac{3}{1-x} + \frac{2x+1}{x^2+9}.$$

Using the above result on the integral, we have

$$\begin{aligned}\int_0^3 \frac{x^2+x+28}{(1-x)(x^2+9)}\,\mathrm{d}x &= \int_0^3 \left(\frac{3}{1-x}+\frac{2x}{x^2+9}+\frac{1}{x^2+9}\right)\mathrm{d}x \\ &= \left[-3\ln|1-x| + \ln\left(x^2+9\right) + \frac{1}{3}\arctan\frac{x}{3}\right]_0^3 \\ &= \frac{\pi}{12} - 2\ln 2.\end{aligned}$$

* * * * *

Problem 7. Find the derivative of $\arcsin x + x\sqrt{1-x^2}$, expressing your answer in its simplest form. Hence, evaluate the exact value of $\int_0^{1/2}\sqrt{1-x^2}\,\mathrm{d}x$.

Solution. We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\arcsin x + x\sqrt{1-x^2}\right] = \frac{1}{\sqrt{1-x^2}} + \left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right) = 2\sqrt{1-x^2}.$$

Hence,

$$\int_0^{1/2}\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^{1/2} 2\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\left[\arcsin x + x\sqrt{1-x^2}\right]_0^{1/2} = \frac{\pi}{12} + \frac{\sqrt{3}}{8}.$$

Assignment B7

Problem 1.

(a) Find $\int \frac{6x^3+2}{x^2+1}\,\mathrm{d}x$.

(b) Evaluate $\int_2^4 x\ln x\,\mathrm{d}x$ exactly.

Solution.

Part (a). Note that

$$\frac{6x^3+2}{x^2+1} = 6x - \frac{6x}{x^2+1} + \frac{2}{x^2+1}.$$

Hence,

$$\int \frac{6x^3+2}{x^2+1}\,\mathrm{d}x = \int \left(6x - \frac{6x}{x^2+1} + \frac{2}{x^2+1}\right)\mathrm{d}x = 3x^2 - 3\ln\big(x^2+1\big) + 2\arctan x + C.$$

Part (b). Consider the substitution $u = x^2$.

$$\int_2^4 x\ln x\,\mathrm{d}x = \frac{1}{2}\int_4^{16} \ln\sqrt{u}\,\mathrm{d}u = \frac{1}{4}\int_4^{16} \ln u\,\mathrm{d}u = \frac{1}{4}\left[u\ln u - u\right]_4^{16} = 14\ln 2 - 3.$$

* * * * *

Problem 2.

(a) Use the derivative of $\cos\theta$ to show that $\frac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta = \sec\theta\tan\theta$.

(b) Use the substitution $x = \sec\theta - 1$ to find the exact value of $\int_{\sqrt{2}-1}^{1} \frac{1}{(x+1)\sqrt{x^2+2x}}\,\mathrm{d}x$.

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta = \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta} = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \sec\theta\tan\theta.$$

Part (b). Consider the substitution $x = \sec\theta - 1 \implies \mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$. When $x = 1$, we have $\theta = \pi/3$. When $x = \sqrt{2}-1$, we have $\theta = \pi/4$. Also note that $x+1 = \sec\theta$. Now observe that

$$x^2 + 2x = (\sec\theta - 1)^2 + 2(\sec\theta - 1) = \sec^2\theta - 1 = \tan^2\theta \implies \sqrt{x^2+2x} = \tan\theta.$$

Thus,

$$\int_{\sqrt{2}-1}^{1} \frac{1}{(x+1)\sqrt{x^2+2x}}\,\mathrm{d}x = \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,\mathrm{d}\theta = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

Problem 3. The expression $\frac{x^2}{9-x^2}$ can be written in the form $A+\frac{B}{3-x}+\frac{C}{3+x}$.

(a) Find the values of constants A, B and C.

(b) Show that $\int_0^2 \frac{x^2}{9-x^2}\,\mathrm{d}x = \frac{3}{2}\ln 5 - 2$.

(c) Hence, find the value of $\int_0^2 \ln\left(9-x^2\right)\mathrm{d}x$, giving your answer in terms of $\ln 5$.

Solution.

Part (a).

$$\frac{x^2}{9-x^2} = -1+\frac{9}{9-x^2} = -1+\frac{9}{(3-x)(3+x)} = -1+\frac{3/2}{3-x}+\frac{3/2}{3+x}.$$

Thus, $A=-1$, $B=3/2$ and $C=3/2$.

Part (b).

$$\int_0^2 \frac{x^2}{9-x^2}\,\mathrm{d}x = \int_0^2 \left(-1+\frac{3/2}{3-x}+\frac{3/2}{3+x}\right)\mathrm{d}x$$
$$= \left[-x-\frac{3}{2}\ln(3-x)+\frac{3}{2}\ln(3+x)\right]_0^2 = \frac{3}{2}\ln 5 - 2.$$

Part (c). Integrating by parts,

	D	I
$+$	$\ln\left(9-x^2\right)$	1
$-$	$-2x/\left(9-x^2\right)$	x

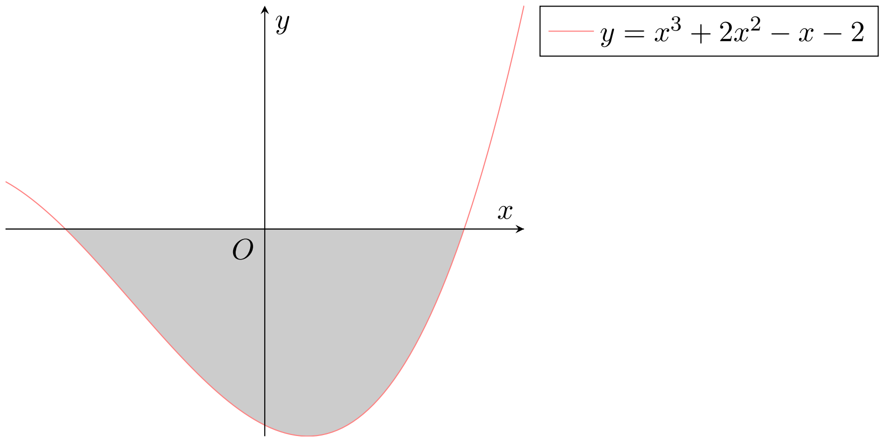
Thus,

$$\int_0^2 \ln\left(9-x^2\right)\,\mathrm{d}x = \left[x\ln\left(9-x^2\right)\right]_0^2 + 2\left(\frac{3}{2}\ln 5 - 2\right) = 5\ln 5 - 4.$$

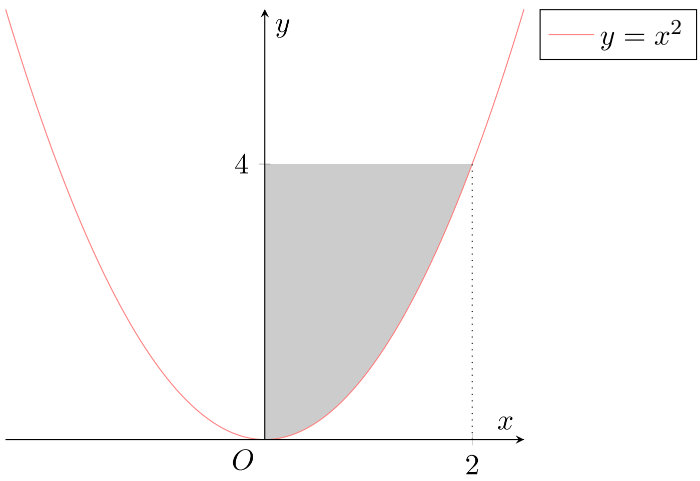
B8 Applications of Integration I - Area and Volume

Tutorial B8

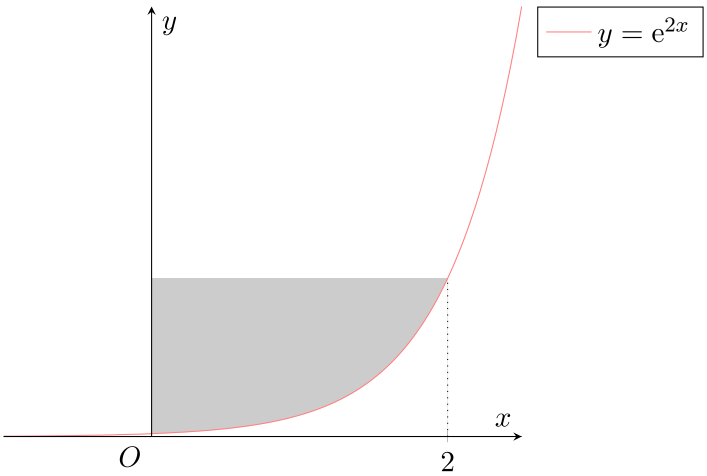
Problem 1. Write down the integral for the area of the shaded region for each of the figure below and use the GC to evaluate it, to 3 significant figures.



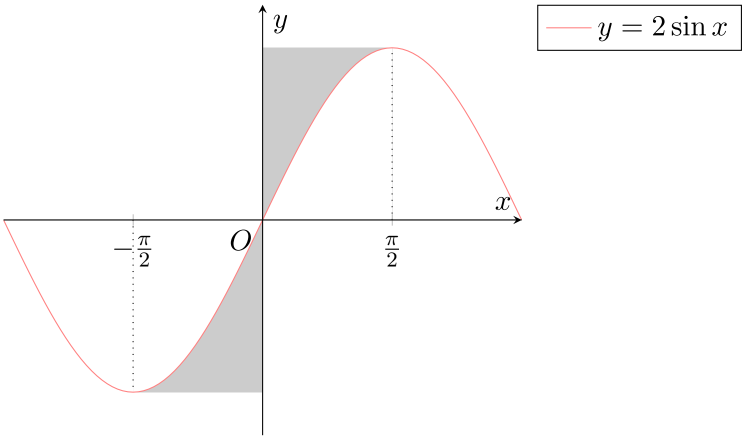
(a)



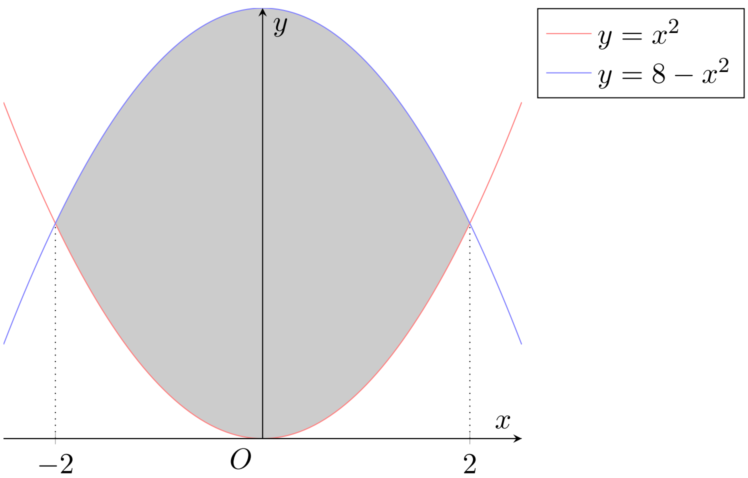
(b)



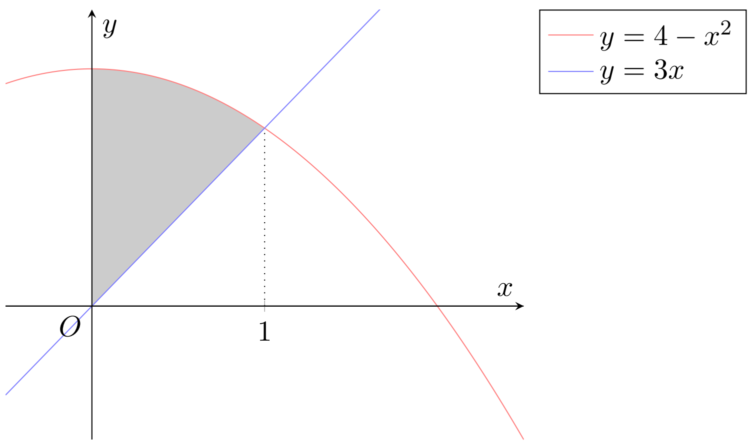
(c)



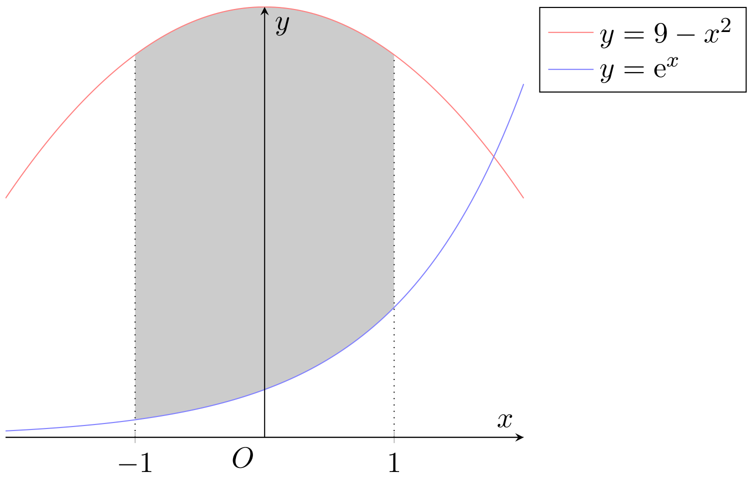
(d)



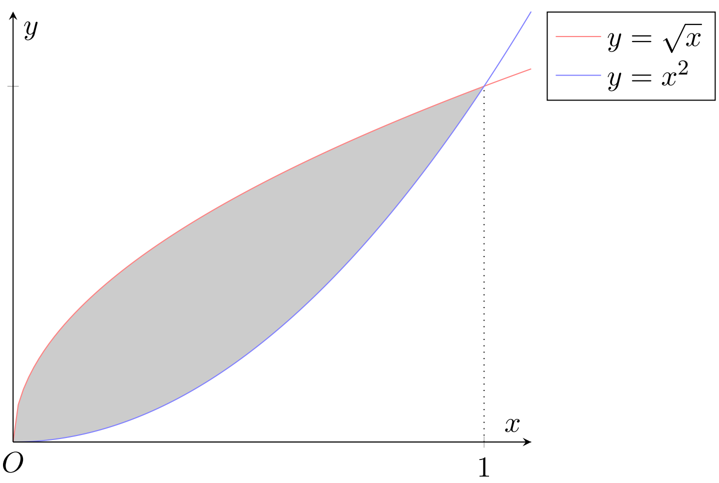
(e)



(f)



(g)



(h)

Solution.

Part (a).

$$\text{Area} = -\int_{-1}^{1} \left(x^3 + 2x^2 - x - 2\right)\, \mathrm{d}x = 2.67 \text{ units}^2 \text{ (3 s.f.)}.$$

Part (b). Note that $y = x^2 \implies x = \sqrt{y}$.

$$\text{Area} = \int_0^4 \sqrt{y}\, \mathrm{d}y = 5.33 \text{ units}^2 \text{ (3 s.f.)}.$$

Part (c). Note that $y = \mathrm{e}^{2x} \implies x = \frac{1}{2}\ln y$. Also, when $x = 0$, we have $y = 1$. Further, when $x = 2$, we have $y = \mathrm{e}^4$. Thus,

$$\text{Area} = \int_0^{\mathrm{e}^4} \frac{1}{2}\ln y \,\mathrm{d}y = 82.4 \text{ units}^2 \text{ (3 s.f.)}.$$

Part (d). Note that when $x = \pi/2$, we have $y = 2$. Thus,

$$\text{Area} = 2\int_0^2 \arcsin\frac{y}{2} \,\mathrm{d}y = 2.28 \text{ units}^2 \text{ (3 s.f.)}.$$

Part (e).

$$\text{Area} = \int_{-2}^{2} \left[\left(8 - x^2\right) - x^2\right] \mathrm{d}x = 21.3 \text{ units}^2 \text{ (3 s.f.)}.$$

Part (f).

$$\text{Area} = \int_0^1 \left[\left(4 - x^2\right) - 3x\right] \mathrm{d}x = 2.17 \text{ units}^2 \text{ (3 s.f.)}.$$

Part (g).

$$\text{Area} = \int_{-1}^{1} \left[\left(9 - x^2\right) - \mathrm{e}^x\right] \mathrm{d}x = 15.0 \text{ units}^2 \text{ (3 s.f.)}.$$

Part (h).

$$\text{Area} = \int_0^1 \left(\sqrt{x} - x^2\right) \mathrm{d}x = 0.333 \text{ units}^2 \text{ (3 s.f.)}.$$

* * * * *

Problem 2.

(a) Write down the integral for the volume of the solid generated when the shaded region is rotated about the x-axis through 2π for questions 1(a), (e), (f) and (h) using the disc method and use the GC to evaluate it.

(b) Write down the integral for the volume of the solid generated when the shaded region is rotated about the y-axis through 2π for questions 1(b), (d) and (f) using the disc method and use the GC to evaluate it.

Solution.

Part (a).

Part (a)(i).

$$\text{Volume} = \pi\int_{-1}^{1} \left(x^3 + 2x^2 - x - 2\right)^2 \mathrm{d}x = 13.9 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (a)(ii).

$$\text{Volume} = \pi\int_{-2}^{2} \left[\left(8 - x^2\right)^2 - x^2\right] \mathrm{d}x = 536 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (a)(iii).

$$\text{Volume} = \pi\int_0^1 \left[\left(4 - x^2\right)^2 - (3x)^2\right] \mathrm{d}x = 33.1 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (a)(iv).

$$\text{Volume} = \pi\int_0^1 \left[\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right] \mathrm{d}x = 0.942 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (b).

Part (b)(i).

$$\text{Volume} = \pi \int_0^4 \left(\sqrt{y}\right)^2 \,\mathrm{d}y = 25.1 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (b)(ii).

$$\text{Volume} = 2\pi \int_0^2 \arcsin^2 \frac{y}{2} \,\mathrm{d}y = 5.87 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (b)(iii).

$$\text{Volume} = \pi \int_3^4 (4-y) \,\mathrm{d}y + \frac{\pi \left(1^2\right)(3)}{3} = 4.71 \text{ units}^3 \text{ (3 s.f.)}.$$

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Problem 3.

(a) Write down the integral for the volume of the solid generated when the shaded region is rotated about the x-axis through 2π for questions 1(e), (f) and (h) using the shell method and use the GC to evaluate it.

(b) Write down the integral for the volume of the solid generated when the shaded region is rotated about the y-axis through 2π for questions 1(b), (d) and (f) using the shell method and use the GC to evaluate it.

Solution.

Part (a).

Part (a)(i). Note that $y = x^2 \implies x = \sqrt{y}$ and $y = 8 - x^2 \implies x = \sqrt{8-y}$ for $x > 0$. Thus,

$$\text{Volume} = 2\left(2\pi \int_0^4 \sqrt{y} \cdot y \,\mathrm{d}y + 2\pi \int_4^8 \sqrt{8-y} \cdot y \,\mathrm{d}y\right) = 536 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (a)(ii). Note that $y = 3x \implies x = y/3$ and $y = 4 - x^2 \implies x = \sqrt{4-y}$ for $x > 0$. Thus,

$$\text{Volume} = 2\pi \int_0^3 \frac{1}{3} y \cdot y \,\mathrm{d}y + 2\pi \int_3^4 \sqrt{4-y} \cdot y \,\mathrm{d}y = 33.1 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (a)(iii). Note that $y = \sqrt{x} \implies x = y^2$ and $y = x^2 \implies x = \sqrt{y}$ for $x > 0$. Thus,

$$\text{Volume} = 2\pi \int_0^1 \left(\sqrt{y} - y^2\right) y \,\mathrm{d}y = 0.942 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (b).

Part (b)(i).

$$\text{Volume} = 2\pi \int_0^2 x \cdot x^2 \,\mathrm{d}x = 25.1 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (b)(ii).

$$\text{Volume} = 2 \cdot 2\pi \int_0^{\pi/2} x\,(2 - 2\sin x) \,\mathrm{d}x = 5.87 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (b)(iii).

$$\text{Volume} = 2\pi \int_0^1 x\left[\left(4-x^2\right)-3x\right]\,\mathrm{d}x = 4.71 \text{ units}^3 \text{ (3 s.f.)}.$$

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Problem 4. Calculate the area enclosed by the petals of the curve $r = \sin 2\theta$ where $r \geq 0$.

Solution. Note that $r \geq 0 \implies \sin 2\theta \geq 0 \implies r \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$. Thus,

$$\text{Area} = 2 \cdot \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta \,\mathrm{d}\theta = \int_0^{\pi/2} \frac{1-\cos 4\theta}{2}\,\mathrm{d}\theta = \frac{1}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{4} \text{ units}^2.$$

* * * * *

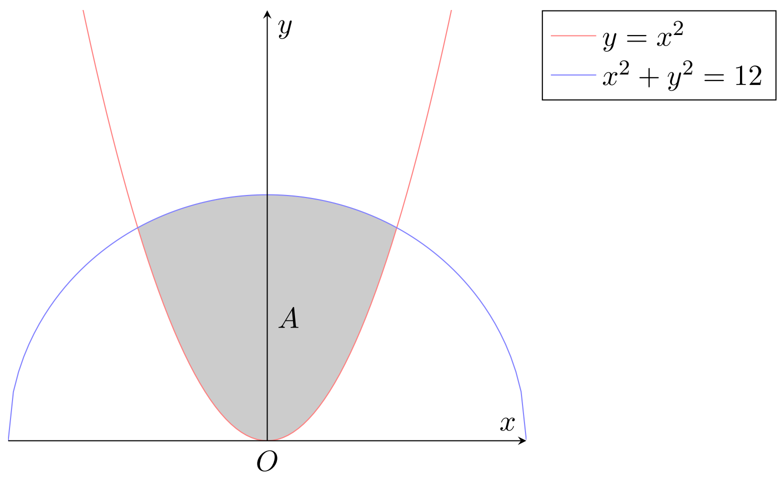
Problem 5. The finite region A is bounded by the curve $y = x^2$ and a minor arc of the circle $x^2 + y^2 = 12$.

(a) Find the numerical value of the area of A, correct to 2 decimal places.

(b) Find the exact volume of the solid obtained when A is rotated about the x-axis through 2π radians.

(c) Find the exact volume of the solid obtained when A is rotated about the y-axis through π radians.

Solution.



Part (a). Consider the intersections between $y = x^2$ and $x^2 + y^2 = 12$.

$$x^2 + y^2 = x^2 + \left(x^2\right)^2 = 12 \implies x^4 + x^2 - 12 = (x^2-3)(x^2+4) = 0$$
$$\implies \left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)\left(x^2+4\right) = 0.$$

Hence, the two curves intersect at $x = -\sqrt{3}$ and $x = \sqrt{3}$. Note that $x^2 + 4 = 0$ has no solution since $x^2 + 4 > 0$. Also note that $x^2 + y^2 = 12 \implies y = \sqrt{12 - x^2}$ for $y > 0$. Thus,

$$\text{Area} = 2\int_0^{\sqrt{3}} \left(\sqrt{12-x^2} - x^2\right)\,\mathrm{d}x = 8.02 \text{ units}^2 \text{ (3 s.f.)}.$$

Part (b). Note that $x^2 + y^2 = 12 \implies y^2 = 12 - x^2$.

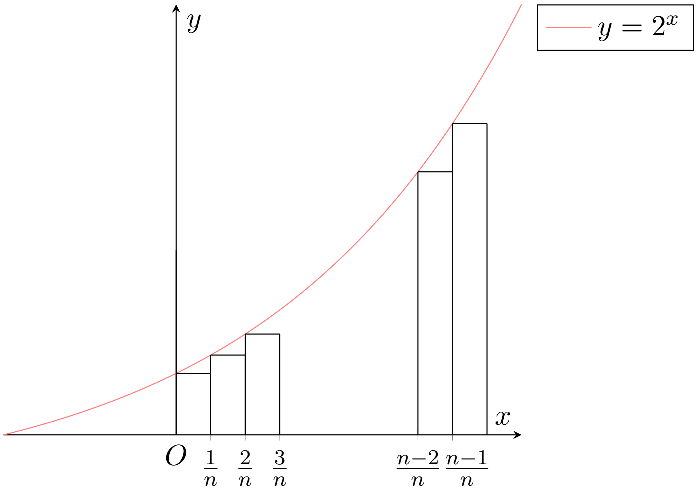
$$\text{Volume} = 2\pi \int_0^{\sqrt{3}} \left[\left(12 - x^2\right) - \left(x^2\right)^2\right] \, dx = 2\pi \left[12x - \frac{x^3}{3} - \frac{x^5}{5}\right]_0^{\sqrt{3}} = \frac{92\sqrt{3}\pi}{5} \text{ units}^3.$$

Part (c). Note that when the curves intersect at $x = \sqrt{3}$, we have $y = 3$. Furthermore, when $x = 0$, we have $y = \sqrt{12}$. Also note that $x^2 + y^2 = 12 \implies x^2 = 12 - y^2$.

$$\text{Volume} = \pi \int_0^3 y \, dy + \pi \int_3^{\sqrt{12}} \left(12 - y^2\right) \, dy = \pi \left(16\sqrt{3} - \frac{45}{2}\right) \text{ units}^3.$$

* * * * *

Problem 6.



(a) The graph of $y = 2^x$, for $0 \leq x \leq 1$ is shown in the diagram. Rectangles, each of width $\frac{1}{n}$, are drawn under the curve. Given that $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$, show that the total area A of all n rectangles is given by $\frac{1}{n}\left(\frac{1}{2^{\frac{1}{n}}-1}\right)$.

(b) Find the limit of A in exact form as $n \to \infty$.

Let V be the volume of all n rectangles rotated about the x-axis.

(c) Find V in terms of n.

(d) State the limit of V in exact form as $n \to \infty$.

Solution.

Part (a).

$$A = \sum_{k=0}^{n-1} \frac{2^{k/n}}{n} = \frac{1}{n} \sum_{k=0}^{n-1} \left(2^{1/n}\right)^k = \frac{1}{n} \cdot \frac{1 - \left(2^{1/n}\right)^n}{1 - 2^{1/n}} = \frac{1}{n}\left(\frac{1-2}{1-2^{1/n}}\right) = \frac{1}{n}\left(\frac{1}{2^{1/n}-1}\right).$$

Part (b).

$$\lim_{n\to\infty} A = \lim_{n\to\infty} \frac{1/n}{2^{1/n}-1} = \lim_{m\to 0} \frac{m}{2^m - 1} = \lim_{m \to 0} \frac{1}{\ln 2 \cdot 2^m} = \frac{1}{\ln 2}.$$

Part (c).

$$V = \pi \sum_{k=0}^{n-1} \frac{1}{n} \left(2^{k/n}\right)^2 = \frac{\pi}{n} \sum_{k=0}^{n-1} \left(2^{2/n}\right)^k = \frac{\pi}{n} \left(\frac{1 - \left(2^{2/n}\right)^n}{1 - 2^{2/n}} \right)$$
$$= \frac{\pi}{n} \left(\frac{1-4}{1 - 2^{2/n}} \right) = \frac{3\pi}{n\left(4^{1/n} - 1\right)}.$$

Part (d).

$$\lim_{n\to\infty} V = \lim_{n\to\infty} \frac{3\pi}{n\left(4^{1/n}-1\right)} = 3\pi \lim_{n\to\infty} \frac{1/n}{4^{1/n}-1} = 3\pi \lim_{m\to 0} \frac{m}{4^m - 1}$$
$$= 3\pi \lim_{m\to 0} \frac{1}{4^m \ln 4} = 3\pi \left(\frac{1}{\ln 4}\right) = \frac{3\pi}{2\ln 2}.$$

* * * * *

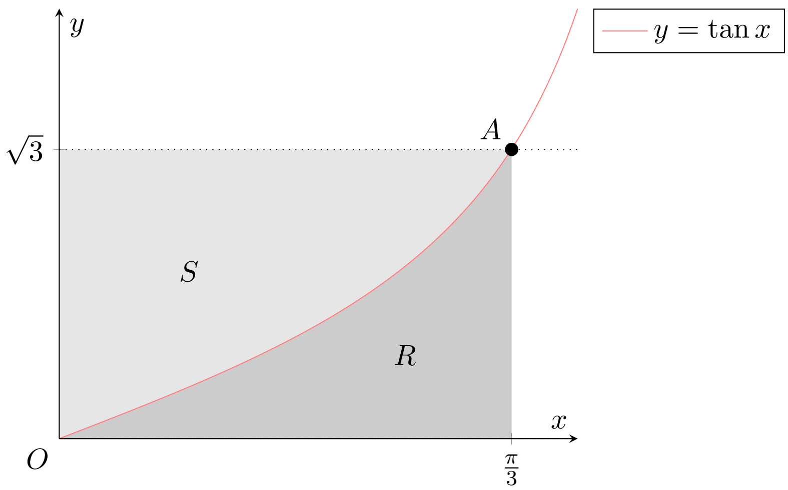
Problem 7. O is the origin and A is the point on the curve $y = \tan x$ where $x = \pi/3$.

(a) Calculate the area of the region R enclosed by the arc OA, the x-axis and the line $x = \pi/3$, giving your answer in an exact form.

(b) The region S is enclosed by the arc OA, the y-axis and the line $y = \sqrt{3}$. Find the volume of the solid of revolution formed when S is rotated through $360°$ about the x-axis, giving your answer in an exact form.

(c) Find $\int_0^{\sqrt{3}} \arctan y \, \mathrm{d}y$ in exact form.

Solution.



Part (a).

$$[R] = \int_0^{\pi/3} \tan x \, \mathrm{d}x = [\ln \sec x]_0^{\pi/3} = \ln 2 \text{ units}^2.$$

Part (b).

$$\text{Volume} = \pi \int_0^{\pi/3} \left[\left(\sqrt{3}\right)^2 - \tan^2 x\right] \mathrm{d}x = \pi \int_0^{\pi/3} \left(3 - \sec^2 x + 1\right) \mathrm{d}x$$
$$= \pi \left[4x - \tan x\right]_0^{\pi/3} = \left(\frac{4\pi^2}{3} - \sqrt{3}\pi\right) \text{ units}^3.$$

Part (c). Observe that $\int_0^{\sqrt{3}} \arctan y \, dy = [S] = [R \cup S] - [R] = (\pi/3) \cdot \sqrt{3} - \ln 2$.

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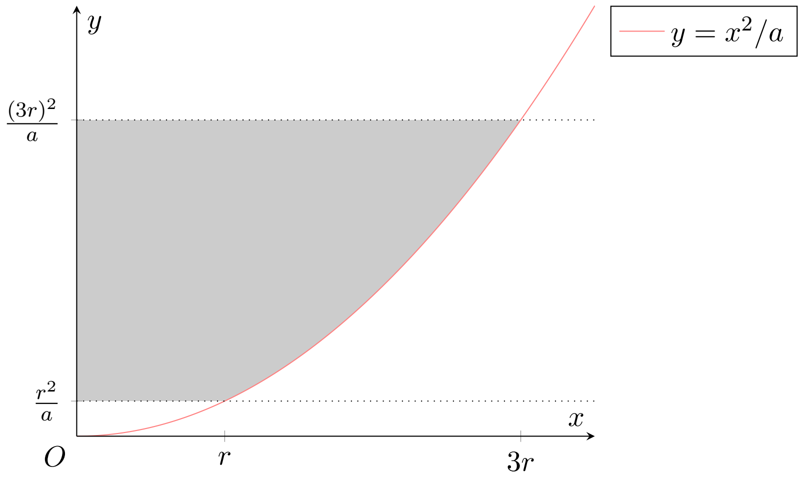
Problem 8. A portion of the curve $ay = x^2$, where a is a positive constant, is rotated about the vertical axis Oy to form the curved surface of an open bowl. The bowl has a horizontal circular base of radius r and a horizontal circular rim of radius $3r$.

(a) Prove that the depth of the bowl is $\frac{8r^2}{a}$.

(b) Find the volume of the bowl in terms of r and a.

(c) Given that the volume of the bowl is $\frac{\pi a^3}{10}$, find the depth of the bowl in terms of a only.

Solution. Note that $ay = x^2 \implies y = \frac{x^2}{a}$.



Part (a).

$$\text{Depth of bowl} = \frac{(3r)^2}{a} - \frac{r^2}{a} = \frac{8r^2}{a} \text{ units.}$$

Part (b).

$$\text{Volume} = \pi \int_{r^2/a}^{9r^2/a} ay \, dy = \pi \left[\frac{a}{2} y^2 \right]_{r^2/a}^{9r^2/a} = \frac{a\pi}{2} \cdot \frac{80r^4}{a^2} = \frac{40\pi r^4}{a} \text{ units}^3.$$

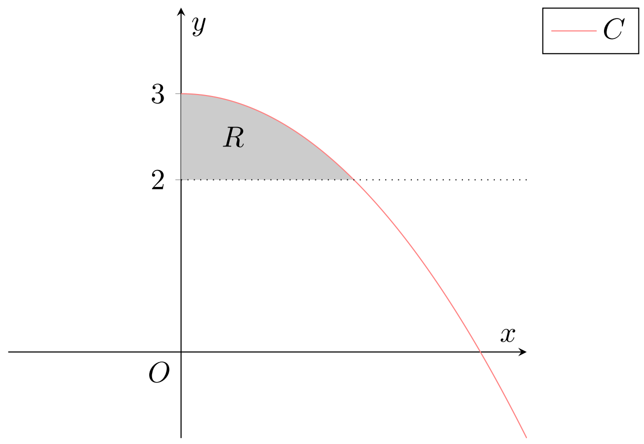
Part (c).

$$\frac{40\pi r^4}{a} = \frac{\pi a^3}{10} \implies 400r^4 = a^4 \implies 20r^2 = a^2 \implies r^2 = \frac{1}{20}a^2.$$

Hence, the depth of the bowl is

$$\frac{8}{a}\left(\frac{1}{20}a^2\right) = \frac{2}{5}a \text{ units.}$$

Problem 9. The diagram shows the region R bounded by part of the curve C with equation $y = 3 - x^2$, the y-axis and the line $y = 2$, lying in the first quadrant.



Write down the equation of the curve obtained when C is translated by 2 units in the negative y-direction.

Hence, or otherwise, show that the volume of the solid formed when R is rotated completely about the line $y = 2$ is given by $\pi \int_0^1 \left(1 - 2x^2 + x^4\right) \mathrm{d}x$ and evaluate this integral exactly.

Solution. Clearly, $C : y = 1 - x^2$.

Note that $3 - x^2 = 2 \implies x = \pm 1$, whence $x = 1$ since $x > 0$.

$$\text{Volume} = \pi \int_0^1 \left(1 - x^2\right)^2 \mathrm{d}x = \pi \int_0^1 \left(1 - 2x^2 + x^4\right) \mathrm{d}x$$
$$= \pi \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 = \frac{8}{15}\pi \text{ units}^3.$$

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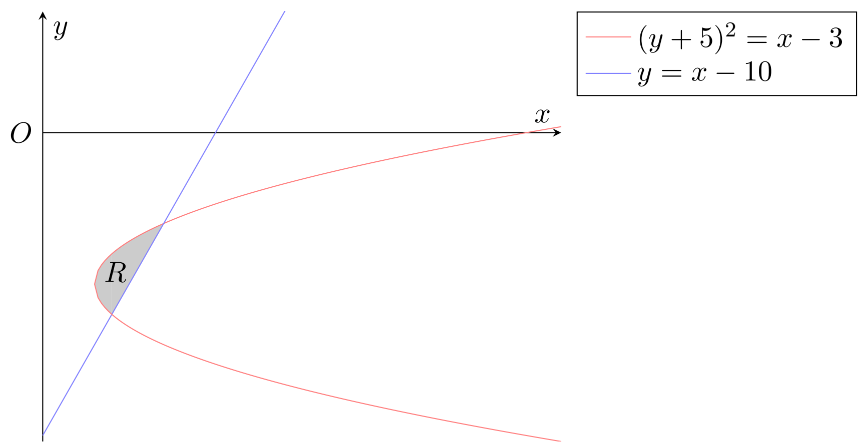
Problem 10. The diagram below shows a region R bounded by the curve $(y+5)^2 = x-3$ and the line $y = x - 10$. Find the volume of solid formed when R is rotated four right angles about

(a) the y-axis, and

(b) the x-axis.



Solution.

Part (a). Consider the intersections between $(y+5)^2 = x-3$ and $y = x-10$.

$$(y+5)^2 = (x-5)^2 = x-3 \implies x^2 - 11x + 28 = (x-4)(x-7) = 0.$$

Hence, $x = 4$ and $x = 7$, whence $y = -6$ and $y = -3$. Thus, the two curves intersect at $(4, -6)$ and $(7, -3)$.

Note that $(y+5)^2 = x-3 \implies x = 3 + (y+5)^2$ and $y = x - 10 \implies x = y + 10$.

$$\text{Volume} = \pi \int_{-6}^{-3} \left[(y+10)^2 - \left(3 + (y+5)^2\right)^2\right] \mathrm{d}y = 130 \text{ units}^3 \text{ (3 s.f.)}.$$

Part (b). Note that

$$(y+5)^2 = x-3 \implies \begin{cases} y = -5 + \sqrt{x-3}, & y \geq -5 \\ y = -5 - \sqrt{x-3}, & y < -5 \end{cases}$$

Thus,

$$\begin{aligned}\text{Volume} = \pi \int_3^4 &\left[\left(-5 - \sqrt{x-3}\right)^2 - \left(-5 + \sqrt{x+3}\right)^2\right] \mathrm{d}x \\ + \pi \int_4^7 &\left[(x-10)^2 - \left(-5 + \sqrt{x-3}\right)^2\right] \mathrm{d}x = 127 \text{ units}^3 \text{ (3 s.f.)}.\end{aligned}$$

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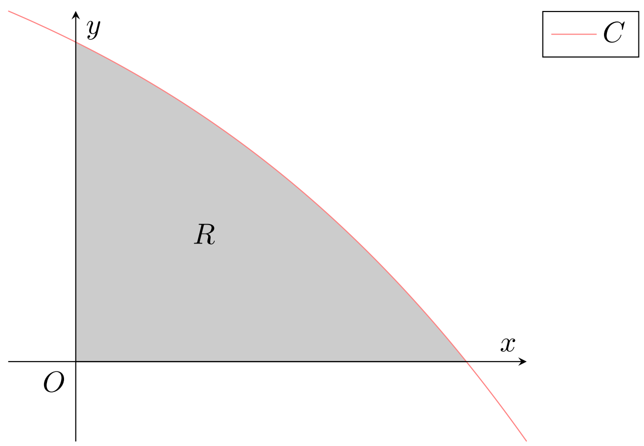
Problem 11. The curve C is defined by the following pair of parametric equations.

$$x = t - \frac{1}{t^2},\ y = 2 - t^2, \qquad t > 0.$$

Find the area of the finite region R enclosed by the curve C and the axes as well as the volume of solid obtained when R is rotated about the x-axis through 4 right-angles.

Solution.



Note that when $x = 0$, we have $t = 1$. Also note that when $y = 0$, we have $t = \sqrt{2}$, whence $x = \sqrt{2} - 1/2$. Thus,

$$\begin{aligned}[R] &= \int_0^{\sqrt{2}-1/2} y \,\mathrm{d}x = \int_1^{\sqrt{2}} \left(2 - t^2\right) \frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t \\ &= \int_1^{\sqrt{2}} \left(2 - t^2\right)\left(1 + \frac{2}{t^3}\right) \mathrm{d}t = 0.526 \text{ units}^2 \text{ (3 s.f.)}.\end{aligned}$$

Also,

$$\text{Volume} = \pi \int_0^{\sqrt{2}-\frac{1}{2}} y^2 \, \mathrm{d}x = \pi \int_1^{\sqrt{2}} \left(2 - t^2\right) \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t$$
$$= \pi \int_1^{\sqrt{2}} \left(2 - t^2\right) \left(1 + \frac{2}{t^3}\right) \mathrm{d}t = 1.19 \text{ units}^3 \text{ (3 s.f.)}.$$

* * * * *

Problem 12. Find the area enclosed by the ellipse $x = a \cos t$, $y = b \sin t$, where a and b are positive constants. Find also the volume of solid obtained when the region enclosed by the ellipse is rotated through π radians about the x-axis.

Solution. By symmetry, we only need to consider the area of the ellipse in the first quadrant. Note that $x = 0 \implies t = \pi/2$ and $x = a \implies t = 0$. Hence,

$$\text{Area} = 4 \int_0^a y \, \mathrm{d}x = 4 \int_{\pi/2}^0 y \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t = 4 \int_{\pi/2}^0 (b \sin t)(-a \sin t) \, \mathrm{d}t = 4ab \int_0^{\pi/2} \sin^2 t \, \mathrm{d}t$$
$$= 4ab \int_0^{\pi/2} \frac{1 - \cos 2t}{2} \, \mathrm{d}t = 2ab \left[t - \frac{\sin 2t}{2} \right]_0^{\pi/2} = \pi ab \text{ units}^2.$$

Also,

$$\text{Volume} = 2\pi \int_0^a y^2 \, \mathrm{d}x = 2\pi \int_{\pi/2}^0 y^2 \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t = 2\pi \int_{\pi/2}^0 (b \sin t)^2 (-a \sin t) \, \mathrm{d}t$$
$$= 2\pi ab^2 \int_0^{\pi/2} \sin^3 t \, \mathrm{d}t = 2\pi ab^2 \int_0^{\pi/2} \frac{3 \sin t - \sin 3t}{4} \, \mathrm{d}t$$
$$= \frac{1}{2} \pi ab^2 \left[-3 \cos t + \frac{1}{3} \cos 3t \right]_0^{\pi/2} = \frac{4\pi}{3} ab^2 \text{ units}^3.$$

* * * * *

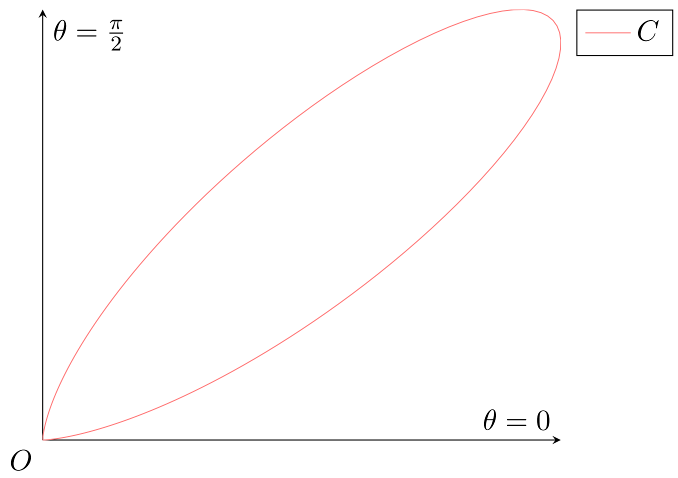
Problem 13. Find the polar equation of the curve C with equation $x^5 + y^5 = 5bx^2y^2$, where b is a positive constant. Sketch the part of the curve C where $0 \le \theta \le \frac{\pi}{2}$. Show, using polar coordinates, that the area A of the region enclosed by this part of the curve is given by

$$A = \frac{25b^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^4 \theta \cos^4 \theta}{(\cos^5 \theta + \sin^5 \theta)^2} \, \mathrm{d}\theta$$

By differentiating $\frac{1}{1+\tan^5 \theta}$ with respect to θ, or otherwise, find the exact value of A in terms of b.

Solution.

$$x^5 + y^5 = 5bx^2y^2 \implies (r \cos \theta)^5 + (r \sin \theta)^5 = 5b(r \cos \theta)^2 (r \sin \theta)^2$$
$$\implies r \left(\cos^5 \theta + \sin^5 \theta\right) = 5b \cos^2 \theta \sin^2 \theta \implies r = \frac{5b \cos^2 \theta \sin^2 \theta}{\cos^5 \theta + \sin^5 \theta}.$$



We have

$$A = \frac{1}{2}\int_0^{\pi/2}\left(\frac{5b\cos^2\theta\sin^2\theta}{\cos^5\theta+\sin^5\theta}\right)^2 \mathrm{d}\theta = \frac{1}{2}\int_0^{\pi/2}\frac{25b^2\cos^4\theta\sin^4\theta}{\left(\cos^5\theta+\sin^5\theta\right)^2}\mathrm{d}\theta$$
$$= \frac{25b^2}{2}\int_0^{\pi/2}\frac{\cos^4\theta\sin^4\theta}{\left(\cos^5\theta+\sin^5\theta\right)^2}\mathrm{d}\theta.$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{1+\tan^5\theta} = -\frac{5\tan^4\theta\sec^2\theta}{(1+\tan^5\theta)^2} = -5\left(\frac{\cos^{10}\theta}{\cos^5\theta+\sin^5\theta}\right)\left(\frac{\sin^4\theta}{\cos^6\theta}\right) = -\frac{5\cos^4\theta\sin^4\theta}{\cos^5\theta+\sin^5\theta}.$$

Hence,

$$A = \frac{-5b^2}{2}\int_0^{\pi/2} -\frac{5\cos^4\theta\sin^4\theta}{\left(\cos^5\theta+\sin^5\theta\right)^2}\mathrm{d}\theta = -\frac{5b^2}{2}\left[\frac{1}{1+\tan^5\theta}\right]_0^{\pi/2} = \frac{5b^2}{2}.$$

* * * * *

Problem 14. The polar equation of a curve is given by $r = \mathrm{e}^\theta$ where $0 \le \theta \le \pi/2$. Cartesian axes are taken at the pole O. Express x and y in terms of θ and hence find the Cartesian equation of the tangent at $\left(\mathrm{e}^{\pi/2}, \pi/2\right)$. The region R is bounded by the polar curve, tangent and the x-axis. Find the exact area of the region R.

Solution. We have $x = \mathrm{e}^\theta\cos\theta$ and $y = \mathrm{e}^\theta\sin\theta$. Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{e^\theta\cos\theta + e^\theta\sin\theta}{-e^\theta\sin\theta+e^\theta\cos\theta} = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}.$$

Hence,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\pi/2} = \frac{\cos(\pi/2)+\sin(\pi/2)}{\cos(\pi/2)-\sin(\pi/2)} = -1.$$

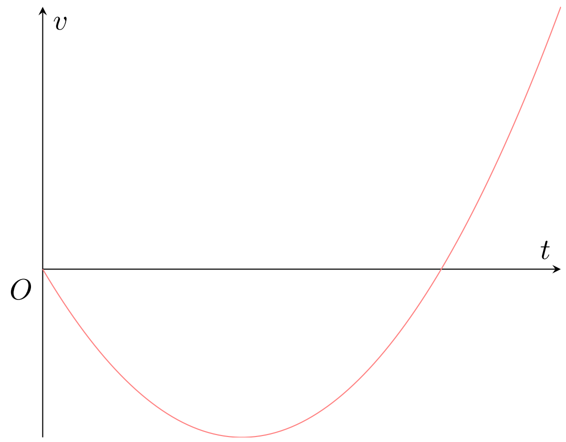
When $\theta = \pi/2$, we have $x = 0$ and $y = \mathrm{e}^{\pi/2}$. Hence, the tangent is given by

$$y - e^{\pi/2} = -(x-0) \implies y = -x + e^{\pi/2}.$$

Thus,

$$[R] = \frac{1}{2}\left(\mathrm{e}^{\pi/2}\right)\left(\mathrm{e}^{\pi/2}\right) - \frac{1}{2}\int_0^{\pi/2}\left(e^\theta\right)^2 \mathrm{d}\theta = \frac{\mathrm{e}^\pi}{2} - \frac{1}{2}\left[\frac{\mathrm{e}^{2\theta}}{2}\right]_0^{\pi/2} = \frac{e^\pi+1}{4}\text{ units}^2.$$

Problem 15.



The diagram shows the velocity-time graph of a particle moving in a straight line. The equation of the curve shown is $v = t(t - 10)$ where t seconds is the time and v ms^{-1} is the velocity. The particle starts at a point A on the line when $t = 0$.

Calculate

(a) the distance travelled by the particle before coming to instantaneous rest, and

(b) the time at which the particle returns to A.

Solution.

Part (a). For instantaneous rest, $v = 0$. Hence, $t(t - 10) = 0$, whence $t = 10$. Note that we reject $t = 0$ since $t > 0$. The distance travelled by the particle before coming to instantaneous rest is hence

$$-\int_0^{10} v \, \mathrm{d}t = -\int_0^{10} t(t-10) \, \mathrm{d}t = -\int_0^{10} \left(t^2 - 10t\right) \, \mathrm{d}t = -\left[\frac{t^3}{3} - \frac{10t^2}{2}\right]_0^{10} = \frac{500}{3} \text{ m}.$$

Part (b). When the particle returns to A, $s = 0$. Let the time at which the particle returns to A be t_0.

$$\int_0^{t_0} v \, \mathrm{d}t = \int_0^{t_0} t(t-10) \, \mathrm{d}t = \left[\frac{t_0^3}{3} - \frac{10t_0^2}{2}\right]_0^{t_0} = \frac{1}{3}t_0^3 - 5t_0^2 = \frac{1}{3}t_0^2 \left(t_0 - 15\right) = 0.$$

Thus, $t_0 = 15$. Note that we reject $t_0 = 0$ since $t_0 > 0$. It hence takes the particle 15 seconds to return to A.

Self-Practice B8

Problem 1.

(a) Find $\int x \sin^2 x \, \mathrm{d}x$.

(b) The region R is bounded by the curve $y = \sqrt{x} \sin x$, the lines $x = 0$ and $x = \pi$, and the x-axis. Find the volume of the solid of revolution formed when R is rotated through 4 right angles about the x-axis.

(c) Hence, calculate the volume of the solid of revolution formed when S is rotated through 4 right angles about the x-axis, where S is the region bounded by the curve $y = \sqrt{x} \sin x$, the lines $x = \pi$ and $y = \sqrt{\pi}$, and the y-axis.

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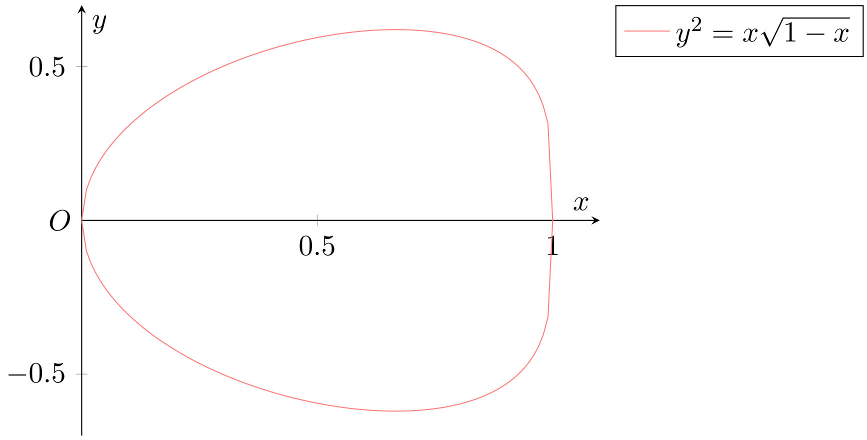
Problem 2. The diagram shows the curve C with the equation $y^2 = x\sqrt{1-x}$. The region enclosed by C is denoted by R.



(a) Write down an integral that gives the area of R, and evaluate this integral numerically.

(b) The part of R above the x-axis is rotated through 2π radians about the x-axis. By using the substitution $u = 1 - x$, or otherwise, find the exact value of the volume obtained.

(c) Find the exact x-coordinate of the maximum point of C.

* * * * *

Problem 3.

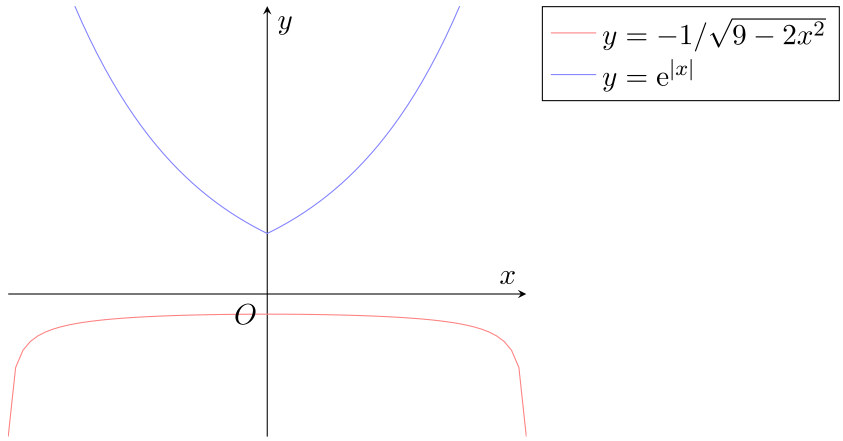
(a) Find the exact value of $\int_0^{5\pi/3} \sin^2 x \, \mathrm{d}x$. Hence, find the exact value of $\int_0^{5\pi/3} \cos^2 x \, \mathrm{d}x$.

(b) The region R is bounded by the curve $y = x^2 \sin x$, the line $x = \frac{1}{2}\pi$ and the part of the x-axis between 0 and $\frac{1}{2}\pi$. Find

  (i) the exact area of R,

  (ii) the numerical value of the volume of revolution formed when R is rotated completely about the x-axis, giving your answer correct to 3 decimal places.

Problem 4. The diagram below shows the graphs of $y = -\frac{1}{\sqrt{9-2x^2}}$ and $y = e^{|x|}$.



(a) The region A is bounded by the curves $y = -\frac{1}{\sqrt{9-2x^2}}$ and $y = e^{|x|}$, and the lines $x = -1$ and $x = 2$. Find the area of A, giving your answer to 3 significant figures.

(b) The region bounded by the curves $y = -\frac{1}{\sqrt{9-2x^2}}$, $y = e^{|x|}$, the y-axis and the line $x = 2$ is rotated through 2π radians about the y-axis. Prove that the volume generated is $2\pi\left(e^2+2\right)$.

* * * * *

Problem 5.

(a) (i) The region S, is enclosed by the x-axis, the line $x = 1$ and the curve given by the parametric equations

$$x = (1+t)^{3/2}, \quad y = (1-t)^{1/2}, \quad t \in [0,1].$$

Find the exact area of S.

(ii) Find also the volume of the solid obtained when the region S is rotated about the y-axis.

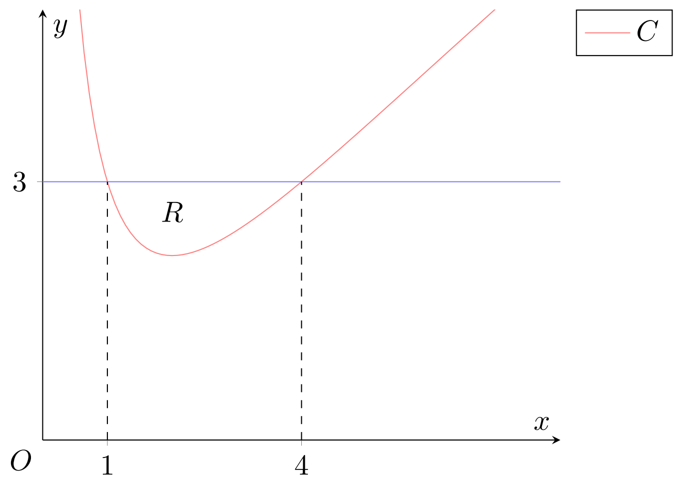
(b) The region R is bounded by the curve $y = \left(\frac{x-2}{4-x}\right)^{1/4}$, the line $x = 2$ and the line $y = 1$. By using the substitution $x = 2\left(1+\cos^2\theta\right)$, or otherwise, find the exact volume of the solid generated when R is rotated through four right angles about the x-axis.

* * * * *

Problem 6. The diagram shows the region R in the first quadrant bounded by the curve C with equation $y = \sqrt{x} + \frac{2}{\sqrt{x}}$ and the line $y = 3$. The line and the curve intersect at the points $(1,3)$ and $(4,3)$. Calculate the exact area of R. Write down the equation of the curve obtained when C is translated by 3 units in the negative y-direction. Hence, or otherwise, show that the volume of the solid formed when R is rotated completely about the line $y = 3$ is given by

$$\pi \int_1^4 \left(x - 6\sqrt{x} + 13 - \frac{12}{\sqrt{x}} + \frac{4}{x}\right) dx,$$

and evaluate this integral exactly.



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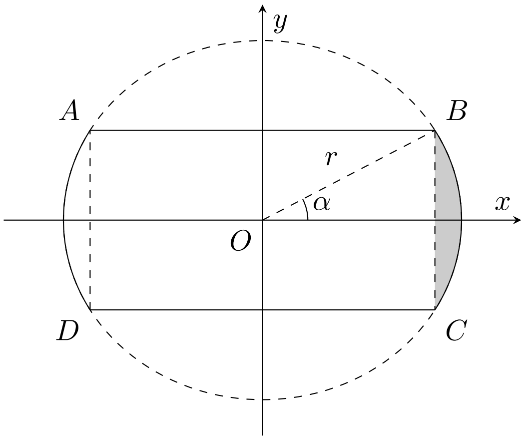
Problem 7. The diagram shows the circle, centre O and radius r, with equation $x^2+y^2=r^2$. The points A, B, C, D on the circle form a rectangle with sides parallel to the axes. $\angle AOD = \angle BOC = 2\alpha$. The region bounded by the line AB, the line DC and the circular arc BC and AD is rotated about the x-axis to form a solid of rotation S.



(a) Show that the volume obtained by rotating the shaded part of the region about the x-axis is $\frac{1}{3}\pi r^3\left(\cos^3\alpha - 3\cos\alpha + 2\right)$.

(b) Show that the total volume of S is $\frac{4}{3}\pi r^3\left(1-\cos^3\alpha\right)$.

(c) Given that the volume of S is half the volume of a sphere of radius r, find the value of α.

* * * * *

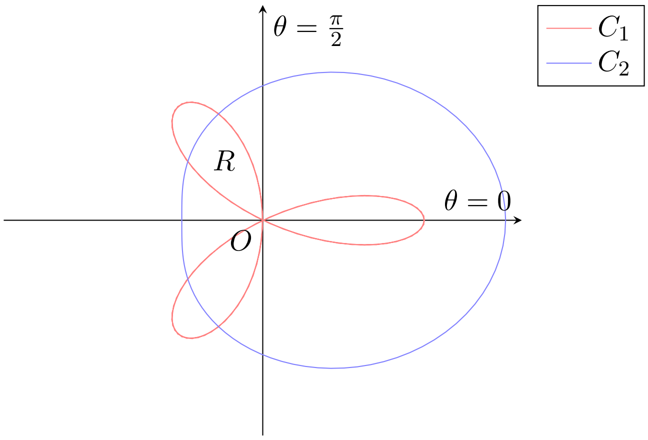
Problem 8. An ellipse E has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are positive constants. Show that the area A of the region enclosed by E is given by $A=\frac{4b}{a}\int_0^a \sqrt{(a^2-x^2)}\,\mathrm{d}x$. By using the substitution $x=a\sin\theta$, or otherwise, find the value of A in terms of a, b, and π. Show on a sketch the region R of points inside the ellipse E such that $x>0$ and $y<x$. Given that $a^2=3b^2$, find the area of R in terms of a and π.

* * * * *

Problem 9. Sketch the polar curve $r=a(1-\sin 2\theta)$, where $a>0$ and $0\le\theta<2\pi$. Prove that the area enclosed by each loop of the curve is $\frac{3}{4}\pi a^2$.

Problem 10. The diagram shows the curves C_1 and C_2 whose respective polar equations are

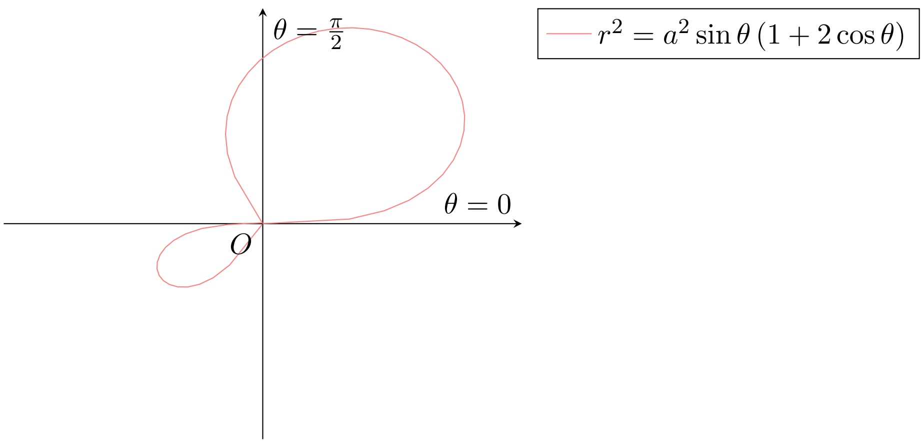
$$r = \cos 3\theta, \qquad (0 \le \theta \le 2\pi)$$
$$r = 1 + \frac{1}{2}\cos\theta. \qquad (0 \le \theta \le 2\pi)$$



R is the region bounded by the curve C_2 and one loop of the curve C_1. Find the area of the region R.

* * * * *

Problem 11. The curve with polar equation $r^2 = a^2 \sin\theta(1 + 2\cos\theta)$, where $r \ge 0$ and a is a positive constant, is shown. Show that the area of the larger loop is nine times that of the smaller loop.



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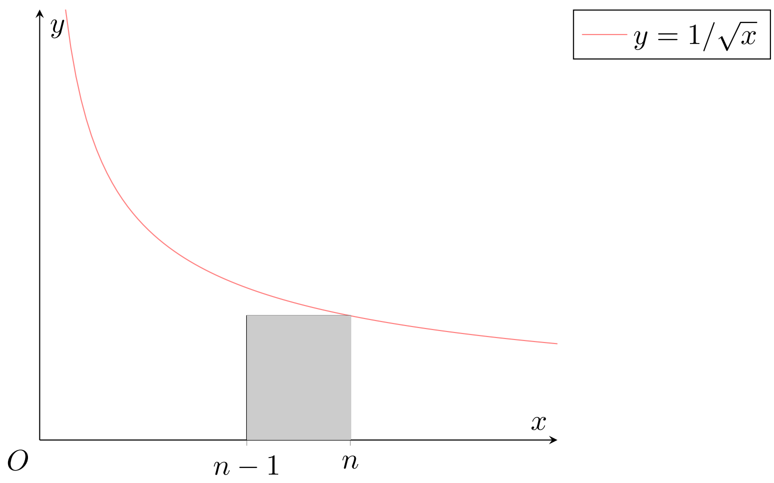
Problem 12. The diagram shows a sketch of the graph of $y = 1/\sqrt{x}$.



By considering the shaded rectangle, and the area of the region between the graph and the x-axis for $n - 1 \leq x \leq n$, where $n \geq 1$, show that

$$\frac{1}{\sqrt{n}} < 2\left(\sqrt{n} - \sqrt{n-1}\right).$$

Deduce that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

Show also that

$$\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - \sqrt{n}\right).$$

Deduce that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2.$$

Hence, find a value of N for which

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{N}} > 1000.$$

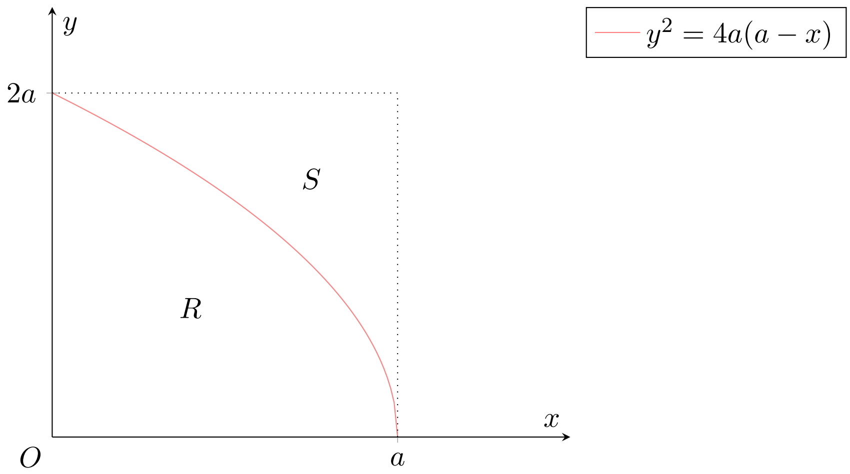
Assignment B8

Problem 1. The diagram shows the region R, which is bounded by the axes and the part of the curve $y^2 = 4a(a-x)$ lying in the first quadrant.

Find, in terms of a, the volume, V_x, of the solid formed when R is rotated completely about the x-axis.

The volume of the solid formed when R is rotated completely about the y-axis is V_y. Show that $V_y = \frac{8}{15}V_x$.

The region S, lying in the first quadrant, is bounded by the curve $y^2 = 4a(a-x)$ and the lines $x = a$ and $y = 2a$. Find, in terms of a, the volume of the solid formed when S is rotated completely about the y-axis.



Solution.

$$V_x = \pi \int_0^a y^2 \,\mathrm{d}x = \pi \int_0^a 4a(a-x)\,\mathrm{d}x = 4\pi a \left[ax - \frac{1}{2}x^2\right]_0^a = 2\pi a^3 \text{ units}^3.$$

Note that

$$x = a - \frac{y^2}{4a} \implies x^2 = \left(a - \frac{y^2}{4a}\right)^2 = a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4.$$

Hence,

$$\begin{aligned} V_y &= \pi \int_0^{2a} x^2 \,\mathrm{d}y = \pi \int_0^{2a} \left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right) \mathrm{d}y \\ &= \pi \left[a^2 y - \frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{1}{16a^2}\left(\frac{y^5}{5}\right)\right]_0^{2a} = \frac{16}{15}\pi a^3 = \frac{8}{15}V_x. \end{aligned}$$

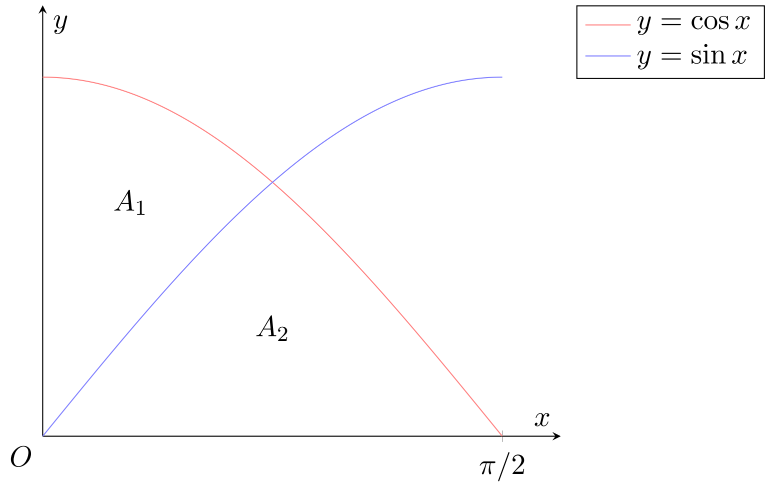
We have

$$\text{Volume} = \text{Volume of cylinder} - V_y = \pi\left(a^2\right)(2a) - \frac{16}{15}\pi a^3 = \frac{14}{15}\pi a^3 \text{ units}^3.$$

Problem 2. The region bounded by the axes and the curve $y = \cos x$ from $x = 0$ to $x = \frac{1}{2}\pi$ is divided into two parts, of areas A_1 and A_2, by the curve $y = \sin x$.

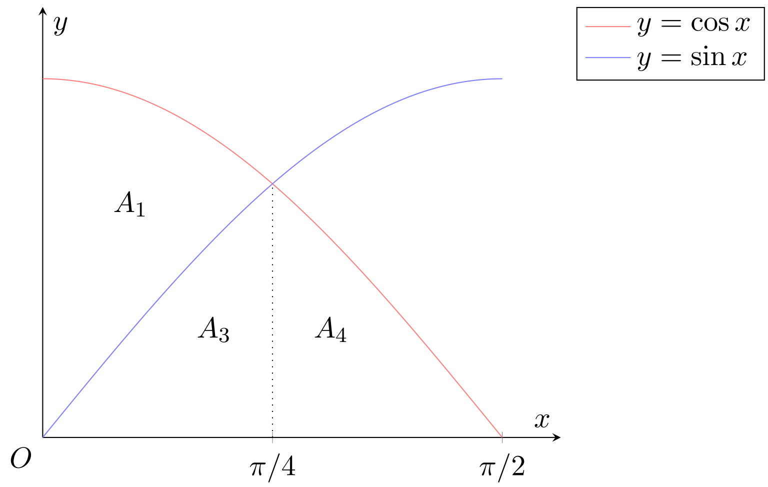
(a) Prove that $A_2 = \sqrt{2}A_1$.

(b) Find the volume of the solid obtained when the region with area A_2 is rotated about the y-axis through 2π radians. Give your answer in exact form.



Solution.

Part (a).



Let A_3 and A_4 be the areas as defined on the diagram above. By the symmetry of $y = \sin x$ and $y = \cos x$ about $x = \pi/4$, we have $A_3 = A_4$.

$$A_3 = \int_0^{\pi/4} \sin x \, \mathrm{d}x = [-\cos x]_0^{\pi/4} = 1 - \frac{\sqrt{2}}{2}.$$

Hence,

$$A_1 = \int_0^{\pi/4} \cos x \, \mathrm{d}x - A_3 = [\sin x]_0^{\pi/4} - \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} - 1 + \frac{\sqrt{2}}{2} = \sqrt{2} - 1.$$

Thus,

$$A_2 = 2A_3 = 2\left(1 - \frac{\sqrt{2}}{2}\right) = \sqrt{2}\left(\sqrt{2} - 1\right) = \sqrt{2}A_1.$$

Part (b). Let V_3 and V_4 be the volumes of the solids obtained when A_3 and A_4 are rotated about the y-axis through 2π radians, respectively.

$$V_3 = 2\pi \int_0^{\pi/4} xy \, \mathrm{d}x = 2\pi \int_0^{\pi/4} x \sin x \, \mathrm{d}x.$$

Integrating by parts,

	D	I
$+$	x	$\sin x$
$-$	1	$-\cos x$
$+$	0	$-\sin x$

Thus,

$$V_3 = 2\pi \left[-x \cos x + \sin x\right]_0^{\pi/4} = \sqrt{2}\pi \left(1 - \frac{\pi}{4}\right).$$

Also,

$$V_4 = 2\pi \int_{\pi/4}^{\pi/2} xy \, \mathrm{d}x = 2\pi \int_{\pi/4}^{\pi/2} x \cos x \, \mathrm{d}x.$$

Integrating by parts,

	D	I
$+$	x	$\cos x$
$-$	1	$\sin x$
$+$	0	$-\cos x$

Thus,

$$V_4 = 2\pi \left[x \sin x + \cos x\right]_{\pi/4}^{\pi/2} = \pi^2 - \sqrt{2}\pi \left(1 + \frac{\pi}{4}\right).$$

Hence, the required volume is

$$V_3 + V_4 = \sqrt{2}\pi \left(1 - \frac{\pi}{4}\right) + \pi^2 - \sqrt{2}\pi \left(1 + \frac{\pi}{4}\right) = \pi^2 \left(1 - \frac{\sqrt{2}}{2}\right) \text{ units}^3.$$

Problem 3. A curve has parametric equations

$$x = \cos^2 t,\ y = \sin^3 t,\ 0 \le t \le \frac{1}{2}\pi.$$

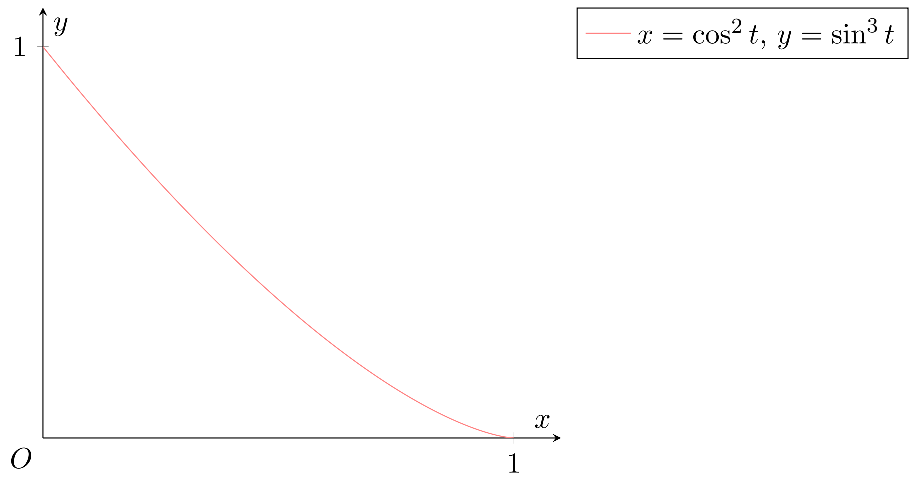
(a) Sketch the curve.

(b) Show that the area under the curve for $0 \le t \le \frac{1}{2}\pi$ is $2\int_0^{\pi/2} \cos t \sin^4 t \,\mathrm{d}t$, and find the exact value of the area.

(c) Find the volume of the solid obtained when the region in (b) is rotated about the y-axis through 2π radians.

Solution.

Part (a).



Part (b). Note that $x = 0 \implies t = \frac{\pi}{2}$ and $x = 1 \implies t = 0$. Hence,

$$\begin{aligned}\text{Area} = \int_0^1 y \,\mathrm{d}x = \int_{\pi/2}^0 y\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_{\pi/2}^0 \sin^3 t(-2\cos t \sin t)\,\mathrm{d}t = 2\int_0^{\pi/2} \cos t \sin^4 t \,\mathrm{d}t \\ = 2\left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = \frac{2}{5} \text{ units}^2.\end{aligned}$$

Part (c).

$$\begin{aligned}\text{Volume} &= 2\pi\int_0^1 xy \,\mathrm{d}x = 2\pi\int_{\pi/2}^0 \cos^2 t \sin^3 t(-2\cos t\sin t)\,\mathrm{d}t = 4\pi\int_0^{\pi/2} \cos^3 t \sin^4 t \,\mathrm{d}t \\ &= 4\pi\int_0^{\pi/2} \sin^4 t\left(1-\sin^2 t\right)\cos t \,\mathrm{d}t = 4\pi\int_0^{\pi/2}\left(\sin^4 t - \sin^6 t\right)\cos t \,\mathrm{d}t \\ &= 4\pi\left[\frac{\sin^5 t}{5} - \frac{\sin^7 t}{7}\right]_0^{\pi/2} = \frac{8\pi}{35} \text{ units}^3.\end{aligned}$$

* * * * *

Problem 4.

(a) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

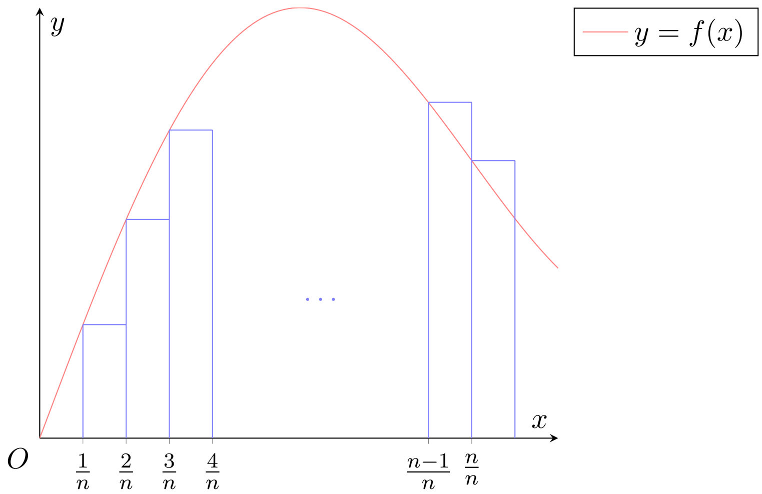
$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right]$$

is $\int_0^1 f(x)\,\mathrm{d}x$.

(b) Hence, evaluate $\lim_{n\to\infty}\frac{1}{n}\left(\frac{\sqrt[3]{1}+\sqrt[3]{2}+\ldots+\sqrt[3]{n}}{\sqrt[3]{n}}\right)$.

Solution.

Part (a).



The area of the rectangles in the above figure is given by

$$\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right].$$

This gives an approximation of the signed area under the curve from $x=\frac{1}{n}$ to $x=\frac{n}{n}=1$. As $n\to\infty$, the widths of the rectangles become smaller and the approximation becomes exact. Hence,

$$\lim_{n\to\infty}\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right]=\int_0^1 f(x)\,\mathrm{d}x.$$

Part (b).

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{\sqrt[3]{1}+\sqrt[3]{2}+\ldots+\sqrt[3]{n}}{\sqrt[3]{n}}\right)=\lim_{n\to\infty}\frac{1}{n}\left[\sqrt[3]{\frac{1}{n}}+\sqrt[3]{\frac{2}{n}}+\ldots+\sqrt[3]{\frac{n}{n}}\right]$$
$$=\int_0^1\sqrt[3]{x}\,\mathrm{d}x=\left[\frac{x^{4/3}}{4/3}\right]_0^1=\frac{3}{4}.$$

* * * * *

Problem 5. The function f satisfies $f'(x)>0$ for $a\le x\le b$, and g is the inverse of f. By making a suitable change of variable, prove that

$$\int_a^b f(x)\,\mathrm{d}x=b\beta-a\alpha-\int_\alpha^\beta g(y)\,\mathrm{d}y$$

where $\alpha=f(a)$ and $\beta=f(b)$. Interpret this formula geometrically by means of a sketch where α and a are positive. Verify this result for the case where $f(x)=\mathrm{e}^{2x}$, $a=0$, $b=1$.

Prove similarly and interpret geometrically the formula

$$2\pi\int_a^b xf(x)\,\mathrm{d}x=\pi(b^2\beta-a^2\alpha)-\pi\int_\alpha^\beta[g(y)]^2\,\mathrm{d}y.$$

Solution. Observe that $y = f(x) \implies \mathrm{d}y = f'(x)\,\mathrm{d}x$. Hence,

$$\int_\alpha^\beta g(y)\,\mathrm{d}y = \int_a^b f^{-1}(f(x))f'(x)\,\mathrm{d}x = \int_a^b xf'(x)\,\mathrm{d}x.$$

Integrating by parts,

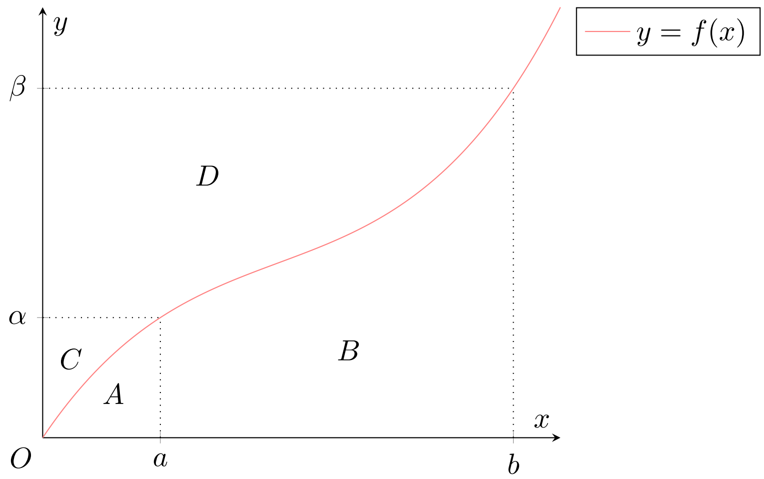
	D	I
$+$	x	$f'(x)$
$-$	1	$f(x)$

Hence,

$$\int_\alpha^\beta g(y)\,\mathrm{d}y = [xf(x)]_a^b - \int_a^b f(x)\,\mathrm{d}x = b\beta - a\alpha - \int_a^b f(x)\,\mathrm{d}x.$$

Thus,

$$\int_a^b f(x)\,\mathrm{d}x = b\beta - a\alpha - \int_\alpha^\beta g(y)\,\mathrm{d}y.$$



Consider the above diagram. We clearly have $[A \cup C] = a\alpha$, $[A \cup B \cup C \cup D] = b\beta$, $[B] = \int_a^b f(x)\,\mathrm{d}x$ and $[D] = \int_\alpha^\beta g(y)\,\mathrm{d}y$. Thus,

$$\int_a^b f(x)\,\mathrm{d}x = [B] = [A \cup B \cup C \cup D] - [A \cup C] - [D] = b\beta - a\alpha - \int_\alpha^\beta g(y)\,\mathrm{d}y.$$

Using the standard way, we get

$$\int_0^1 \mathrm{e}^{2x}\,\mathrm{d}x = \left[\frac{1}{2}\mathrm{e}^{2x}\right]_0^1 = \frac{\mathrm{e}^2 - 1}{2}.$$

We now use the formula. Let $f(x) = e^{2x}$. Then $g(x) = \frac{1}{2}\ln x$. Hence, $\alpha = g(0) = 1$ and $\beta = g(1) = e^2$. Invoking the above formula,

$$\int_0^1 \mathrm{e}^{2x}\,\mathrm{d}x = 1\left(\mathrm{e}^2\right) - 0(1) - \int_1^{\mathrm{e}^2} \frac{1}{2}\ln x\,\mathrm{d}x = \mathrm{e}^2 - \frac{1}{2}\left[x\ln x - x\right]_1^{\mathrm{e}^2} = \frac{\mathrm{e}^2 - 1}{2}.$$

Hence, the formula holds for the above case.

Similar to the above part, we have

$$\int_\alpha^\beta [g(y)]^2 \, \mathrm{d}y = \int_\alpha^\beta \left[f^{-1}(f(x))\right]^2 f'(x) \, \mathrm{d}x = \int_a^b x^2 f'(x) \, \mathrm{d}x.$$

Integrating by parts,

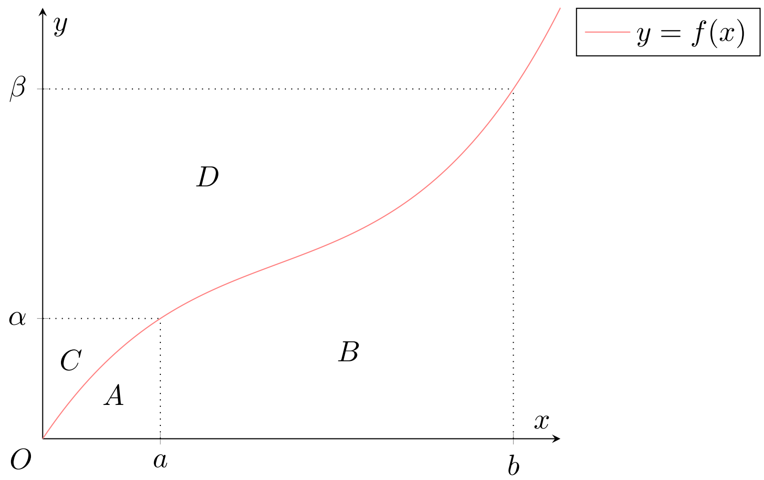
	D	I
$+$	x^2	$f'(x)$
$-$	$2x$	$f(x)$

Thus,

$$\int_\alpha^\beta [g(y)]^2 \, \mathrm{d}y = \left[x^2 f(x)\right]_a^b - 2\int_a^b x f(x) \, \mathrm{d}x = b^2\beta - a^2\alpha - 2\int_a^b x f(x) \, \mathrm{d}x.$$

Rearranging,

$$2\pi \int_a^b x f(x) \, \mathrm{d}x = \pi\left(b^2\beta - a^2\alpha\right) - \pi \int_\alpha^\beta [g(y)]^2 \, \mathrm{d}y.$$



Let $V(R)$ represent the volume of the solid obtained when a region R is rotated completely about the y-axis.

We clearly have $V(A \cup B \cup C \cup D) = \pi b^2 \beta$, $V(A \cup C) = \pi a^2 \alpha$, $V(B) = 2\pi \int_a^b x f(x) \, \mathrm{d}x$ (using the shell method), and $V(D) = \pi \int_\alpha^\beta [g(y)]^2 \, \mathrm{d}y$ (using the disc method). Thus,

$$2\pi \int_a^b x f(x) \, \mathrm{d}x = V(B) = V(A \cup B \cup C \cup D) - V(A \cup C) - V(D)$$
$$= \pi b^2 \beta - \pi a^2 \alpha - \pi \int_\alpha^\beta [g(y)]^2 \, \mathrm{d}y = \pi\left(b^2\beta - a^2\alpha\right) - \pi \int_\alpha^\beta [g(y)]^2 \, \mathrm{d}y.$$

B9 Applications of Integration II - Arc Length and Surface Area

Tutorial B9

Problem 1. Calculate the exact length of each of the arcs of the following curves.

(a) $y^3 = x^2$ for $-1 \leq x \leq 1$.

(b) $x = t^2 - 1$, $y = t^3 + 1$ from $t = 0$ to $t = 1$.

(c) $r = a\cos\theta$ from $\theta = 0$ to $\theta = \pi/2$.

Solution.
Part (a). Note that

$$y^3 = x^2 \implies y = x^{2/3} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{3}x^{-1/3}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \left(\frac{2}{3}x^{-1/3}\right)^2} = \sqrt{1 + \frac{4}{9}x^{-2/3}}.$$

Thus,

$$\begin{aligned}\text{Length} &= \int_{-1}^{1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{-1}^{1} \sqrt{1 + \frac{4}{9}x^{-2/3}}\,\mathrm{d}x = 2\int_{0}^{1} \sqrt{1 + \frac{4}{9}x^{-2/3}}\,\mathrm{d}x \\ &= 3\int_0^1 \frac{2}{3}x^{-1/3}\sqrt{x^{2/3} + \frac{4}{9}}\,\mathrm{d}x = 3\left[\frac{2}{3}\left(x^{2/3} + \frac{4}{9}\right)^{3/2}\right]_0^1 = \frac{2}{27}\left(13\sqrt{13} - 8\right) \text{ units.}\end{aligned}$$

Part (b). Since the arc length of a curve is invariant under translation, it suffices to find the arc length of the curve with parametric equations $x = t^2, y = t^3$, $0 \leq t \leq 1$. The Cartesian equation of this curve is $y = x^{3/2}$, $0 \leq x \leq 1$, which is the inverse of $y = x^{2/3}$, $0 \leq x \leq 1$. From part (a), the required arc length is

$$\frac{1}{2} \cdot \frac{2}{27}\left(13\sqrt{13} - 8\right) = \frac{1}{27}\left(13\sqrt{13} - 8\right) \text{ units.}$$

Part (c). Since $r = a\cos\theta$, $0 \leq \theta \leq \pi/2$ describes the top half of a circle with centre $(a/2, 0)$ and diameter a, the arc length of the curve is $\pi a/2$ units.

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Problem 2. Find the exact areas of the surfaces generated by completely rotating the following arcs about the (i) x-axis and (ii) y-axis.

(a) The line $2y = x$ between the origin and the point $(4, 2)$.

(b) The curve $x = t^3 - 3t + 2$, $y = 3\left(t^2 - 1\right)$, $t \in \mathbb{R}$ from $t = 1$ to $t = 2$.

Solution.

Part (a).

Part (a)(i). When rotated about the x-axis, the curve forms a cone with slant height $\sqrt{4^2+2^2} = 2\sqrt{5}$ and radius 2. Hence, the required surface area is $\pi(2)(2\sqrt{5}) = 4\sqrt{5}\pi$ units^2.

Part (a)(ii). When rotated about the y-axis, the curve forms a cone with slant height $\sqrt{4^2+2^2} = 2\sqrt{5}$ and radius 4. Hence, the required surface area is $\pi(4)(2\sqrt{5}) = 8\sqrt{5}\pi$ units^2.

Part (b). Note that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 - 3, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 6t.$$

Hence,

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \sqrt{(3t^2-3)^2 + (6t)^2} = \sqrt{(3t^2+3)^2} = 3t^2 + 3.$$

Part (b)(i).

$$\begin{aligned}\text{Area} &= 2\pi \int_1^2 y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = 2\pi \int_1^2 3\left(t^2-1\right)\left(3t^2+3\right)\,\mathrm{d}t \\ &= 18\pi \int_1^2 \left(t^4-1\right)\,\mathrm{d}t = 18\pi\left[\frac{1}{5}t^5 - t\right]_1^2 = \frac{468}{5}\pi \text{ units}^2.\end{aligned}$$

Part (b)(ii).

$$\text{Area} = 2\pi \int_1^2 x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = 2\pi \int_1^2 \left(t^3-3t+2\right)\left(3t^2+3\right)\,\mathrm{d}t$$

$$= 6\pi \int_1^2 \left(t^5 - 2t^3 + 2t^2 - 3t + 2\right)\,\mathrm{d}t = 6\pi\left[\frac{1}{6}t^6 - \frac{2}{4}t^4 - \frac{2}{3}t^3 - \frac{3}{2}t^2 + 2t\right]_1^2 = 31\pi \text{ units}^2.$$

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Problem 3. The section of the curve $y = \mathrm{e}^x$ between $x=0$ and $x=1$ is rotated through one revolution about

(a) the x-axis.

(b) the y-axis.

Find the numerical values of the areas of the surfaces obtained.

Solution.

Part (a).

$$\text{Area} = 2\pi \int_0^1 y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = 2\pi \int_0^1 \mathrm{e}^x \sqrt{1+\mathrm{e}^{2x}}\,\mathrm{d}x = 22.9 \text{ units}^2 \text{ (3 s.f.)}.$$

Part (b). Note that $y = \mathrm{e}^x \implies x = \ln y$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \implies \frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{e}^{-x}$.

$$\text{Area} = 2\pi \int_1^{\mathrm{e}} x \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y = 2\pi \int_0^1 \ln y \sqrt{1+\mathrm{e}^{-2x}}\,\mathrm{d}x = 7.05 \text{ units}^2 \text{ (3 s.f.)}.$$

Problem 4. The curve $y^2 = \frac{1}{3}x(1-x)^2$ has a loop between $x = 0$ and $x = 1$. Prove that the total length of the loop is $\frac{4\sqrt{3}}{3}$.

Solution. Since the curve is even with respect to y, it is symmetric about the x-axis. We thus only consider the part of the curve above the x-axis, i.e. $y \geq 0$, where $y = (1-x)\sqrt{x/3}$. Differentiating,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{3}}\left(-\sqrt{x} + \frac{1-x}{2\sqrt{x}}\right) = \frac{1-3x}{2\sqrt{3x}} \implies 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \frac{(1-3x)^2}{12x} = \frac{(1+3x)^2}{12x}.$$

Thus,

$$\text{Length} = 2\int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = 2\int_0^1 \frac{1+3x}{\sqrt{12x}}\,\mathrm{d}x = \frac{1}{\sqrt{12}}\left[\frac{x^{1/2}}{1/2} + \frac{3x^{3/2}}{3/2}\right]_0^1 = \frac{4\sqrt{3}}{3} \text{ units.}$$

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Problem 5. The tangent at a point P on the curve $x = a\left(t - \frac{1}{3}t^3\right)$, $y = at^2$ cuts the x-axis at T. Prove that the distance of the point T from the origin O is half the length of the arc OP.

Solution. Let P be the point on the curve with parameter $t = t_P$. Note that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a\left(1 - t^2\right), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2at.$$

Thus,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \left[a\left(1-t^2\right)\right]^2 + (2at)^2 = a^2\left(t^2+1\right)^2.$$

Thus,

$$\begin{aligned}\text{Length of arc } OP &= \int_0^{t_P} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = a\int_0^{t_P} \left(t^2+1\right)\,\mathrm{d}t \\ &= a\left[\frac{t^3}{3} + t\right]_0^{t_P} = a\left(\frac{t_P^3}{3} + t_P\right) \text{ units.}\end{aligned}$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{2at}{a\left(1-t^2\right)} = \frac{2t}{1-t^2}.$$

Hence, the equation of the tangent at P is given by

$$y - at_P^2 = \frac{2t_P}{1-t_P^2}\left[x - a\left(t_P - \frac{t_P^3}{3}\right)\right].$$

At T, $x = OT$ and $y = 0$. Hence,

$$0 - at_P^2 = \frac{2t_P}{1-t_P^2}\left[OT - a\left(t_P - \frac{t_P^3}{3}\right)\right],$$

whence

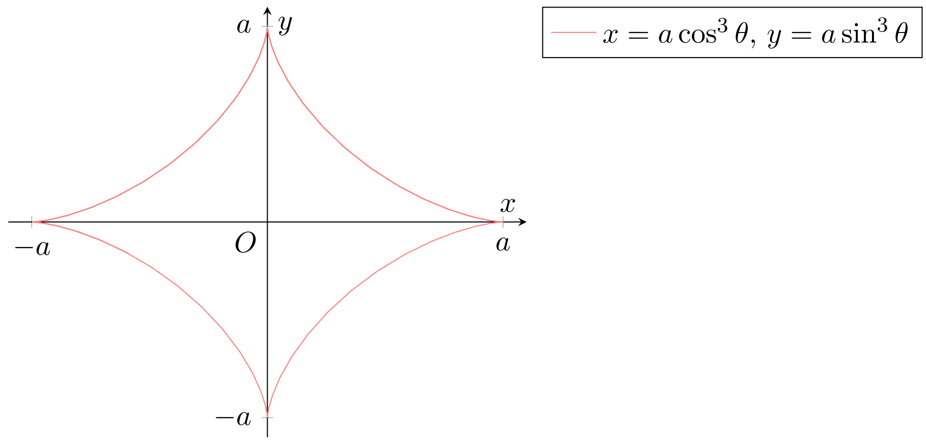
$$\begin{aligned}OT = \frac{-at_P^2\left(1-t_P^2\right)}{2t_P} + a\left(t_P - \frac{t_P^3}{3}\right) &= \frac{a}{2}\left[\left(-t_P + t_P^3\right) + \left(2t_P - \frac{2t_P^3}{3}\right)\right] \\ &= \frac{a}{2}\left(\frac{t_P^3}{3} + t_P\right) = \frac{OP}{2}.\end{aligned}$$

Problem 6. Sketch the curve whose parametric equations are $x = a\cos^3\theta$, $y = a\sin^3\theta$, $a > 0$.

(a) Find the total length of the curve.

(b) The portion of the curve in the first quadrant is revolved through four right angles about the x-axis. Prove that the area of the surface thus formed is $\frac{6}{5}\pi a^2$.

Solution.



Part (a). By symmetry, we only consider the length of the curve in the first quadrant. Note that $x = 0 \implies \theta = \pi/2$ and $x = a \implies \theta = 0$. Also,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = (-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2$$
$$= 9a^2\left(\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta\right) = (3a\cos\theta\sin\theta)^2.$$

Thus,

$$\text{Length} = 4\int_0^{\pi/2}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = 12a\int_0^{\pi/2}\cos\theta\sin\theta\,\mathrm{d}\theta$$
$$= 12a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 6a \text{ units}.$$

Part (b).

$$\text{Area} = 2\pi\int_0^{\pi/2} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = 2\pi\int_0^{\pi/2} a\cos^3\theta\,(3a\cos\theta\sin\theta)\,\mathrm{d}t$$
$$= 6\pi a^2\int_0^{\pi/2}\sin\theta\cos^4\theta\,\mathrm{d}\theta = 6\pi a^2\left[-\frac{\cos^5\theta}{5}\right]_0^{\pi/2} = \frac{6}{5}\pi a^2 \text{ units}^2.$$

Problem 7. The parametric equations of a curve are given by

$$x = \tan t - t,\ y = \ln \sec t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

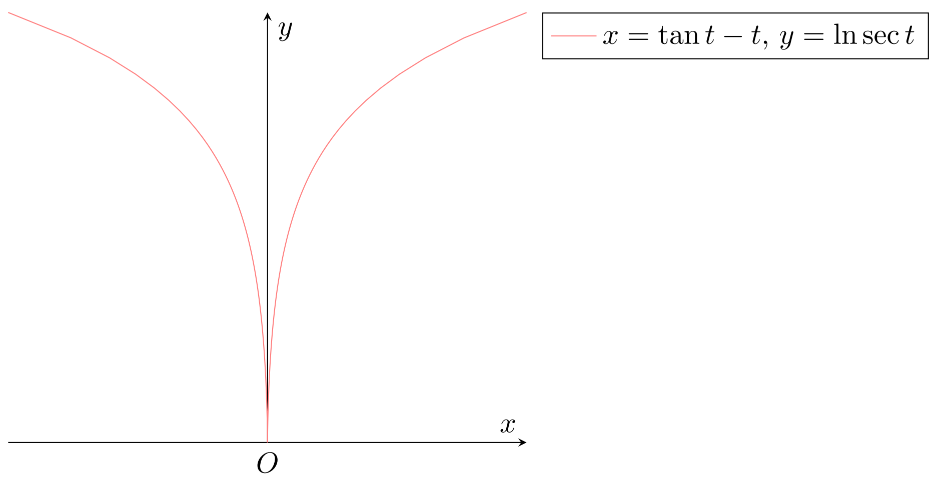
(a) Sketch the curve.

(b) Prove that the arc length of the curve measured from the origin to the point $\left(1 - \frac{\pi}{4}, \frac{1}{2}\ln 2\right)$ is $\sqrt{2} - 1$.

(c) The arc in (b) is rotated about the x-axis through an angle of 360°. Find the exact surface area formed.

Solution.

Part (a).



Part (b). Note that $x = 0 \implies t = 0$ and $x = 1 - \pi/4 \implies = t = \pi/4$. Further,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sec^2 t - 1 = \tan^2 t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \tan t.$$

Thus,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \left(\tan^2 t\right)^2 + (\tan t)^2 = \tan^2 t \left(\tan^2 t + 1\right) = \tan^2 t \sec^2 t.$$

Hence,

$$\text{Length} = \int_0^{\pi/4} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^{\pi/4} \tan t \sec t \,\mathrm{d}t = [\sec t]_0^{\pi/4} = \sqrt{2} - 1 \text{ units.}$$

Part (c). We have

$$\text{Area} = 2\pi \int_0^{\pi/4} y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = 2\pi \int_0^{\pi/4} \ln \sec t \cdot \tan t \sec t \,\mathrm{d}t.$$

Integrating by parts,

	D	I
$+$	$\ln \sec t$	$\tan t \sec t$
$-$	$\tan t$	$\sec t$

Thus,

$$\text{Area} = 2\pi \left[[\sec t \ln \sec t]_0^{\pi/4} - \int_0^{\pi/4} \tan t \sec t \, \mathrm{d}t \right] = 2\pi \left[\sqrt{2} \ln \sqrt{2} - \left(\sqrt{2} - 1 \right) \right]$$
$$= \sqrt{2}\pi \left(\ln 2 - 2 + \sqrt{2} \right) \text{ units}^2.$$

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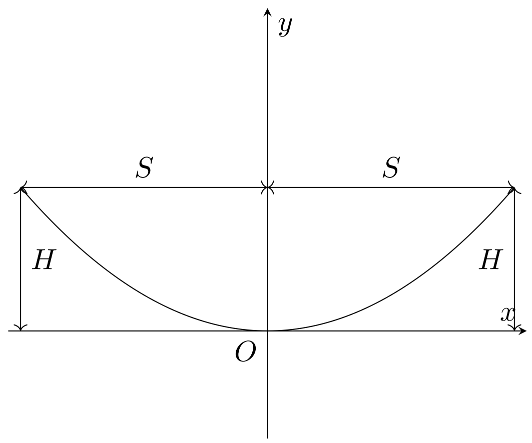
Problem 8.



The diagram shows a cable for a suspension bridge, which has the shape of a parabola with equation $y = kx^2$. The suspension bridge has a total span $2S$ and the height of the cable relative to the lowest point is H at each end. Show that the total length of the cable is $L = 2\int_0^S \sqrt{1 + \frac{4H^2}{S^4}x^2} \, \mathrm{d}x$.

(a) Engineers from country A proposed a suspension bridge across a strait of 8 km wide to country B. The plan included suspension towers 380 m high at each end. Find the length of the parabolic cable for this proposed bridge to the nearest metre.

(b) By using the result $\frac{\mathrm{d}}{\mathrm{d}x} \ln\left(x + \sqrt{a^2 + x^2}\right) = \frac{1}{\sqrt{a^2+x^2}}$ or otherwise, find L in terms of S and H.

Solution. By symmetry, we only need to consider the length of the curve where $x \geq 0$. Since (S, H) is on the curve, $H = kS^2 \implies k = \frac{H}{S^2}$. Note that

$$y = kx^2 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 2kx = \frac{2H}{S^2}x.$$

Hence,

$$L = 2\int_0^S \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x = 2\int_0^S \sqrt{1 + \frac{4H^2}{S^4}x^2} \, \mathrm{d}x.$$

Part (a). Note that $2S = 8000 \implies S = 4000$ and $H = 380$. Hence,

$$L = 2\int_0^{4000} \sqrt{1 + \frac{4(380)^2}{(4000)^4}x^2} \, \mathrm{d}x = 8048 \text{ (to the nearest integer).}$$

The bridge is thus 8048 m long.

Part (b). Consider the integral $I = \int \sqrt{1+(kx)^2}\,\mathrm{d}x$. Under the substitution $kx = \tan\theta$, we get

$$I = \int \sqrt{1+(kx)^2}\,\mathrm{d}x = \frac{1}{k}\int \sqrt{1+\tan^2\theta}\sec^2\theta\,\mathrm{d}\theta = \frac{1}{k}\int \sec^3\theta\,\mathrm{d}\theta.$$

Integrating by parts,

	D	I
$+$	$\sec\theta$	$\sec^2\theta$
$-$	$\sec\theta\tan\theta$	$\tan t$

Hence,

$$kI = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,\mathrm{d}\theta = \sec\theta\tan\theta - \int \sec\theta\left(\sec^2\theta - 1\right)\,\mathrm{d}\theta$$
$$= \sec\theta\tan\theta - \int \sec^3\theta\,\mathrm{d}\theta + \int \sec\theta\,\mathrm{d}\theta = \sec\theta\tan\theta - kI + \ln|\sec\theta + \tan\theta|\,.$$

Thus,

$$I = \frac{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|}{2k} + C = \frac{1}{2k}\left[kx\sqrt{(kx)^2+1} + \ln\left|\sqrt{(kx)^2+1}+kx\right|\right] + C.$$

In our case, $k = \frac{2H}{S^2} > 0$. Hence,

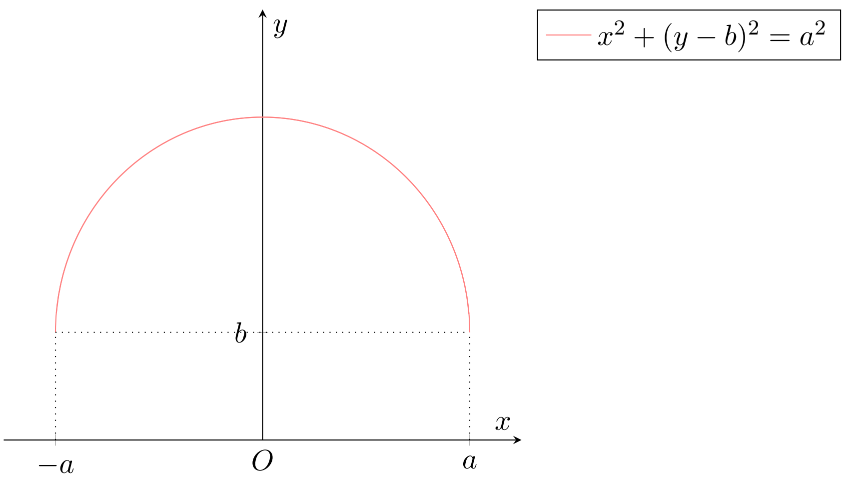
$$L = 2\left[\frac{1}{2}\left(\frac{S^2}{2H}\right)\left[\left(\frac{2H}{S^2}x\right)\sqrt{\left(\frac{2H}{S^2}x\right)^2+1} + \ln\left(\sqrt{\left(\frac{2H}{S^2}x\right)^2+1} + \frac{2H}{S^2}x\right)\right]\right]_0^S$$
$$= \frac{S^2}{2H}\left[\left(\frac{2H}{S}\right)\sqrt{\left(\frac{2H}{S}\right)^2+1} + \ln\left(\sqrt{\left(\frac{2H}{S}\right)^2+1} + \frac{2H}{S}\right)\right]$$
$$= \sqrt{4H^2+S^2} + \frac{S^2}{2H}\ln\left(\frac{\sqrt{4H^2+S^2}+2H}{S}\right).$$

* * * * *

Problem 9. Sketch the semicircle with equation $x^2+(y-b)^2 = a^2$, $y \geq b$ where a and b are positive constants.

A solid is formed by rotating the region bounded by the semicircle and its diameter on the line $y = b$ about the x-axis through 4 right angles. Find the total surface area of the solid.

Solution.



Observe that

$$x^2 + (y-b)^2 = a^2 \implies y = b + \sqrt{a^2 - x^2},$$

whence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{2\sqrt{a^2-x^2}} = -\frac{x}{\sqrt{a^2-x^2}}.$$

Thus,

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \frac{x^2}{a^2-x^2} = \frac{a^2}{a^2-x^2}.$$

Hence,

$$\text{Area} = 2\pi \int_{-a}^{a} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x + 2\pi(b)(2a) = 4\pi \int_0^a y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x + 4\pi ab$$

$$= 4\pi \int_0^a \left(b+\sqrt{a^2-x^2}\right)\left(\frac{a}{\sqrt{a^2-x^2}}\right)\mathrm{d}x + 4\pi ab = 4\pi a \int_0^a \left(\frac{b}{\sqrt{a^2-x^2}}+1\right)\mathrm{d}x + 4\pi ab$$

$$= 4\pi a\left[b\arcsin\frac{x}{a} + x\right]_0^a + 4\pi ab = \left(2\pi^2 ab + 4\pi a^2 + 4\pi ab\right)\ \text{units}^2$$

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Problem 10. Using polar coordinates with pole O, the curve C has the equation $r = a\mathrm{e}^{\theta/k}$, where a and k are positive constants and $0 \le \theta \le 2\pi$. The points A and B on the curve corresponds to $\theta = 0$ and $\theta = \beta$ respectively where $0 < \beta < \pi$. The length of the arc AB is denoted by q and the area of the sector OAB is denoted by Q.

(a) Show that $Q = \frac{1}{4}ka^2\left(\mathrm{e}^{2\beta/k} - 1\right)$.

(b) Show that $q = a(1+k^2)^{1/2}\left(\mathrm{e}^{\beta/k} - 1\right)$.

(c) Deduce from the results of parts (a) and (b) that, for large values of k, $\frac{Q}{q} \approx \frac{1}{2}a$.

(d) Draw a sketch of C for the case where k is large and explain how the result in part (c) can be deduced from the sketch.

Solution.

Part (a).

$$Q = \frac{1}{2}\int_0^\beta r^2\,\mathrm{d}\theta = \frac{a^2}{2}\int_0^\beta \mathrm{e}^{2\theta/k}\,\mathrm{d}\theta = \frac{a^2}{2}\left[\frac{\mathrm{e}^{2\theta/k}}{2/k}\right]_0^\beta = \frac{a^2k}{4}\left(\mathrm{e}^{2\beta/k}-1\right).$$

Part (b). Note that

$$r = a\mathrm{e}^{\theta/k} \implies \frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{a\mathrm{e}^{\theta/k}}{k} = \frac{r}{k}.$$

Hence,

$$q = \int_0^\beta \sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = \int_0^\beta \sqrt{r^2+\frac{r^2}{k^2}}\,\mathrm{d}\theta = \sqrt{1+k^{-2}}\int_0^\beta r\,\mathrm{d}\theta$$

$$= \sqrt{1+k^{-2}}\int_0^\beta a\mathrm{e}^{\theta/k}\,\mathrm{d}\theta = a\sqrt{1+k^{-2}}\left[\frac{\mathrm{e}^{\theta/k}}{1/k}\right]_0^\beta = a\sqrt{k^2+1}\left(\mathrm{e}^{\beta/k}-1\right).$$

Part (c).

$$\lim_{k\to\infty}\frac{Q}{q}=\lim_{k\to\infty}\frac{\frac{1}{4}a^2k\left(\mathrm{e}^{2\beta/k}-1\right)}{a\sqrt{k^2+1}\left(\mathrm{e}^{\beta/k}-1\right)}=\frac{a}{4}\lim_{k\to\infty}\left(\frac{k}{\sqrt{k^2+1}}\right)\lim_{k\to\infty}\left(\frac{\mathrm{e}^{2\beta/k}-1}{\mathrm{e}^{\beta/k}-1}\right).$$

Now observe that

$$\lim_{k\to\infty}\left(\frac{k}{\sqrt{k^2+1}}\right)=\lim_{k\to\infty}\frac{1}{1+k^{-2}}=1,$$

and by the difference of squares identity,

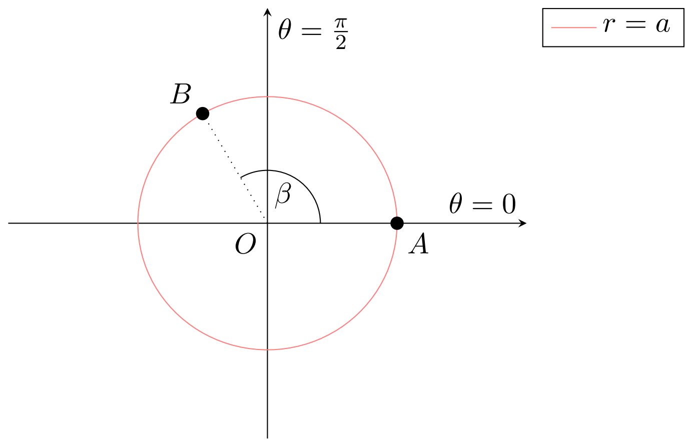
$$\lim_{k\to\infty}\left(\frac{e^{2\beta/k}-1}{e^{\beta/k}-1}\right)=\lim_{k\to\infty}\left(\mathrm{e}^{\beta/k}+1\right)=2.$$

Hence,

$$\lim_{k\to\infty}\frac{Q}{q}=\frac{a}{2}.$$

Part (d). Note that

$$\lim_{k\to\infty}r=\lim_{k\to\infty}a\mathrm{e}^{\theta/k}=a.$$



As $k\to\infty$, the curve becomes a circle. Hence, Q is the area of a sector with angle β, and q is the arc length of a sector with angle β. Thus,

$$\frac{Q}{q}=\left(\frac{\beta}{2\pi}\cdot\pi a^2\right)\Bigg/\left(\frac{\beta}{2\pi}\cdot 2\pi a\right)=\frac{a}{2}.$$

Self-Practice B9

Problem 1. The arc of the curve $y^2 = 4ax$, for which $y \geq 0$ and $0 \leq x \leq a$, is rotated through 2π radians about the x-axis. Prove that the area of the surface so generated is $\frac{8}{3}(2\sqrt{2} - 1)\pi a^2$.

* * * * *

Problem 2. The area bounded by the ellipse with parametric equations $x = 3\cos\theta, y = 2\sqrt{2}\sin\theta$ and the positive x- and y-axis is rotated completely about the y-axis. Find the curved surface area of the solid.

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Problem 3. A curve is defined parametrically by $x = 2\sqrt{2}a\sin\theta,\ y = \frac{1}{2}a\sin 2\theta$.
Show that
$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = a^2\left(2 + \cos 2\theta\right)^2.$$
The portion of the curve from $\theta = 0$ to $\theta = \pi/3$ is rotated completely about the x-axis. Find the exact surface area generated.

* * * * *

Problem 4. A curve is defined parametrically by $x = t^2 - 2\ln t,\ y = 4(t-1)$, where $t \in \mathbb{R},\ t \geq 1$.

(a) The points A and B on the curve are given by $t = 1$ and $t = 2$ respectively. Show that the length of the arc AB of the curve is $3 + 2\ln 2$.

(b) The arc AB is rotated through one revolution about the x-axis. Show that the area of the curved surface generated is $\frac{8}{3}\pi(11 - 6\ln 2)$.

* * * * *

Problem 5. The curve Γ has polar equation $r = ke^{\theta}$, where k is a positive constant and $0 \leq \theta \leq \pi$. The points P and Q on Γ correspond to $\theta = \alpha$ and $\theta = \beta$ respectively ($\beta > \alpha$). The area of the region bounded by the lines $\theta = \alpha$, $\theta = \beta$ and the arc PQ is denoted by A. The length of the arc PQ is denoted by s.

(a) Find expressions for A and s in terms of α, β and k.

(b) Deduce that
$$\frac{A}{s^2} = \frac{1}{8}\left(\frac{\mathrm{e}^{\beta} + \mathrm{e}^{a}}{\mathrm{e}^{b} - \mathrm{e}^{a}}\right).$$

Assignment B9

Problem 1. The curve C is defined parametrically by $x = a(2\cos\theta + \cos 2\theta)$, $y = a(2\sin\theta + \sin 2\theta)$ where $0 \le \theta \le \pi$ and a is a positive constant.

(a) Find the coordinates of the points at which C meets the x-axis.

(b) Sketch C.

(c) Find the exact total length of C.

(d) Find the exact area of the curve surface generated when C is rotated through 2π radians about the x-axis.

Solution.

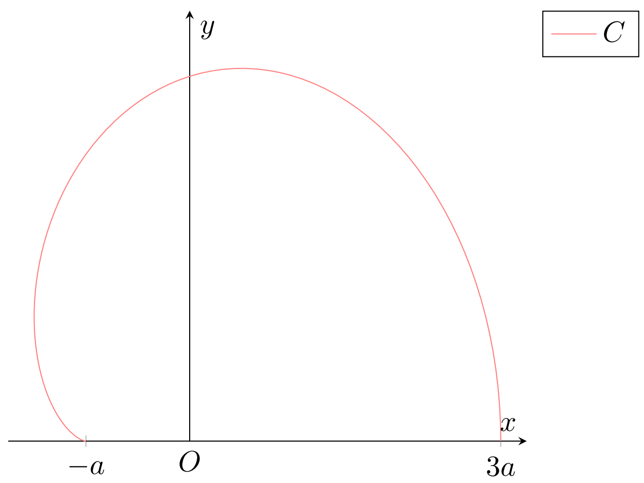
Part (a). When C meets the x-axis, $y = 0$.

$$y = a(2\sin\theta + \sin 2\theta) = a\left(2\sin\theta + 2\sin\theta\cos\right) = 2a\sin\theta\left(1 + \cos\theta\right) = 0.$$

Thus, $\theta = 0$ or $\theta = \pi$.

At $\theta = 0$, $x = 3a$. At $\theta = \pi$, $x = -a$. Hence, C meets the x-axis at $(3a, 0)$ and $(-a, 0)$.

Part (b).



Part (c). Note that

$$\frac{dx}{d\theta} = -2a(\sin\theta + \sin 2\theta), \quad \frac{dy}{d\theta} = 2a(\cos\theta + \cos 2\theta).$$

Hence,

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (2a)^2\left(2 + 2\sin\theta\sin 2\theta + 2\cos\theta\cos 2\theta\right)$$
$$= (2a)^2\left(2 + 2\cos\theta\right) = (2a)^2\left[2 + \left(4\cos^2\frac{\theta}{2} - 2\right)\right] = \left(4a\cos\frac{\theta}{2}\right)^2.$$

Thus,

$$\text{Length} = \int_0^\pi \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = 4a\int_0^\pi \cos\frac{\theta}{2}\, dt = 4a\left[2\sin\frac{\theta}{2}\right]_0^\pi = 8a \text{ units}.$$

Part (d).

$$\text{Area} = 2\pi \int_0^\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = 16\pi a^2 \int_0^\pi \sin\theta\,(1+\cos\theta)\cos\frac{\theta}{2}\,\mathrm{d}\theta$$

$$= 16\pi a^2 \int_0^\pi \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\cos^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\,\mathrm{d}\theta = 64\pi a^2 \int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,\mathrm{d}\theta$$

$$= -128\pi a^2 \left[-\frac{\cos^5(\theta/2)}{5}\right]_0^\pi = \frac{128}{5}\pi a^2 \text{ units}^2.$$

* * * * *

Problem 2. The curve C is given by the equation $y = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$.

(a) Sketch the curve C.

(b) Find the exact area bounded by C, the lines $x = 2$ and $x = -2$ and the x-axis.

(c) Points A and B are on C where $x = 2$ and $x = -2$ respectively. Find the exact length of the arc AB.

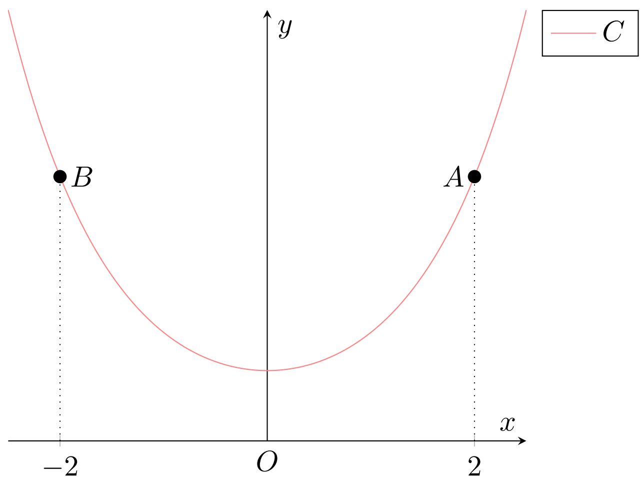
A solid, made of a certain material, is of the shape obtained by rotating the region bounded by C, the lines $x = 2$ and $x = -2$ and the x-axis about the y-axis through π radians.

(d) Find the exact amount of material required to make this solid if x is measured in cm.

(e) The solid is painted with a brush that uses 2 cm^3 of paint for every cm^2 of surface painted. Find the exact amount of paint required.

Solution.

Part (a).



Part (b). Note that $y = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = \cosh x$ is an even function. Hence,

$$\text{Area} = \int_{-2}^{2} y\,\mathrm{d}x = 2\int_0^2 \cosh x\,\mathrm{d}x = 2\left[\sinh x\right]_0^2$$

$$= 2(\sinh 2 - \sinh 0) = 2\left(\frac{e^2 - e^{-2}}{2} - 0\right) = \mathrm{e}^2 - \mathrm{e}^{-2} \text{ units}^2.$$

Part (c). Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x,$$

whence

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x$$

Hence,

$$\text{Length} = \int_{-2}^{2}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{-2}^{2}\cosh x\,\mathrm{d}x = 2\int_{0}^{2}\cosh x\,\mathrm{d}x = \mathrm{e}^2 - \mathrm{e}^{-2}\text{ units.}$$

Part (d). We have

$$\text{Volume} = 2\pi\int_0^2 xy\,\mathrm{d}x = 2\pi\int_0^2 x\cosh x\,\mathrm{d}x.$$

Integrating by parts,

	D	I
$+$	x	$\cosh x$
$-$	1	$\sinh x$
$+$	0	$\cosh x$

Thus,

$$\begin{aligned}\text{Volume} &= 2\pi\left[x\sinh x - \cosh x\right]_0^2 = 2\pi\left[(2\sinh 2 - \cosh 2) - (0\sinh 0 - \cosh 0)\right]\\ &= 2\pi\left[2\left(\frac{e^2-e^2}{2}\right) - \frac{e^2+e^{-2}}{2} + 1\right] = \pi\left(e^2 - 3e^{-2} + 2\right).\end{aligned}$$

Thus, $\pi\left(e^2 - 3e^{-2} + 2\right)$ cm^3 of material is required.

Part (e).

$$\begin{aligned}&\text{Area} = \text{Area of curved surface} + \text{Area of side} + \text{Area of bottom}\\ &= 2\pi\int_0^2 x\cosh x\,\mathrm{d}x + 2^2\pi + 2^2\pi\cosh 2 = \pi\left(e^2 - 3e^{-2} + 2\right) + 4\pi + 4\pi\left(\frac{e^2+e^{-2}}{2}\right)\\ &= \pi\left[3e^2 - e^{-2} + 6\right].\end{aligned}$$

Thus, $2\pi\left(3e^2 - e^{-2} + 6\right)$ cm^3 of paint is required.

B10 Applications of Integration III - Trapezium and Simpson's Rule

Tutorial B10

Problem 1. Estimate, using the trapezium rule, the values of the following definite integrals, taking the number ordinates given in each case.

(a) $\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta$, 3 ordinates

(b) $\int_{-0.4}^{0.2} \frac{x^2-4x+1}{4x-4}$, 4 ordinates

Solution.
Part (a). Let $f(\theta) = \frac{1}{1+\cos\theta}$.

$$\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta \approx \frac{1}{2}\cdot\frac{0-(-\pi/2)}{3-1}\cdot\Big[f\Big(-\frac{\pi}{2}\Big)+2f\Big(-\frac{\pi}{4}\Big)+f(0)\Big] = 1.05.$$

Part (b). Let $f(x) = \frac{x^2-4x+1}{4x-4}$.

$$\int_{-0.4}^{0.2} \frac{x^2-4x+1}{4x-4}\,\mathrm{d}x \approx \frac{1}{2}\cdot\frac{0.2-(-0.4)}{4-1}\cdot\Big[f(-0.4)+2\big[f(-0.2)+f(0)\big]+f(0.2)\Big] = -0.183.$$

* * * * *

Problem 2. Use the trapezium rule with intervals of width 0.5 to obtain an approximation to $\int_{2}^{3.5} \ln\frac{1}{x}\,\mathrm{d}x$, giving your answer to 2 decimal places.

Solution.

$$\int_{2}^{3.5} \ln\frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2}\cdot\frac{3.5-2}{4-1}\cdot\left[\ln\frac{1}{2}+2\left(\ln\frac{1}{2.5}+\ln\frac{1}{3}\right)+\ln\frac{1}{3.5}\right] = -1.49 \text{ (2 d.p.)}.$$

* * * * *

Problem 3. Estimate, using Simpson's rule, the values of the following definite integrals, taking the number of ordinates given in each case.

(a) $\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta$, 3 ordinates

(b) $\int_{0}^{0.4} \sqrt{1-x^2}\,\mathrm{d}x$, 5 ordinates

Solution.
Part (a). Let $f(\theta) = \frac{1}{1+\cos\theta}$.

$$\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta \approx \frac{1}{3}\cdot\frac{0-(-\pi/2)}{3-1}\cdot\Big[f\Big(-\frac{\pi}{2}\Big)+4f\Big(-\frac{\pi}{4}\Big)+f(0)\Big] = 1.01.$$

Part (b). Let $f(x) = \sqrt{1-x^2}$.

$$\int_0^{0.4} \sqrt{1-x^2}\,\mathrm{d}x \approx \frac{1}{3}\cdot\frac{0.4-0}{5-1}\cdot\Big[f(0)+4f(0.1)+2f(0.2)+4f(0.3)+f(0.4)\Big] = 0.389.$$

* * * * *

Problem 4. Show, by means of substitution $u = \sqrt{x}$, that

$$\int_0^{0.25} \frac{1}{\sqrt{x}}\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^{0.5} 2\mathrm{e}^{-u^2}\,\mathrm{d}u$$

Use the trapezium rule, with ordinates at $u = 0$, $u = 0.1$, $u = 0.2$, $u = 0.3$, $u = 0.4$ and $u = 0.5$, to estimate the value of $I = \int_0^{0.5} 2\mathrm{e}^{-u^2}\,\mathrm{d}u$, giving three decimal places in your answer.

Explain briefly why the trapezium rule cannot be used directly to estimate the value of $\int_0^{0.25} \frac{1}{\sqrt{x}}\mathrm{e}^{-x}\,\mathrm{d}x$.

By using the first four terms of the expansion of e^{-x}, obtain an estimate for the integral $\int_0^{0.25} \frac{1}{\sqrt{x}}\mathrm{e}^{-x}\,\mathrm{d}x$, giving three decimal places in your answer.

Solution. Note that

$$u = \sqrt{x} \implies u^2 = x \implies 2u\,\mathrm{d}u = \mathrm{d}x.$$

Furthermore,

$$x = 0 \implies u = 0, \quad x = 0.25 \implies u = 0.5.$$

Hence,

$$\int_0^{0.25} \frac{1}{\sqrt{x}}\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^{0.5} \frac{1}{u}\mathrm{e}^{-u^2}\cdot 2u\,\mathrm{d}u = \int_0^{0.5} 2\mathrm{e}^{-u^2}\,\mathrm{d}u.$$

Let $f(u) = 2\mathrm{e}^{-u^2}$. Using the trapezium rule,

$$I \approx \frac{1}{2}\cdot\frac{0.5-0}{5}\Big[f(0)+2\big[f(0.1)+f(0.2)+f(0.3)+f(0.4)\big]+f(0.5)\Big] = 0.921 \text{ (3 d.p.)}.$$

At $x = 0$, $\frac{1}{\sqrt{x}}\mathrm{e}^{-x}$ is undefined. Hence, the trapezium rule cannot be used.

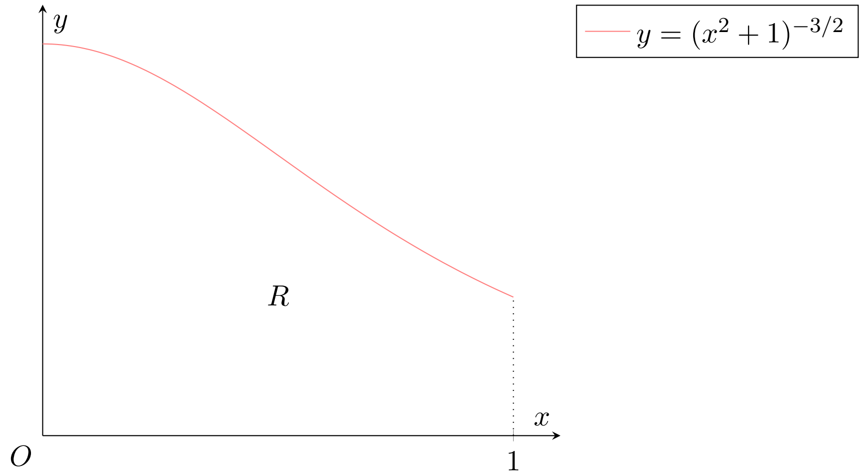
Recall that

$$\mathrm{e}^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots.$$

Hence,

$$\int_0^{0.25} \frac{1}{\sqrt{x}}\mathrm{e}^{-x}\,\mathrm{d}x \approx \int_0^{0.25} \frac{1}{\sqrt{x}}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right)\mathrm{d}x = 0.923 \text{ (3 d.p.)}.$$

Problem 5.



The diagram (not to scale) show the region R bounded by the axes, the curve $y = (x^2+1)^{-3/2}$ and the line $x = 1$. The integral $\int_0^1 (x^2+1)^{-3/2}$ is denoted by I.

(a) Use the trapezium rule and Simpson's rule, with ordinates at $x = 0$, $x = 0.5$ and $x = 1$, to estimate the value of I correct to 4 significant figures.

(b) Use the substitution $x = \tan\theta$ to show that $I = \frac{1}{2}\sqrt{2}$. Comment on the approximations using the 2 rules and give a reason why one gives a better approximation than the other.

(c) By using the trapezium rule, with the same ordinates as in part (a), or otherwise, estimate the volume of the solid formed when R is rotated completely about the x-axis, giving your answer to 2 significant figures.

Solution.

Part (a). Let $f(x) = (x^2+1)^{-3/2}$. Using the trapezium rule,

$$I \approx \frac{1}{2}\cdot\frac{1-0}{3-1}\cdot\Big[f(0)+2f(0.5)+f(1)\Big] = 0.6962 \text{ (4 s.f.)}.$$

Using Simpson's rule,

$$I \approx \frac{1}{3}\cdot\frac{1-0}{3-1}\cdot\Big[f(0)+4f(0.5)+f(1)\Big] = 0.7026 \text{ (4 s.f.)}.$$

Part (b). Using the substitution $x = \tan\theta$, we get

$$\int_0^1 (x^2+1)^{-3/2} = \int_0^{\pi/4}\left(\tan^2\theta+1\right)^{-3/2}\sec^2\theta\,\mathrm{d}\theta = \int_0^{\pi/4}(\sec^2\theta)^{-3/2}\sec^2\theta\,\mathrm{d}\theta$$
$$= \int_0^{\pi/4}(\sec\theta)^{-1}\,\mathrm{d}\theta = \int_0^{\pi/4}\cos\theta\,\mathrm{d}\theta = [\sin\theta]_0^{\pi/4} = \frac{1}{2}\sqrt{2}.$$

The approximation given by Simpson's rule is closer to the actual value than the approximation given by the trapezium rule. This is because Simpson's rule accounts for the concavity of the curve, which produces a better estimate.

Part (c). Let $g(x) = (x^2+1)^{-3}$.

$$\text{Volume} = \pi\int_0^1 y^2\,\mathrm{d}x = \pi\int_0^1 (x^2+1)^{-3}\,\mathrm{d}x$$
$$\approx \pi\left(\frac{1}{2}\cdot\frac{1-0}{3-1}\Big[g(0)+2g(0.5)+g(1)\Big]\right) = 1.7 \text{ units}^3 \text{ (2 s.f.)}.$$

Problem 6. It is given that $f(x) = \frac{1}{\sqrt{1+\sqrt{x}}}$, and the integral $\int_0^1 f(x)\,\mathrm{d}x$ is denoted by I.

(a) Using the trapezium rule, with four trapezia of equal width, obtain an approximation I_1 to the value of I, giving 3 decimal places in your answer.

(b) Explain, with the aid of a sketch, why $I < I_1$.

(c) Evaluate I_2, where $I_2 = \frac{1}{3}\sum_{r=1}^{3} f\left(\frac{1}{3}r\right)$, giving 3 decimal places in your answer, and use the sketch in (b) to justify the inequality $I > I_2$.

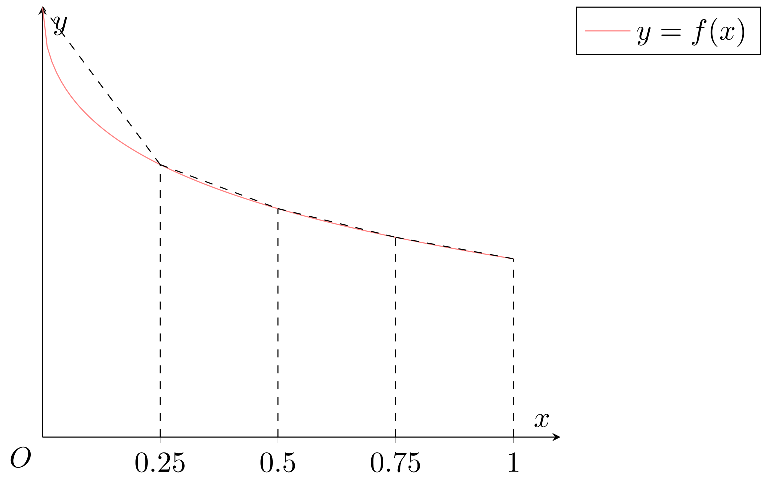
(d) By means of a substitution $\sqrt{x} = u - 1$, show that the value of I is $\frac{4}{3}(2-\sqrt{2})$.

Solution.

Part (a).

$$I_1 = \frac{1}{2} \cdot \frac{1-0}{4}\Big[f(0) + 2\big[f(0.25) + f(0.5) + f(0.75)\big] + f(1)\Big] = 0.792 \text{ (3 d.p.)}.$$

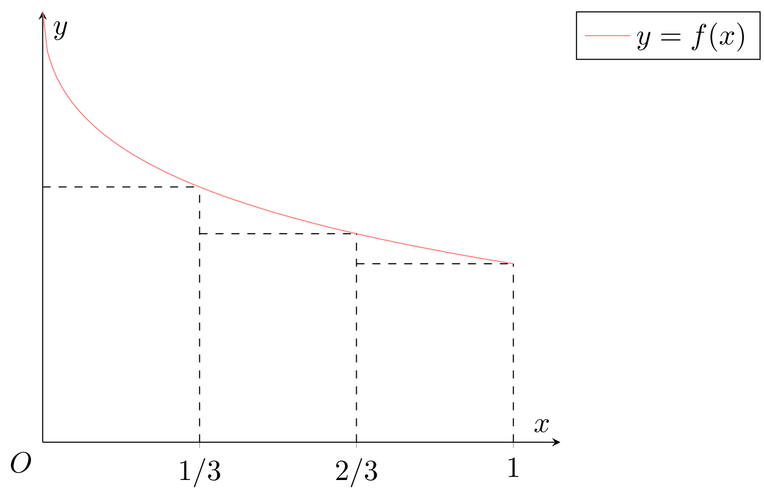
Part (b).



I is the area under the curve $y = f(x)$, while I_1 is the sum of the areas of the trapeziums. Hence, from the sketch, $I_1 > I$.

Part (c).

$$I_2 = \frac{1}{3}\sum_{r=1}^{3} f\left(\frac{1}{3}r\right) = \frac{1}{3}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f(1)\right] = 0.748 \text{ (3 d.p.)}.$$



I is the area under the curve $y = f(x)$, while I_2 is the sum of the areas of the rectangles. Hence, from the sketch, $I_2 < I$.

Part (d). Note

$$\sqrt{x} = u - 1 \implies x = u^2 - 2u + 1 \implies \mathrm{d}x = (2u - 2)\,\mathrm{d}u.$$

Furthermore,

$$x = 0 \implies u = 1, \quad x = 1 \implies u = 2.$$

Thus,

$$\int_0^1 \frac{1}{\sqrt{1+\sqrt{x}}}\,\mathrm{d}x = 2\int_1^2 \frac{u-1}{\sqrt{u}}\,\mathrm{d}u = 2\left[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right]_1^2 = \frac{4}{3}(2-\sqrt{2}).$$

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Problem 7. For $0 < x < \pi$, the curve C has the equation $y = \ln\sin x$. The region of the plane bounded by C, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.

Show that the surface area of the solid generated in this way is given by S, where

$$S = 2\pi \int_{\pi/4}^{\pi/2} \left|\frac{\ln\sin x}{\sin x}\right|\,\mathrm{d}x$$

Use Simpson's rule with 5 ordinates to find an approximate value of S, giving your answer to 3 decimal places.

Solution. Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{\sin x} = \cot x \implies 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \cot^2 x = \csc^2 x.$$

Thus,

$$\begin{aligned} S &= 2\pi \int_{\pi/4}^{\pi/2} |y|\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = 2\pi\int_{\pi/4}^{\pi/2} |\ln\sin x|\,|\csc x|\,\mathrm{d}x \\ &= 2\pi\int_{\pi/4}^{\pi/2} |\ln\sin x|\left|\frac{1}{\sin x}\right|\,\mathrm{d}x = 2\pi\int_{\pi/4}^{\pi/2}\left|\frac{\ln\sin x}{\sin x}\right|\,\mathrm{d}x. \end{aligned}$$

Let $f(x) = \left|\frac{\ln\sin x}{\sin x}\right|$.

$$S \approx \frac{2\pi}{3}\cdot\frac{\pi/4}{4}\left[f\left(\frac{4\pi}{16}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{6\pi}{16}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{8\pi}{16}\right)\right] = 0.670 \text{ (3 d.p.)}.$$

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Problem 8. The value of the integral $\int_{0.2}^{0.4} f(x)\,\mathrm{d}x$ is to be estimated from information in the table below.

x	0.2	0.3	0.4
$f(x)$	1.2030	1.2441	1.2777

(a) Find the best possible estimate for the integral using the trapezium rule.

(b) Using the table of values above, find an approximate value for $f''(0.3)$ and use your answer to explain why the estimate found in part (a) is likely to be smaller than the actual value.

(c) Estimate the integral using Simpson's rule and determine the equation of the curve used in this method.

Solution.

Part (a).

$$\int_{0.2}^{0.4} f(x)\,\mathrm{d}x \approx \frac{1}{2}\cdot\frac{0.4-0.2}{3-1}\Big[f(0.2)+2f(0.3)+f(0.4)\Big] = 0.248.$$

Part (b). Note that $f'(0.25) \approx \frac{f(0.3)-f(0.2)}{0.3-0.2} = 0.411$ and $f'(0.35) \approx \frac{f(0.4)-f(0.3)}{0.4-0.3} = 0.336$. Hence,

$$f''(0.30) \approx \frac{f'(0.35)-f'(0.25)}{0.35-0.25} = -0.75.$$

Since $f''(0.3) < 0$, $f(x)$ is concave downwards around $x = 0.3$. Hence, the estimate is likely to be smaller than the actual value.

Part (c).

$$\int_{0.2}^{0.4} f(x)\,\mathrm{d}x \approx \frac{1}{3}\cdot\frac{0.4-0.2}{3-1}\cdot\Big[f(0.2)+4f(0.3)+f(0.4)\Big] = 0.249.$$

Let the equation of the quadratic used be $P(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$. Since $P(0.2) = f(0.2)$, $P(0.3) = f(0.3)$ and $P(0.4) = f(0.4)$, we obtain the system

$$\begin{cases} (0.2)^2a + 0.2b + c = 1.2030 \\ (0.3)^2a + 0.3b + c = 1.2441 \\ (0.4)^2a + 0.4b + c = 1.2777 \end{cases}$$

which has the unique solution $a = -0.375$, $b = 0.5985$, $c = 1.0983$. Thus, the required equation is

$$y = -0.375x^2 + 0.5985x + 1.0983.$$

* * * * *

Problem 9. The curve C is given by $y = \frac{1}{x}$, where $x > 0$.

(a) Apply the trapezium rule with ordinates at unit intervals to the function $f : x \mapsto \frac{1}{x}$, $x \in \mathbb{R}^+$, to show that $\ln n < \frac{1}{2} + \frac{1}{2n} + \sum_{r=2}^{n-1} \frac{1}{r}$ where $n \geq 3$.

(b) Obtain the area of the trapezium bounded by the axis, the lines $x = r \pm \frac{1}{2}$, and the tangent to the curve $y = \frac{1}{x}$ at the point $\left(r, \frac{1}{r}\right)$.

Hence, show that $\sum_{r=2}^{n-1} \frac{1}{r} < \ln\left(\frac{2n-1}{3}\right)$, where $n \geq 3$.

(c) From these results, obtain numerical values between which the value of $\sum_{r=2}^{99} \frac{1}{r}$ lies, and show that $4.110 < \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{100} < 4.205$.

Solution.

Part (a). Applying the trapezium rule,

$$\int_1^n \frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2}\left[\frac{1}{1} + 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\right) + \frac{1}{n}\right] = \frac{1}{2} + \frac{1}{2n} + \sum_{r=2}^{n-1} \frac{1}{r}.$$

Note that $\mathrm{d}^2y/\mathrm{d}x^2 = 2x^{-3} > 0$ for $x > 0$. Hence, $y = 1/x$ is concave upwards. Thus,

$$\frac{1}{2} + \frac{1}{2n} + \sum_{r=2}^{n-1} \frac{1}{r} > \int_1^n \frac{1}{x}\,\mathrm{d}x = \ln n.$$

Part (b). Since $\mathrm{d}y/\mathrm{d}x = -x^{-2}$, the equation of the tangent at $x = r$ is given by

$$y - \frac{1}{r} = -\frac{1}{r^2}(x - r) \implies y = -\frac{x}{r^2} + \frac{2}{r}.$$

The area of the trapezium centred at r is hence given by

$$\int_{r-1/2}^{r+1/2} \left(-\frac{x}{r^2} + \frac{2}{r}\right)\mathrm{d}x = \left[-\frac{1}{r^2}\left(\frac{x^2}{2}\right) + \frac{2x}{r}\right]_{r-1/2}^{r+1/2} = \frac{1}{r} \text{ units}^2.$$

Observe that the area of the trapezium centred at r is less than the area under the curve $y = \frac{1}{x}$ from $r - \frac{1}{2}$ to $r + \frac{1}{2}$. That is,

$$\frac{1}{r} < \int_{r-1/2}^{r+1/2} \frac{1}{x}\,\mathrm{d}x = \ln\left(r + \frac{1}{2}\right) - \ln\left(r - \frac{1}{2}\right).$$

Summing from $r = 2$ to $n - 1$,

$$\sum_{r=2}^{n-1} \frac{1}{r} < \sum_{r=2}^{n-1} \left[\ln\left(r + \frac{1}{2}\right) - \ln\left(r - \frac{1}{2}\right)\right] = \ln\left(n - \frac{1}{2}\right) - \ln\left(2 - \frac{1}{2}\right) = \ln\left(\frac{2n-1}{3}\right).$$

Part (c). Taking $n = 100$, we have

$$\frac{1}{2} + \frac{1}{2(100)} + \sum_{r=2}^{100-1} \frac{1}{r} > \ln 100 \implies \sum_{r=2}^{99} \frac{1}{r} > \ln 100 - \frac{1}{2} - \frac{1}{200} = 4.100$$

We also have

$$\sum_{r=2}^{100-1} \frac{1}{r} < \ln\left(\frac{2(100)-1}{3}\right) \implies \sum_{r=2}^{100-1} \frac{1}{r} < \ln\left(\frac{199}{3}\right) = 4.195$$

Putting both inequalities together, we obtain

$$4.100 < \sum_{r=2}^{99} \frac{1}{r} < 4.195.$$

Adding $\frac{1}{100} = 0.01$ to all sides of the inequality, we see that

$$4.110 < \sum_{r=2}^{100} \frac{1}{r} < 4.205.$$

Self-Practice B10

Problem 1. Use the trapezium rule, with 6 intervals to estimate the value of $\int_0^3 \ln(1+x)\,\mathrm{d}x$, showing your working. Give your answer to 3 significant figures. Hence, write down an approximate value for $\int_0^3 \ln\sqrt{1+x}\,\mathrm{d}x$.

* * * * *

Problem 2. Use the trapezium rule with 5 intervals to estimate the value of

$$\int_0^{0.5} \sqrt{1+x^2}\,\mathrm{d}x,$$

showing your working. Give your answer to 2 decimal places.

By expanding $(1+x^2)^{1/2}$ in powers of x as far as the term in x^4, obtain a second estimate for the value of $\int_0^{0.5} \sqrt{1+x^2}\,\mathrm{d}x$ giving this answer also correct to 2 decimal places.

* * * * *

Problem 3. The trapezium rule, with 2 intervals of equal width, is to be used to find an approximate value for $\int_0^1 \mathrm{e}^{-x}\,\mathrm{d}x$. Explain, with the aid of a sketch, why the approximation will be greater than the exact value of the integral. Calculate the approximate value and the exact value, giving each answer correct to 3 decimal places.

Another approximation to $\int_0^1 \mathrm{e}^{-x}\,\mathrm{d}x$ is to be calculated by using two trapezia of unequal width. The first trapezium has width h and the second has width $1-h$, so that the three ordinates are at $x=0$, $x=h$ and $x=1$. Show that the total area T of these two trapezia is given by

$$T = \frac{1}{2}\left[\mathrm{e}^{-1} + h\left(1-\mathrm{e}^{-1}\right) + \mathrm{e}^{-h}\right].$$

Show that the value of h for which T is a minimum is given by $h = \ln\frac{\mathrm{e}}{\mathrm{e}-1}$.

* * * * *

Problem 4. Derive Simpson's rule with 2 strips for evaluating $\int_a^b f(x)\,\mathrm{d}x$.

Use Simpson's composite rule with 4 strips to obtain an estimate of $\int_2^3 \cos(x-2)\ln x\,\mathrm{d}x$, giving your answer to 5 decimal places.

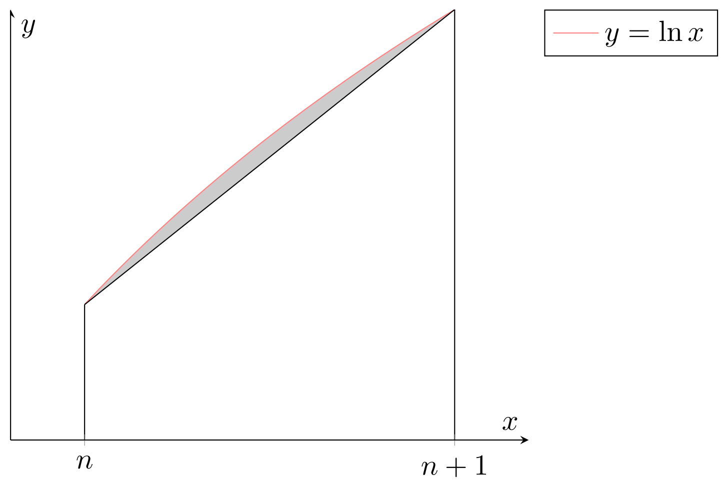
* * * * *

Problem 5.

(a) Show that $\int_n^{n+1} \ln x\,\mathrm{d}x = (n+1)\ln(n+1) - n\ln n - 1$.

(b) The diagram below shows the graph of $y = \ln x$ between $x = n$ and $x = n+1$. The area of the shaded region represents the error when the value of the integral in part (a) is approximated by using a single trapezium. Show that the area of the shaded region is

$$\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) - 1.$$



(c) Use a series expansion to show that if n is large enough for $\frac{1}{n^3}$ and higher powers of $\frac{1}{n}$ to be neglected, then the area in part (b) is approximately equal to $\frac{k}{n^2}$, where k is a constant to be determined.

Assignment B10

Problem 1. Given that $y = \mathrm{e}^{-x}\cos x$, show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\left(y + \frac{\mathrm{d}y}{\mathrm{d}x}\right)$. By further differentiation, find the series expansion of y, in ascending powers of x, up to and including the term in x^3. Use the series to obtain an approximate value for $\int_0^{0.2} \frac{\cos x^2}{\mathrm{e}^{x^2}}\,\mathrm{d}x$, giving your answer correct to 4 decimal places.

Using the trapezium rule with 4 trapezia of equal width, find another approximation for $\int_0^{0.2} \frac{\cos x^2}{\mathrm{e}^{x^2}}\,\mathrm{d}x$, giving your answer correct to 4 decimal places.

Solution. Differentiating with respect to x, we get

$$y' = -\mathrm{e}^{-x}\sin x - \mathrm{e}^{-x}\cos x \implies y' = -\mathrm{e}^{-x}\sin x - y.$$

Differentiating once more,

$$y'' = -\mathrm{e}^{-x}\cos x + \mathrm{e}^{-x}\sin x - y' = -y + \left(-y' - y\right) - y' = -2\left(y + y'\right).$$

Further differentiating, we obtain $y''' = -2(y' + y'')$. Evaluating y and its derivatives at $x = 0$, we get

$$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 0, \quad y'''(0) = 2.$$

Thus,

$$\mathrm{e}^{-x}\cos x = 1 - x + \frac{x^3}{3} + \cdots.$$

Hence,

$$\int_0^{0.2} \frac{\cos x^2}{\mathrm{e}^{x^2}}\,\mathrm{d}x = \int_0^{0.2} \mathrm{e}^{-x^2}\cos x^2\,\mathrm{d}x \approx \int_0^{0.2} \left(1 - \left(x^2\right) + \frac{\left(x^2\right)^3}{3}\right)\mathrm{d}x = 0.1973 \text{ (4 d.p.)}.$$

Let $g(x) = \frac{\cos x^2}{\mathrm{e}^{x^2}}$. By the trapezium rule, we have

$$\int_0^{0.2} \frac{\cos x^2}{\mathrm{e}^{x^2}}\,\mathrm{d}x \approx \frac{1}{2} \cdot \frac{0.2}{4}\Big[g(0) + 2\big[g(0.05) + g(0.1) + g(0.15)\big] + g(0.2)\Big] = 0.1973 \text{ (4 d.p.)}.$$

* * * * *

Problem 2. The curve C has equation $y^2 = \frac{x}{\sqrt{1+x^2}}$, $y \geq 0$.

The finite region R is bounded by C, the x-axis and the lines $x = 0$ and $x = 2$. R is rotated through 2π radians about the x-axis.

(a) Find the exact volume of the solid formed.

An estimate for the volume in (a) is found using the trapezium rule with 7 ordinates.

(b) Find the percentage error resulting from using this estimate, giving your answer to 3 decimal places.

Explain, with the help of a sketch, why the estimate given by the trapezium rule is less than the actual value.

Solution.

Part (a).

$$\text{Volume} = \pi \int_0^2 y^2 \, \mathrm{d}x = \pi \int_0^2 \frac{2x}{2\sqrt{1+x^2}} \, \mathrm{d}x = \pi \left[\sqrt{1+x^2}\right]_0^2 = \pi(\sqrt{5}-1) \text{ units}^3.$$

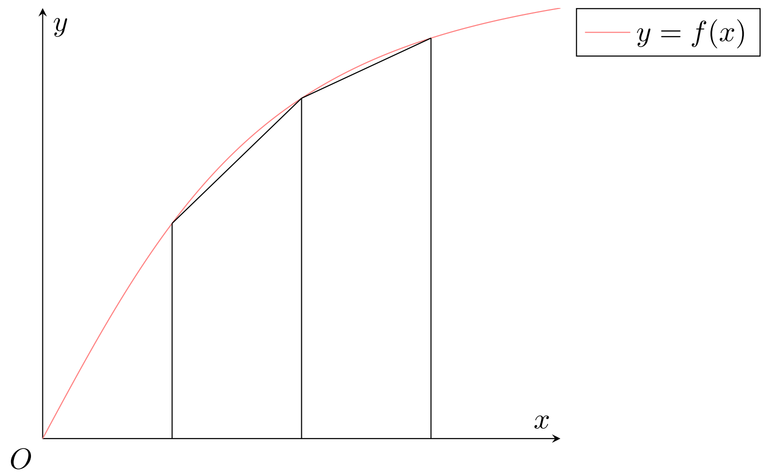
Part (b). Let $f(x) = \frac{x}{\sqrt{1+x^2}}$. By the trapezium rule,

$$\text{Volume} = \pi \int_0^2 f(x) \, \mathrm{d}x \approx \pi \cdot \frac{1}{2} \cdot \frac{2-0}{6} \sum_{n=0}^{5} \left[f\left(\frac{n}{3}\right) + f\left(\frac{n+1}{3}\right)\right] = 3.8566 \text{ (5 s.f.)}.$$

Hence, the percentage error is

$$\left|\frac{\pi(\sqrt{5}-1) - 3.8566}{\pi(\sqrt{5}-1)}\right| = 0.686\% \text{ (3 d.p.)}.$$

Consider the following graph of $y = f(x)$.



From the graph, the curve $y = f(x)$ is clearly concave downwards. Hence, the approximation given by the trapezium rule is an underestimate and is thus less than the actual value.

* * * * *

Problem 3. Prove that $\int_{-h}^{h} f(x) \, \mathrm{d}x = \frac{1}{3}h(y_{-1}+4y_0+y_1)$, where $y = f(x)$ is the quadratic curve passing through the points $(-h, y_{-1})$, $(0, y_0)$ and (h, y_1).

Use Simpson's rule with 5 ordinates to find an approximation to

$$\int_{-3}^{1} \left(x^4 - 7x^3 + 3x^2 + 6x + 4\right)^{1/3} \mathrm{d}x$$

Find another approximation to the same integral using the trapezium rule with 5 ordinates.

Which of these approximations would you expect to be more accurate? Justify your answer.

Solution. Let $f(x) = ax^2 + bx + c$ be the quadratic such that the graph $y = f(x)$ passes through the points $(-h, y_{-1})$, $(0, y_0)$ and (h, y_1).

Note that we have $y_0 = f(0) = c$. We also have

$$y_{-1} + y_1 = f(-h) + f(h) = \left[a(-h)^2 + b(-h) + c\right] + \left[ah^2 + bh + c\right] = 2ah^2 + 2c.$$

Hence,

$$\int_{-h}^{h} f(x)\,\mathrm{d}x = \int_{-h}^{h} (ax^2+bx+c)\,\mathrm{d}x = \left[\frac{1}{3}x^3+\frac{1}{2}bx^2+cx\right]_{-h}^{h} = \frac{1}{3}h\left(2h^2+6c\right)$$
$$= \frac{1}{3}h\left(2h^2+2c+4c\right) = \frac{1}{3}h\left(y_{-1}+y_1+4y_0\right) = \frac{1}{3}h\left(y_{-1}+4y_0+y_1\right).$$

Let $f(x) = \left(x^4-7x^3+3x^2+6x+4\right)^{1/3}$. By Simpson's rule,

$$\int_{-3}^{1} \left(x^4-7x^3+3x^2+6x+4\right)^{1/3}\,\mathrm{d}x$$
$$\approx \frac{1}{3}\cdot\frac{1-(-3)}{4}\Big[f(-3)+4f(-2)+2f(-1)+4f(0)+f(1)\Big] = 11.977 \text{ (5 s.f.)}$$

By the trapezium rule,

$$\int_{-3}^{1} \left(x^4-7x^3+3x^2+6x+4\right)^{1/3}\,\mathrm{d}x$$
$$\approx \frac{1}{2}\cdot\frac{1-(-3)}{4}\Big[f(-3)+2f(-2)+2f(-1)+2f(0)+f(1)\Big] = 12.142 \text{ (5 s.f.)}$$

The approximation given by Simpson's rule should be more accurate as Simpson's rule accounts for the concavity of the curve $y = f(x)$.

* * * * *

Problem 4.

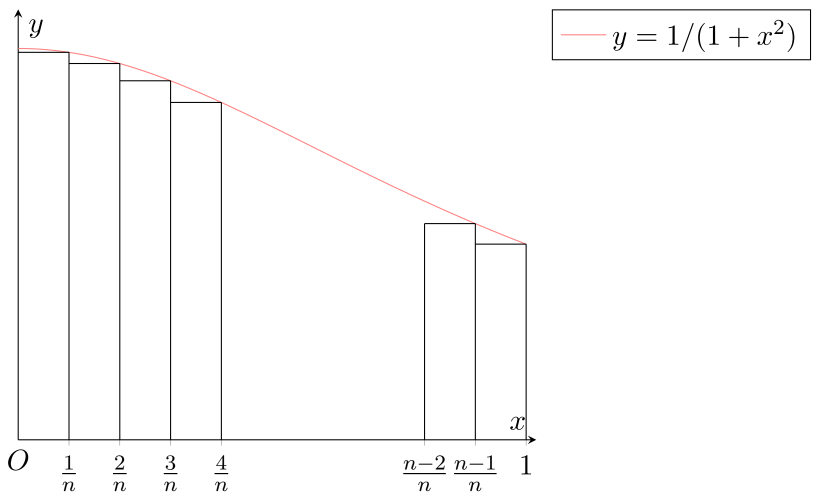
(a) Find the exact value of $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$.

(b) The graph of $y = \frac{1}{1+x^2}$ is shown in the diagram below. Rectangles, each of width $\frac{1}{n}$, are drawn under the curve.

Show that the total area A of all n rectangles is given by

$$A = \frac{1}{n}\left[\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\frac{1}{1+\left(\frac{3}{n}\right)^2}+\ldots+\frac{1}{2}\right]$$

State the limit of A as $n \to \infty$.



(c) It is given that

$$B = \frac{1}{n}\left[\frac{1}{1+\left(\frac{1}{n}\right)^4} + \frac{1}{1+\left(\frac{2}{n}\right)^4} + \frac{1}{1+\left(\frac{3}{n}\right)^4} + \ldots + \frac{1}{2}\right]$$

Find an approximation for the limit of B as $n \to \infty$ by considering an appropriate graph and using the trapezium rule with 5 intervals. Given your answer correct to 2 decimal places.

Solution.

Part (a).

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = [\arctan x]_0^1 = \frac{\pi}{4}.$$

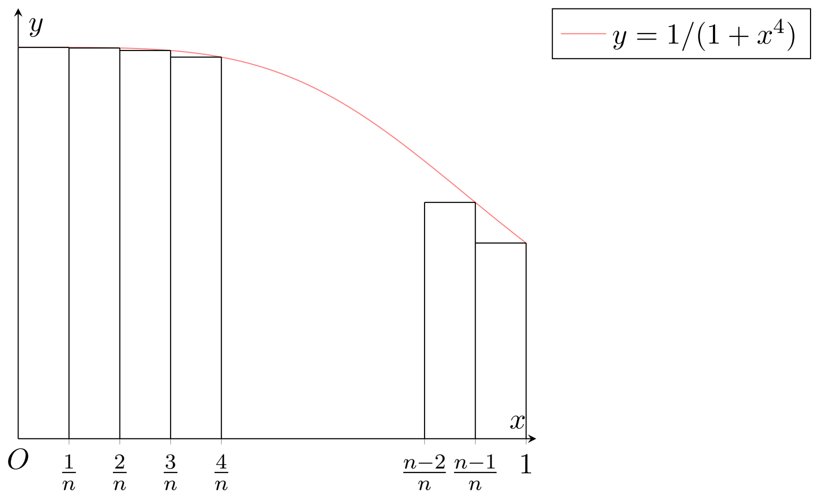
Part (b). Observe that the kth rectangle has height $\frac{1}{1+(k/n)^2}$ and width $1/n$. Hence,

$$\begin{aligned} A &= \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{1}{1+(k/n)^2} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+(k/n)^2} \\ &= \frac{1}{n}\left[\frac{1}{1+\left(\frac{1}{n}\right)^2} + \frac{1}{1+\left(\frac{2}{n}\right)^2} + \frac{1}{1+\left(\frac{3}{n}\right)^2} + \ldots + \frac{1}{1+\left(\frac{n}{n}\right)^2}\right] \\ &= \frac{1}{n}\left[\frac{1}{1+\left(\frac{1}{n}\right)^2} + \frac{1}{1+\left(\frac{2}{n}\right)^2} + \frac{1}{1+\left(\frac{3}{n}\right)^2} + \ldots + \frac{1}{2}\right] \end{aligned}$$

Thus,

$$\lim_{n\to\infty} A = \int_0^1 \frac{1}{1+x^2} = \frac{\pi}{4}.$$

Part (c). Consider the following graph of $y = \frac{1}{1+x^4}$.



Using a similar line of logic presented in part (b), we have that B is the total area of the rectangles above. Hence,

$$\lim_{n\to\infty} B = \int_0^1 \frac{1}{1+x^4}\,\mathrm{d}x.$$

Let $f(x) = \frac{1}{1+x^4}$. Using the trapezium rule,

$$\lim_{n\to\infty} B \approx \frac{1}{2}\cdot\frac{1}{5}\Big[f(0) + 2\big[f(0.2) + f(0.4) + f(0.6) + f(0.8)\big] + f(1)\Big] = 0.86 \text{ (2 d.p.)}.$$

B11 Functions of Two Variables

Tutorial B11

Problem 1. Find the natural domain of the function f for the following:

(a) $f(x,y) = \sqrt{1-x^2-y^2}$

(b) $f(x,y) = \ln\left(x^2-y\right)$

(c) $f(x,y) = \arcsin(x+y)$

(d) $f(x,y) = \frac{1}{x^2-y^2}$

Solution.

Part (a). Observe that the argument of the square root must be non-negative. Hence, $1-x^2-y^2 \geq 0 \implies x^2+y^2 \leq 1$. Thus,

$$D_f = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}.$$

Part (b). Observe that the argument of the natural log must be positive. Hence, $x^2-y > 0 \implies y < x^2$. Thus,

$$D_f = \{(x,y) \in \mathbb{R}^2 : y < x^2\}.$$

Part (c). Observe that the argument of arcsin must be within the range of sin, i.e. between -1 and 1 inclusive. Hence, $-1 \leq x+y \leq 1$. Thus,

$$D_f = \{(x,y) \in \mathbb{R}^2 : -1 \leq x+y \leq 1\}.$$

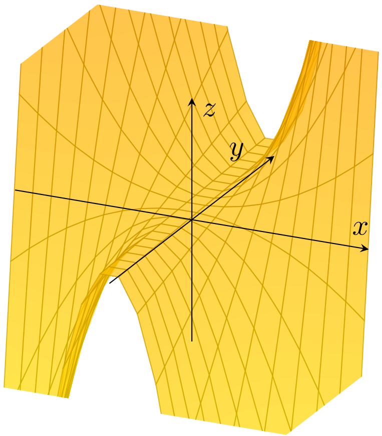
Part (d). Observe that the denominator must be non-zero. Hence, $x^2-y^2 \neq 0 \implies y^2 \neq x^2 \implies y \neq x$ or $y \neq -x$. Thus,

$$D_f = \{(x,y) \in \mathbb{R}^2 : y \neq x \text{ or } y \neq -x\}.$$

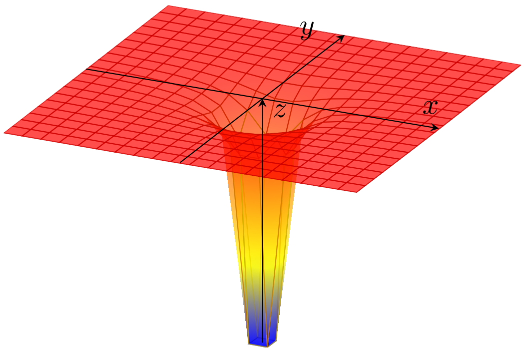
* * * * *

Problem 2. Identify the correct equations of the following surfaces in 3-D space.

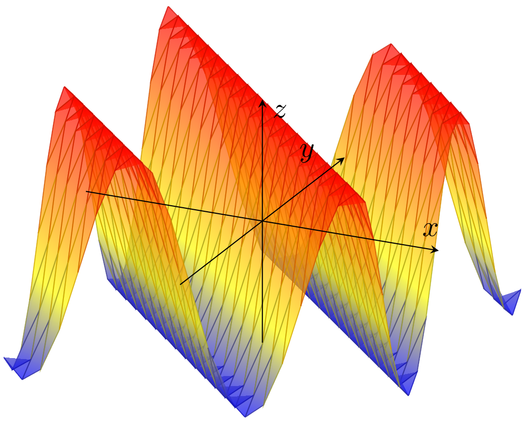
- $z = \cos(x+y)$
- $z = x^2y+1$
- $z = 3-x+y$
- $z = -\frac{1}{\sqrt{x^2+y^2}}$



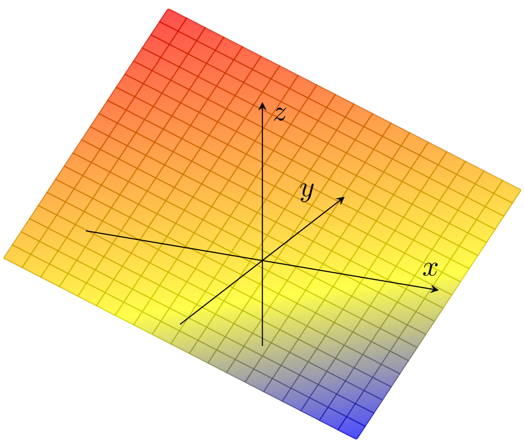
(a)



(b)



(c)



(d)

Solution.

Part (a). $z = x^2y + 1$

Part (b). $z = -\frac{1}{\sqrt{x^2+y^2}}$

Part (c). $z = \cos(x + y)$
Part (d). $z = 3 - x + y$

* * * * *

Problem 3. Let $f(x, y) = x^2 - 2x^3 + 3xy$. Find an equation of the level curve that passes through the point

(a) $(-1, 1)$

(b) $(2, -1)$

(c) $(1, 5)$

Solution.
Part (a). Note that $f(-1, 1) = 0$. Hence, the level curve is given by

$$x^2 - 2x^3 + 3xy = 0.$$

Part (b). Note that $f(2, -1) = -18$. Hence, the level curve is given by

$$x^2 - 2x^3 + 3xy = -18.$$

Part (c). Note that $f(1, 5) = 14$. Hence, the level curve is given by

$$x^2 - 2x^3 + 3xy = 14.$$

* * * * *

Problem 4. If $V(x, y)$ is the voltage or potential at a point (x, y) in the xy-plane, then the level curves of V are called equipotential curves. Along such a curve, the voltage remains constant. Given that

$$V(x, y) = \frac{8}{\sqrt{16 + x^2 + y^2}}$$

find an equation of the equipotential curves at which

(a) $V = 2.0$

(b) $V = 1.0$

(c) $V = 0.5$

Solution. Rearranging the given equation, we have

$$x^2 + y^2 = \frac{64}{V^2} - 16.$$

Part (a). When $V = 2.0$, we have $x^2 + y^2 = \frac{64}{2.0^2} - 16 = 0$, whence

$$x = 0 \text{ and } y = 0.$$

Part (b). When $V = 1.0$, we have

$$x^2 + y^2 = \frac{64}{1.0^2} - 16 = 48.$$

Part (c). When $V = 0.5$, we have

$$x^2 + y^2 = \frac{64}{0.5^2} - 16 = 240.$$

* * * * *

Problem 5. Given that $f(x, y) = x^4 \sin(xy^3)$, find $f_x(x, y)$, $f_y(x, y)$, $f_{xy}(x, y)$ and $f_{yx}(x, y)$.

Solution. Differentiating f with respect to x,

$$f_x(x, y) = 4x^3 \sin(xy^3) + x^4 y^3 \cos(xy^3).$$

Differentiating f with respect to y,

$$f_y(x, y) = 3x^5 y^2 \cos(xy^3).$$

Differentiating f_x with respect to y,

$$\begin{aligned} f_{xy}(x, y) &= 12x^4 y^2 \cos(xy^3) + x^4 \left[3y^2 \cos(xy^3) - 3xy^5 \sin(xy^3)\right] \\ &= 15x^4 y^2 \cos(xy^3) - 3x^5 y^5 \sin(xy^3). \end{aligned}$$

Differentiating f_y with respect to x,

$$\begin{aligned} f_{yx}(x, y) &= 3y^2 \left[5x^4 \cos(xy^3) - x^5 y^3 \sin(xy^3)\right] \\ &= 15x^4 y^2 \cos(xy^3) - 3x^5 y^5 \sin(xy^3). \end{aligned}$$

* * * * *

Problem 6. Given that $z = x^2 y$, $x = t^2$, $y = t^3$, use the chain rule to find $\frac{\mathrm{d}z}{\mathrm{d}t}$ in terms of t.

Solution.

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = 2xy\,(2t) + x^2\left(3t^2\right) = 2t^2 t^3 \cdot 2t + \left(t^2\right)^2 \cdot 3t^2 = 7t^6.$$

* * * * *

Problem 7. Find the gradient of $f(x, y) = 3x^2 y$ at the point $(1, 2)$ and use it to calculate the directional derivative of f at $(1, 2)$ in the direction of the vector $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$.

Solution. Note that $f_x(x, y) = 6xy$ and $f_y(x, y) = 3x^2$. Hence, ∇f at $(1, 2)$ is $\langle 12,\, 3\rangle$.

Observe that the directional derivative of f in the direction of $\mathbf{u}$ at $(1, 2)$ is given by

$$\nabla f \cdot \hat{\mathbf{u}} = \begin{pmatrix} 12 \\ 3 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{48}{5}.$$

Thus, the instantaneous rate of change at $(1, 2)$ in the direction of $\mathbf{u}$ is $48/5$.

* * * * *

Problem 8. Suppose that a point moves along the intersection of the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x = \frac{2}{3}$. Find the rate of z with respect to y when the point is at $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Solution. Note that $x^2+y^2+z^2=1 \implies z=\pm\sqrt{1-x^2-y^2}$. Given that the object we want (the rate of change of z with respect to y) will later be evaluated when $z=\frac{2}{3}>0$, we consider only the positive branch. Let $f(x,y)=\sqrt{1-x^2-y^2}$. Then $f_y(x,y)=\frac{-y}{\sqrt{1-x^2-y^2}}$. Evaluating at the desired point, we get,

$$f_y\left(\frac{2}{3},\frac{1}{3}\right)=\frac{-1/3}{\sqrt{1-(2/3)^2-(1/3)^2}}=-\frac{1}{2}.$$

* * * * *

Problem 9.

(a) The Cauchy-Riemann equations are such that $\partial u/\partial x=\partial v/\partial y$ and $\partial u/\partial y=-\partial v/\partial x$ for $u(x,y)$ and $v(x,y)$. Show that $u=\mathrm{e}^x\cos y$, $v=\mathrm{e}^x\sin y$ satisfy the Cauchy-Riemann equations.

(b) Show that the function $f(x,y)=\mathrm{e}^x\sin y+\mathrm{e}^y\cos x$ satisfies that Laplace equation, i.e. $\partial^2 f/\partial x^2+\partial^2 f/\partial y^2=0$.

(c) If $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations, state the conditions for both u and v to satisfy the Laplace equation.

Solution.

Part (a). Differentiating u with respect to x, we get $\partial u/\partial x=\mathrm{e}^x\cos y$. Differentiating v with respect to y, we get $\partial v/\partial y=\mathrm{e}^x\cos y$. Hence, $\partial u/\partial x=\partial v/\partial y$.

Differentiating u with respect to y, we get $\partial u/\partial y=-\mathrm{e}^x\sin y$. Differentiating v with respect to x, we get $\partial v/\partial x=\mathrm{e}^x\sin y$. Hence, $\partial u/\partial y=-\partial v/\partial x$.

Thus, u and v satisfy the Cauchy-Riemann equations.

Part (b). Differentiating f twice with respect to x,

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(\mathrm{e}^x\sin y-\mathrm{e}^y\sin x)=\mathrm{e}^x\sin y-\mathrm{e}^y\cos x.$$

Differentiating f twice with respect to y,

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(\mathrm{e}^x\cos y+\mathrm{e}^y\cos x)=-\mathrm{e}^x\sin y+\mathrm{e}^y\cos x.$$

Hence,

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=(\mathrm{e}^x\sin y-\mathrm{e}^y\cos x)+(-\mathrm{e}^x\sin y+\mathrm{e}^y\cos x)=0.$$

Thus, $f(x,y)=\mathrm{e}^x\sin y+\mathrm{e}^y\cos x$ satisfies the Laplace equation.

Part (c). Suppose $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations. This gives

$$\begin{cases} u_x=v_y \\ u_y=-v_x \end{cases}$$

Differentiating with respect to x and y, we obtain

$$\begin{cases} u_{xx}=v_{yx} \\ u_{yx}=-v_{xx} \end{cases} \quad \text{and} \quad \begin{cases} u_{xy}=v_{yy} \\ u_{yy}=-v_{xy} \end{cases}$$

This gives

$$\begin{cases} u_{xx}+u_{yy}=v_{yx}-v_{xy} \\ v_{xx}+v_{yy}=-u_{yx}+u_{xy} \end{cases}$$

Hence, if u and v both satisfy the Laplace equation, we require

$$\begin{cases} v_{yx} - v_{xy} = 0 \\ -u_{yx} + u_{xy} = 0 \end{cases}$$

which gives the conditions $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$.

* * * * *

Problem 10. Find the equation of the tangent plane to the surface $z = x^2 y$ at the point $(2, 1, 4)$. Hence, state the normal vector of the tangent plane.

Solution. Let $f(x, y) = x^2 y$. Then $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$. Hence, the equation of the tangent plane at $(2, 1, 4)$ is given by

$$z = 4 + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = 4 + 4(x - 2) + 4(y - 1) = 4x + 4y - 8.$$

Rearranging,

$$4x + 4y - z = 8,$$

whence the normal vector of the tangent plane is $\langle 4,\ 4,\ -1\rangle$.

* * * * *

Problem 11. The volume of a right-circular cone of radius r cm and height h cm is denoted by V. If h increases from 10 cm to 10.01 cm and r decreases from 12 cm to 11.95 cm, use a linear approximation to estimate the volume of the cone after the changes.

Solution. Let $V(r, h) = \frac{1}{3}\pi r^2 h$ be the volume of the cone. We have $V_r(r, h) = \frac{2}{3}\pi r h$ and $V_h(r, h) = \frac{1}{3}\pi r^2$. The equation of the tangent plane at $r = 12$ and $h = 10$ is given by

$$\begin{aligned} v &= V(12, 10) + V_r(12, 10)(r - 12) + V_h(12, 10)(h - 10) \\ &= \frac{1}{3}\pi(12^2)(10) + \frac{2}{3}\pi(12)(10)(r - 12) + \frac{1}{3}\pi(12^2)(h - 10) = 16\pi(5r + 3h - 60). \end{aligned}$$

Evaluating at $r = 11.95$ and $h = 10.01$, we have

$$v = 16\pi\,[5(11.95) + 3(10.01) - 60] = 476.48\pi.$$

The volume of the cone after the changes is hence approximately 476.48π cm^3.

* * * * *

Problem 12. The radius of a right-circular cylinder is measured with an error of at most 2%, and the height is measured with an error of at most 4%. Approximate the maximum possible percentage error in the volume of the cylinder calculated from these measurements.

Solution. Let the volume of the cylinder be $V = \pi r^2 h$. By the chain rule, we have

$$\mathrm{d}V = \frac{\partial V}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial h}\,\mathrm{d}h = 2\pi r h\,\mathrm{d}r + \pi r^2\,\mathrm{d}h.$$

Dividing throughout by $V = \pi r^2 h$,

$$\frac{\mathrm{d}V}{V} = \frac{2\pi r h\,\mathrm{d}r + \pi r^2\,\mathrm{d}h}{\pi r^2 h} = 2\frac{\mathrm{d}r}{r} + \frac{\mathrm{d}h}{h}.$$

Note that $\mathrm{d}V/V$ measures the percentage error of the volume V, while $\mathrm{d}r/r$ and $\mathrm{d}h/h$ measure the percentage error of the radius and height respectively. Hence,

$$\max\frac{\mathrm{d}V}{V} = 2(2\%) + 4\% = 8\%.$$

Problem 13. On a certain mountain, the elevation z above a point (x, y) in a horizontal xy-plane that lies at sea level is $z = 2000 - 2x^2 - 4y^2$ ft. The positive x-axis points east, and the positive y-axis points north. A climber is at the point $(-20, 5, 1100)$.

(a) If the climber uses a compass reading to walk due northeast, will he ascend or descend? Find this rate.

(b) Find the direction where the climber should walk to travel a level path.

Solution.

Part (a). Let $f(x, y) = 2000 - 2x^2 - 4y^2$. Then $f_x(x, y) = -4x$ and $f_y(x, y) = -8y$. Hence,

$$\nabla f = \begin{pmatrix} -4x \\ -8y \end{pmatrix} = -4 \begin{pmatrix} x \\ 2y \end{pmatrix}$$

Note that the vector $\langle 1,\, 1 \rangle$ points northeast.

$$\nabla f \cdot \widehat{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = -4 \begin{pmatrix} x \\ 2y \end{pmatrix} \cdot \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2\sqrt{2}\,(x + 2y)\,.$$

Evaluating at $(-20, 5, 1100)$, the instantaneous rate of change of the climber's altitude would be $-2\sqrt{2}\,(-20 + 2 \cdot 5) = 20\sqrt{2}$ ft/s. That is, the climber would ascend at a rate of $20\sqrt{2}$ feet per second.

Part (b). For a level path, the instantaneous rate of change of the climber's altitude should be 0. Let the direction of the climber be $u = \langle a,\, b \rangle$.

$$D_{\mathbf{u}} f(x, y)\big|_{(-20,5)} = -4 \begin{pmatrix} -20 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} \implies -2a + b = 0 \implies b = 2a.$$

We hence have

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 2a \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus, the climber should walk in the direction of $\langle 1,\, 2 \rangle$.

* * * * *

Problem 14. Find the absolute maximum and minimum values of $f(x, y) = 3xy - 6x - 3y + 7$ on the closed triangular region R with vertices $(0, 0)$, $(3, 0)$ and $(0, 5)$.

Solution. Note that $f_x(x, y) = 3y - 6$ and $f_y(x, y) = 3x - 3$, whence $f_{xx}(x, y) = f_{yy}(x, y) = 0$ and $f_{xy} = 3$. For stationary points,

$$\nabla f = \mathbf{0} \implies \begin{pmatrix} 3y - 6 \\ 3x - 3 \end{pmatrix} = \mathbf{0} \implies x = 1,\; y = 2.$$

Consider the nature of the stationary point at $(1, 2)$. We have

$$D = f_{xx}(1, 2) f_{yy}(1, 2) - [f_{xy}(1, 2)]^2 = -9 < 0$$

Hence, by the second derivative test, we see that $f(x, y)$ has a saddle point at $(1, 2)$. Thus, the extrema of $f(x, y)$ must occur along its boundary.

Note that the boundary of $f(x, y)$ is given by

- $x = 0,\; y \in [0, 5]$
- $x \in [0, 3],\; y = 0$

- $x \in [0,3]$, $y = 5 - \frac{5}{3}x$

Case 1: $x = 0$, $y \in [0,5]$. We have that $f(0,y) = -3y + 7$, which clearly attains a maximum of 7 at $y = 0$ and a minimum of -8 at $y = 5$.

Case 2: $x \in [0,3]$, $y = 0$. We have that $f(x,0) = -6x + 7$, which clearly attains a maximum of 7 at $x = 0$ and a minimum of -11 at $x = 3$.

Case 3: $x \in [0,3]$, $y = 5 - \frac{5}{3}x$. Observe that

$$f\left(x, 5 - \frac{5}{3}x\right) = 3x\left(5 - \frac{5}{3}x\right) - 6x - 3\left(5 - \frac{5}{3}x\right) + 7 = -(x-2)(5x-4)$$

is concave down and has a turning point at $x = 1.4$. Hence, the function clearly attains a maximum of 1.8 when $x = 1.4$ and a minimum of -11 when $x = 3$ (note that at $x = 0$, the function returns -8).

Hence, the maximum of $f(x,y)$ is 7, while the minimum is -11.

* * * * *

Problem 15. Find the dimensions of a rectangular box, open at the top, having a volume of 32 cm^3, and requiring the least amount of material for its construction.

Solution. Let the box have side lengths of x, y and z cm. Given that the volume of the box is fixed at 32 cm^3, we have

$$xyz = 32 \implies z = \frac{32}{xy}$$

Let the surface area of the box be measured by $f(x,y)$. Then

$$f(x,y) = xy + 2yz + 2xz = xy + 2y\left(\frac{32}{xy}\right) + 2x\left(\frac{32}{xy}\right) = xy + 64x^{-1} + 64y^{-1}.$$

Note that

$$\nabla f = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix} = \begin{pmatrix} y - 64x^{-2} \\ x - 64y^{-2} \end{pmatrix}.$$

For stationary points, $\nabla f = \mathbf{0}$. We hence obtain

$$\begin{cases} y = 64x^{-2} \\ x = 64y^{-2} \end{cases} \implies \begin{cases} yx^2 = 64 \\ xy^2 = 64 \end{cases} \implies x^3y^3 = 64^2 \implies xy = 16$$

Hence,

$$x = \frac{x^2y}{xy} = \frac{64}{16} = 4,$$

whence $y = 4$ and $z = 2$. Thus, $f(x,y)$ has a stationary point at $(4,4,2)$.

We now consider the nature of this stationary point. Note that

$$f_{xx}(x,y) = 128x^{-3}, \quad f_{yy} = 128y^{-3}, \quad f_{xy} = 1.$$

Hence,

$$D = f_{xx}(4,4)f_{yy}(4,4) - [f_{xy}(4,4)]^2 = 3$$

Since $D > 0$ and $f_{xx}(4,4) = 2 > 0$, by the second derivative test, $f(x,y)$ attains a minimum at $(4,4,2)$. Thus, the amount of material required is lowest for a box of dimension $4 \times 4 \times 2$.

Problem 16. Find the quadratic approximation of $f(x, y) = x^2y + xy^2$ around the point $(1, 1)$.

Solution. Taking partial derivatives, we have

$$f_x(x, y) = 2xy + y^2, \quad f_y(x, y) = 2xy + x^2$$
$$f_{xx}(x, y) = 2y, \quad f_{xy}(x, y) = 2x + 2y, \quad f_{yy}(x, y) = 2x.$$

Hence, the required quadratic approximation $Q(x, y)$ is given by

$$\begin{aligned} Q(x, y) &= f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \\ &\quad + \frac{1}{2}f_{xx}(1, 1)(x - 1)^2 + f_{xy}(1, 1)(x - 1)(y - 1) + \frac{1}{2}f_{yy}(1, 1)(y - 1)^2 \\ &= 2 + 3(x - 1) + 3(y - 1) + (x - 1)^2 + 4(x - 1)(y - 1) + (y - 1)^2 \\ &= 2 - 3x - 3y + 4xy + x^2 + y^2 \end{aligned}$$

Self-Practice B11

Problem 1. At what rate is the area of a rectangle changing if its length is 15 units and increasing at 3 units/s while its width is 6 units and increasing at 2 units/s?

Solution. Let $l(t)$ and $w(t)$ be the length and width of the rectangle respectively, where t is the time in seconds. Let $A = lw$ be the area of the rectangle. Note that

$$\frac{\partial A}{\partial l} = w \quad \text{and} \quad \frac{\partial A}{\partial w} = l.$$

Also,

$$\frac{\mathrm{d}l}{\mathrm{d}t} = 3 \quad \text{and} \quad \frac{\mathrm{d}w}{\mathrm{d}t} = 2.$$

Hence,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial l}\frac{\mathrm{d}l}{\mathrm{d}t} + \frac{\partial A}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t} = 3w + 2l.$$

When $l = 15$ and $w = 6$, we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 3(6) + 2(15) = 48.$$

Thus, the area of the rectangle is increasing at a rate of 48 units^2/s.

* * * * *

Problem 2. A particle moving along a metal plate in the xy-plane has the velocity $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$ cm/s at the point $(3, 2)$. If the temperature of the plate at points in the xy-plane is $T(x, y) = y^2 \ln x$ where $x \geq 1$, in degrees Celsius, find $\mathrm{d}T/\mathrm{d}t$ at $(3, 2)$.

Solution. Note that

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{y^2}{x}\right)(1) + (2y\ln x)(-4) = \frac{y^2}{x} - 9y\ln x.$$

At $(3, 2)$, we have

$$\left.\frac{\mathrm{d}T}{\mathrm{d}t}\right|_{(3,2)} = \frac{2^2}{3} - 9(2)\ln 3 = -16.2 \text{ (3 d.p.)}.$$

* * * * *

Problem 3. Given that $f(x, y) = x^2 \mathrm{e}^y$, find the maximum value of a directional derivative at $(-2, 0)$ and give a unit vector in the direction in which the maximum value occurs.

Solution. At $(-2, 0)$, we have

$$\nabla f = \begin{pmatrix} 2x\mathrm{e}^y \\ x^2\mathrm{e}^y \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

Note that

$$D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta,$$

where θ is the angle between ∇f and $\mathbf{u}$. Hence, the maximum value of the directional derivative is

$$\left|\begin{pmatrix} -4 \\ 4 \end{pmatrix}\right| = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}.$$

This occurs when $\theta = 0$, i.e. when $\mathbf{u}$ is the same direction as ∇f. Hence,

$$\mathbf{u} = \frac{1}{4\sqrt{2}}\begin{pmatrix} -4 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Problem 4. If the electric potential at a point (x, y) in the xy-plane is $V(x, y)$, where $V(x, y) = \mathrm{e}^{-2x} \cos 2y$, find the direction where V decreases most rapidly at $(0, \pi/6)$.

Solution. At $(0, \pi/6)$, we have

$$\nabla V = \begin{pmatrix} -2\mathrm{e}^{-2x} \cos 2y \\ -2\mathrm{e}^{-2x} \sin 2y \end{pmatrix} = -\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}.$$

Note that

$$D_{\mathbf{u}} V(x, y) = \nabla V \cdot \mathbf{u} = |\nabla V| \cos \theta,$$

where θ is the angle between ∇V and $\mathbf{u}$. Hence, V decreases the most when $\theta = \pi$, i.e. when $\mathbf{u}$ is in the opposite direction as ∇V. Thus, the desired direction is $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

* * * * *

Problem 5. Find all local extrema and saddle points of $f(x, y) = 4xy - x^4 - y^4$.

Solution. Note that

$$\nabla f = \begin{pmatrix} 4y - 4x^3 \\ 4x - 4y^3 \end{pmatrix}.$$

Setting this equal to the zero vector, we have the system

$$\begin{cases} -4x^3 + \ \ 4y = 0 \\ \quad 4x - 4y^3 = 0 \end{cases}.$$

From the first equation, we get $y = x^3$. Substituting this into the second equation yields

$$x - x^9 = x\left(1 - x^8\right) = 0.$$

Note that $x^8 - 1$ factors as $\left(x^4 + 1\right)\left(x^2 + 1\right)(x + 1)(x - 1)$. We thus have $x = -1, 0, 1$, which correspond to the points $(-1, -1)$, $(0, 0)$ and $(1, 1)$.

Let

$$D = f_{xx} f_{yy} - f_{xy}^2 = \left(-12x^2\right)\left(-12y^2\right) - (4)^2 = 144x^2y^2 - 16.$$

Case 1. At $(-1, -1)$, we have

$$D = 144(-1)^2(-1)^2 - 16 = 128 > 0.$$

Since $f_{xx} = -12(-1)^2 = -12 < 0$, by the second partial derivative test, $(-1, -1)$ is a maximum point.

Case 2. At $(0, 0)$, we have

$$D = 144(0)^2(0)^2 - 16 = -16 < 0.$$

By the second partial derivative test, $(0, 0)$ is a saddle point.

Case 3. At $(1, 1)$, we have

$$D = 144(1)^2(1)^2 - 16 = 128 > 0.$$

Since $f_{xx} = -12(1)^2 = -12 < 0$, by the second partial derivative test, $(1, 1)$ is a maximum point.

* * * * *

Problem 6. Find the absolute extrema of $f(x, y) = x^2 + 2y^2 - x$ such that the domain of this function f is the circular region $x^2 + y^2 \leq 4$.

Solution. Note that

$$\nabla f = \begin{pmatrix} 2x-1 \\ 4y \end{pmatrix}.$$

Setting this equal to the zero vector, we see that f has only one stationary point at $(1/2, 0)$, which is in the domain. At this point,

$$f\left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}\right)^2 + 2(0)^2 - \frac{1}{2} = -\frac{1}{4}.$$

We now consider the points along the boundary of D_f, which is given by the equation $x^2 + y^2 = 4$. Substituting $y^2 = 4 - x^2$ into the definition of $f(x, y)$, we get the univariate function $g(x)$:

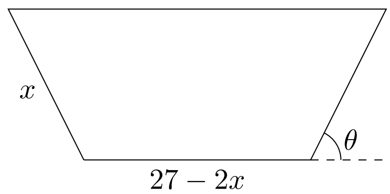
$$g(x) = x^2 + 2\left(4 - x^2\right) - x = -\left(x + \frac{1}{2}\right)^2 + \frac{33}{4}.$$

Also note that $x \in [-2, 2]$. Clearly, $g(x)$ attains a maximum of $33/4$ at $x = -1/2$ and a minimum of 2 at $x = 2$.

Thus, the absolute maximum of $f(x, y)$ is $33/4$, while the absolute minimum of $f(x, y)$ is $-1/4$.

* * * * *

Problem 7. A length of sheet metal 27 cm wide is to be made into a water trough by bending up two sides as shown in the figure below. Find the values of x and θ such that the trapezoid-shaped cross-section has a maximum area.



Solution 1. Take the mirror image of the figure and place it on top of the original image. We get a hexagon with perimeter 54, and we are tasked with maximizing its area. It is a well-known result that for n-sided polygons with fixed perimeters, the regular n-gon encloses the largest area. Hence, we have $x = 54/6 = 9$ and $\theta = 2\pi/6 = \pi/3$.

Solution 2. Observe that the longer side of the trapezium is given by $(27-2x)+2(x\cos\theta)$, while the height of the trapezium is given by $x \sin\theta$. The area A of the trapezium is thus given by

$$A = \frac{(27-2x) + (27 - 2x + 2x\cos\theta)}{2}(x\sin\theta) = 27x\sin\theta + x^2\sin\theta\left(\cos\theta - 2\right).$$

Observe that

$$\nabla A = \begin{pmatrix} A_x \\ A_\theta \end{pmatrix} = \begin{pmatrix} 27\sin\theta + 2x\sin\theta\left(\cos\theta - 2\right) \\ 27x\cos\theta + x^2\left(\cos^2\theta - \sin^2\theta - 2\cos\theta\right) \end{pmatrix}.$$

Setting this equal to the zero vector, we get the following system:

$$\begin{cases} 27\sin\theta + 2x\sin\theta\left(\cos\theta - 2\right) & = 0, \\ 27x\cos\theta + x^2\left(\cos^2\theta - \sin^2\theta - 2\cos\theta\right) & = 0. \end{cases}$$

From the first equation, we get

$$x = \frac{-27}{2\left(\cos\theta - 2\right)}.$$

Substituting this into the second equation,

$$27\left(\frac{-27}{2\left(\cos\theta-2\right)}\right)\cos\theta+\left(\frac{-27}{2\left(\cos\theta-2\right)}\right)^2\left(\cos^2\theta-\sin^2\theta-2\cos\theta\right)=0.$$

Clearing denominators and simplifying, we get

$$-2\cos\theta\left(\cos\theta-2\right)+\left(\cos^2\theta-\sin^2\theta-2\cos\theta\right)=0.$$

Expanding, we have

$$-\cos^2\theta+2\cos\theta-\sin^2\theta=0,$$

from which we immediately get $\cos\theta=1/2$, whence $\theta=\pi/3$ and $x=9$.

We now calculate the second partial derivatives of A at $\theta=\pi/3$ and $x=9$. First, we have

$$A_{xx}=2\sin\theta\left(\cos\theta-2\right)=-2.5981\ (5\text{ s.f.}).$$

Secondly, we have

$$A_{\theta\theta}=-27x\sin\theta+x^2\left(-2\cos\theta\sin\theta-2\sin\theta\cos\theta+2\sin\theta\right)=-210.44\ (5\text{ s.f.}).$$

Lastly, we have

$$A_{\theta x}=27\cos\theta+2x\left(\cos^2\theta-\sin^2\theta-2\cos\theta\right)=-13.5.$$

Since

$$D=A_{xx}A_{\theta\theta}-A_{\theta x}^2=(-2.5981)(-210.44)-(-13.5)^2=364.5>0$$

and $A_{xx}=-2.5981<0$, by the second partial derivative test, A attains a maximum when $x=9$ and $\theta=\pi/3$.

* * * * *

Problem 8. A Further Maths student smiled when a question asked for him to find the quadratic approximation for the function of $f(x,y)=xy-3y-x$ around the point $(2,3)$. Explain why he is so delighted.

Solution. The function $f(x,y)=xy-3y-x$ is already a quadratic, hence the quadratic approximation to $f(x,y)$ is simply $f(x,y)$ itself.

* * * * *

Problem 9. A company produces two products, A and B, which require different amounts of two resources, Resource 1 and Resource 2. The profit generated by selling product A is \$10 per unit, and the profit from selling product B is \$15 per unit. Each unit of product A requires 2 units of Resource 1 and 1 unit of Resource 2. Each unit of product B requires 1 unit of Resource 1 and 3 units of Resource 2. The company has a total of 100 units of Resource 1 and 90 units of Resource 2. What should the company produce in order to maximize its profitability?

Solution. We use linear programming to solve this problem.

Let n and m be the amount of product A and B produced by the company. Let Π be the total revenue earned, i.e.

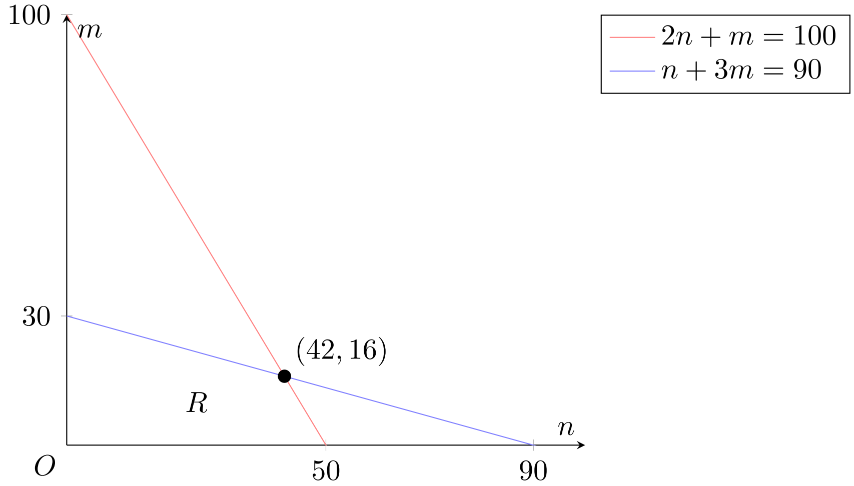
$$\Pi=10n+15m.$$

Due to resource constraints, we have the inequalities

$$\begin{cases}2n+m & \leq 100,\\ n+3m & \leq 90.\end{cases}$$

Additionally, we have $n, m \geq 0$.

We can visualize these inequalities graphically:

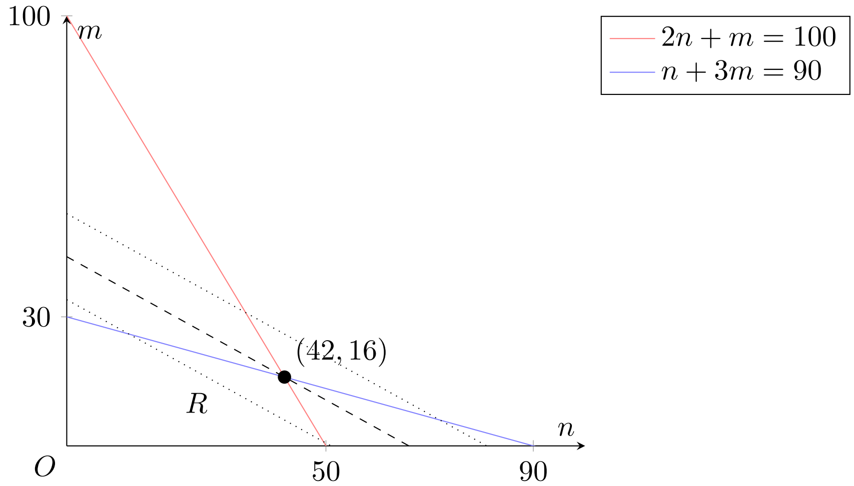


Here, R represents the feasible zone, i.e. the region where both inequalities are satisfied. This is the region where the company can produce. Also note that the two "boundaries" intersect at $(42, 16)$.

Now, recall that $\Pi = 10n + 15m$. Rearranging,

$$m = \frac{\Pi}{15} - \frac{2}{3}n.$$

If we plot this, we get a line with m-intercept $\Pi/15$ and gradient $-2/3$. Our goal of maximizing Π can be restated as "find the largest value $\Pi/15$ such that a line with gradient $-2/3$ intersects the region R once".



From the figure above, it is easy to see that the "optimal line" will only intersect the region R at only one point: $(42, 16)$. Hence, the company should produce 42 units of product A and 16 units of product B.

Assignment B11

Problem 1. Show that if f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is perpendicular to the level curve of f through (x_0, y_0).

Solution. Let $f(x, y) = (x(t), y(t))$. Let the level curve at (x_0, y_0) have equation $f(x, y) = c$. Implicitly differentiating this with respect to t, we get

$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \end{pmatrix} = \nabla f \cdot \mathbf{u} = 0,$$

where $\mathbf{u}$ is the tangent to the level curve at (x_0, y_0). Since both ∇f and $\mathbf{u}$ are non-zero vectors, they must be perpendicular to each other.

* * * * *

Problem 2. Find the quadratic approximation of $f(x, y) = \mathrm{e}^{x^2+y^2}$ around the point $\left(\frac{1}{2}, 0\right)$.

Solution. Observe that we have

$$f_x(x, y) = 2x\mathrm{e}^{x^2+y^2}, \quad f_y(x, y) = 2y\mathrm{e}^{x^2+y^2}$$
$$f_{xx}(x, y) = 2\mathrm{e}^{x^2+y^2}(2x^2 + 1), \quad f_{xy}(x, y) = 4xy\mathrm{e}^{x^2+y^2}, \quad f_{yy}(x, y) = 2\mathrm{e}^{x^2+y^2}(2y^2 + 1).$$

Evaluating $f(x, y)$ and the above partial derivatives at $\left(\frac{1}{2}, 0\right)$, we obtain

$$f(x, y) = \mathrm{e}^{1/4}, \quad f_x(x, y) = \mathrm{e}^{1/4}, \quad f_y(x, y) = 0$$
$$f_{xx}(x, y) = 3\mathrm{e}^{1/4}, \quad f_{xy}(x, y) = 0, \quad f_{yy}(x, y) = 2\mathrm{e}^{1/4}.$$

The quadratic approximation $Q(x, y)$ to $f(x, y)$ at $\left(\frac{1}{2}, 0\right)$ is hence

$$Q(x, y) = \mathrm{e}^{1/4} + \mathrm{e}^{1/4}\left(x - \frac{1}{2}\right) + 3\mathrm{e}^{1/4}\left(x - \frac{1}{2}\right)^2 + \mathrm{e}^{1/4}y^2.$$

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Problem 3. A common problem in experimental work is to obtain a mathematical relationship between two variables x and y by "fitting" a curve to points in the plane corresponding to various experimentally determines values of x and y, say

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n).$$

Based on theoretical considerations, or simply on the pattern of the points, one decides on the general form of the curve to be fitted. Often, the "curve" to be fitted is a straight line, $y = ax + b$. One criterion for selecting a line of "best fit" is to choose a and b to minimize the function

$$f(a, b) = \sum_{k=1}^{n}(ax_k + b - y_k)^2.$$

Geometrically, $|ax_k + b - y_k|$ is the vertical distance between the data point (x_k, y_k) and the line $y = ax + b$, so in effect, minimizing $f(a, b)$ minimizes the sum of the squares of the vertical distances. This procedure is called the method of least squares.

(a) Show that the conditions $\partial f/\partial a = 0$ and $\partial f/\partial b = 0$ result in the equations

$$\left(\sum_{k=1}^{n} x_k^2\right) a + \left(\sum_{k=1}^{n} x_k\right) b = \sum_{k=1}^{n}(x_k y_k)$$
$$\left(\sum_{k=1}^{n} x_k\right) a + nb = \sum_{k=1}^{n} y_k$$

(b) Solve the equations for a and b to show that

$$a = \frac{n\sum_{k=1}^{n}(x_k y_k) - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right)}{n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2}$$

and

$$b = \frac{\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k\right) - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n}(x_k y_k)\right)}{n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2}.$$

(c) Given that $\bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$, show that $n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2 > 0$.

(d) Find $f_{aa}(a,b)$, $f_{bb}(a,b)$ and $f_{ab}(a,b)$.

(e) Show that f has a relative minimum at the critical point found in (b).

Solution.

Part (a). Observe that

$$\frac{\partial f}{\partial a} = \frac{\partial}{\partial a}\sum_{k=1}^{n}(ax_k + b - y_k)^2 = \sum_{k=1}^{n} 2x_k(ax_k + b - y_k) = 2\sum_{k=1}^{n}(ax_k^2 + bx_k - x_k y_k).$$

Hence,

$$\frac{\partial f}{\partial a} = 2\sum_{k=1}^{n}(ax_k^2 + bx_k - x_k y_k) = 0 \implies a\sum_{k=1}^{n} x_k^2 + b\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} x_k y_k.$$

Observe that

$$\frac{\partial f}{\partial b} = \frac{\partial}{\partial b}\sum_{k=1}^{n}(ax_k + b - y_k)^2 = \sum_{k=1}^{n} 2(ax_k + b - y_k) = 2\left[\sum_{k=1}^{n}(ax_k - y_k) + bn\right].$$

Hence,

$$\frac{\partial f}{\partial b} = 2\left[\sum_{k=1}^{n}(ax_k - y_k) + bn\right] = 0 \implies a\sum_{k=1}^{n} x_k + bn = \sum_{k=1}^{n} y_k.$$

Part (b). Let

$$A = \sum_{k=1}^{n} x_k^2, \quad B = \sum_{k=1}^{n} x_k, \quad C = \sum_{k=1}^{n}(x_k y_k), \quad D = n, \quad E = \sum_{k=1}^{n} y_k.$$

The above equations transform into

$$\begin{cases} Aa + Bb = C \\ Ba + Db = E \end{cases}.$$

One can easily solve the system for a and b, yielding

$$a = \frac{CD - BE}{AD - B^2}, \quad b = \frac{AE - BC}{AD - B^2}.$$

Thus,

$$a = \frac{n\sum_{k=1}^{n}(x_k y_k) - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right)}{n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2}$$

and

$$b = \frac{\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k\right) - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n}(x_k y_k)\right)}{n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2}.$$

Part (c). Observe that

$$\bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k \implies \sum_{k=1}^{n} x_k = n\bar{x}.$$

Consider $n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2$.

$$\begin{aligned} n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2 &= n\left(\sum_{k=1}^{n} x_k^2 - n\bar{x}^2\right) = n\left(\sum_{k=1}^{n} x_k^2 - 2n\bar{x}^2 + n\bar{x}^2\right) \\ &= n\left[\sum_{k=1}^{n} x_k^2 - 2n\bar{x}\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) + \sum_{k=1}^{n} \bar{x}^2\right] = n\left(\sum_{k=1}^{n} x_k^2 - \sum_{k=1}^{n} 2x_k\bar{x} + \sum_{k=1}^{n} \bar{x}^2\right) \\ &= n\sum_{k=1}^{n}\left(x_k^2 - 2x_k\bar{x} + \bar{x}^2\right) = n\sum_{k=1}^{n}\left(x_k - \bar{x}\right)^2. \end{aligned}$$

Given that the RHS is a sum of squares, it must be greater than or equal to 0. We thus have the inequality

$$n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2 \geq 0.$$

However, if $n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2 = 0$, then both a and b would be undefined. Thus, we must have a strict inequality, which gives

$$n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2 > 0.$$

Part (d). From (a), we have

$$f_a(a,b) = 2a\sum_{k=1}^{n} x_k^2 + 2b\sum_{k=1}^{n} x_k - 2\sum_{k=1}^{n} x_k y_k$$

and

$$f_b(a,b) = 2a\sum_{k=1}^{n} x_k + 2nb - 2\sum_{k=1}^{n} y_k.$$

Thus,

$$f_{aa}(a,b) = 2\sum_{k=1}^{n} x_k^2, \qquad f_{ab}(a,b) = 2\sum_{k=1}^{n} x_k, \qquad f_{bb}(a,b) = 2n.$$

Part (e). Let $D = f_{aa}(a,b)f_{bb}(a,b) - [f_{ab}(a,b)]^2$. From part (d), we have

$$D = 4\left[n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2\right],$$

which is clearly positive from part (c). Furthermore, $f_{aa}(a,b) = 2\sum_{k=1}^{n} x_k^2$ is clearly positive (note that we reject the equality for the reason stated in part (c)). Thus, by the second partial derivative test, the critical point found in part (b) must be a minimum point.

B12 Separable Differential Equations

Tutorial B12

Problem 1. Given that $y = 1$ when $x = 1$, find the particular solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x}$.

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x} \implies \frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} \implies \int \frac{1}{y^2}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x$$
$$\implies -\frac{1}{y} = \ln|x| + C_1 \implies y = \frac{1}{C - \ln|x|}, \quad C = -C_1.$$

Since $y(1) = 1$, we have

$$1 = \frac{1}{C - \ln|1|} \implies C = 1 \implies y = \frac{1}{1 - \ln|x|}.$$

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Problem 2. Two variables x and t are connected by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{kx}{10-x}$, where $0 < x < 10$ and where k is a constant. It is given that $x = 1$ when $t = 0$ and that $x = 2$ when $t = 1$. Find the value of t when $x = 5$, given your answer to three s.f.

Solution.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{kx}{10-x} \implies \frac{10-x}{x}\frac{\mathrm{d}x}{\mathrm{d}t} = k$$
$$\implies \int \frac{10-x}{x}\,\mathrm{d}x = \int k\,\mathrm{d}t \implies 10\ln x - x = kt + C.$$

Evaluating at $x = 1$ and $t = 0$,

$$10\ln(1) - 1 = k(0) + C \implies C = -1.$$

Evaluating at $x = 2$ and $t = 1$,

$$10\ln(2) - 2 = k(1) - 1 \implies k = 10\ln 2 - 1.$$

Hence, evaluating at $x = 5$, we get

$$10\ln(5) - 5 = (10\ln 2 - 1)\,t - 1 \implies t = 2.04 \text{ (3 s.f.)}.$$

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Problem 3. Use the substitution $y = u - 2x$ to find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{8x+4y+1}{4x+2y+1}$.

Solution. Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{8x+4y+1}{4x+2y+1} = -2 + \frac{1}{4x+2y+1}.$$

Also note that under the substitution $y = u - 2x$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 2.$$

Thus,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{4x+2y+1} = \frac{1}{4x+2(u-2x)+1} = \frac{1}{2u+1} \implies (2u+1)\frac{\mathrm{d}u}{\mathrm{d}x} = 1.$$

Integrating with respect to x,

$$\int (2u+1)\ \mathrm{d}u = \int 1\,\mathrm{d}x \implies u^2 + u = x + C \implies (y+2x)^2 + y + x = C.$$

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Problem 4. By using the substitution $z = y\mathrm{e}^{2x}$, find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x\mathrm{e}^{-2x}$.

Find the particular solution of the differential equation given that $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Solution. Note that

$$z = ye^{2x} \implies \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{2x} + 2y\mathrm{e}^{2x} = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{2x} + 2z \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x}\mathrm{e}^{-2x} - 2y.$$

Substituting this into the given differential equation,

$$\frac{\mathrm{d}z}{\mathrm{d}x}\mathrm{e}^{-2x} - 2y + 2y = x\mathrm{e}^{-2x} \implies \frac{\mathrm{d}z}{\mathrm{d}x} = x.$$

Integrating with respect to x, we easily see that

$$y\mathrm{e}^{2x} = z = \frac{z^2}{2} + C \implies y = \frac{x^2}{2\mathrm{e}^{2x}} + \frac{C}{\mathrm{e}^{2x}}.$$

Since $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$, we have

$$1 + 2y = 0 \implies y = -\frac{1}{2} \implies C = -\frac{1}{2}.$$

The desired particular solution is hence

$$y = \frac{x^2 - 1}{2\mathrm{e}^{2x}}.$$

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Problem 5. Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 6xy^3$.

Find its particular solution given that $y = 0.5$ when $x = 0$.

Determine the interval of validity for the particular solution.

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6xy^3 \implies \frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x} = 6x \implies \int \frac{1}{y^3}\,\mathrm{d}y = \int 6x\,\mathrm{d}x$$
$$\implies -\frac{1}{2y^2} = 3x^2 + C_1 \implies y^2 = \frac{1}{C - 6x^2}.$$

Since $y(0) = 0.5$, we have

$$(0.5)^2 = \frac{1}{C - 6(0)^2} \implies C = 4.$$

Thus, the particular solution is

$$y^2 = \frac{1}{4 - 6x^2}.$$

For the solution to be valid, we require $4 - 6x^2 > 0$, whence $x \in \left(-\sqrt{2/3}, \sqrt{2/3}\right)$.

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Problem 6.

(a) Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x}{x^2+1}$.

(b) What can you say about the gradient of every solution as $x \to \pm\infty$?

(c) Find the particular solution of the differential equation for which $y = 2$ when $x = 0$. Hence, sketch the graph of this solution.

Solution.

Part (a).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x}{x^2+1} = \frac{3}{2}\left(\frac{2x}{x^2+1}\right) \implies y = \frac{3}{2}\ln\left(x^2+1\right) + C.$$

Part (b).

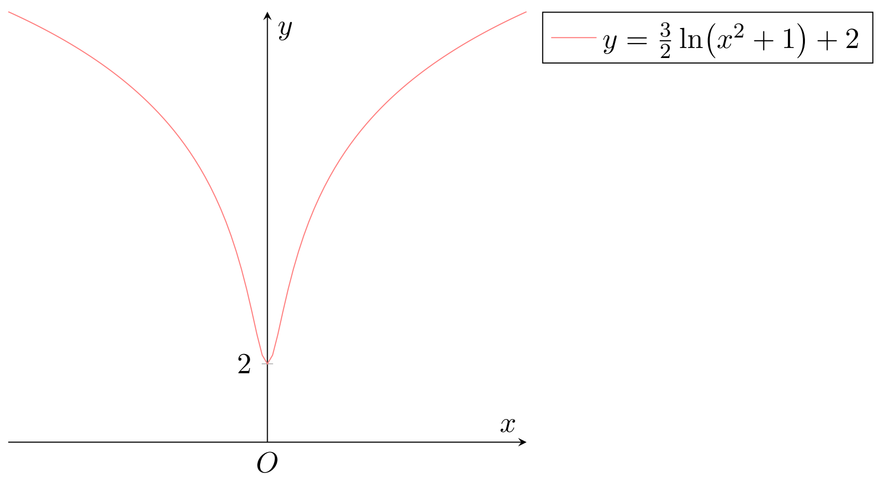
$$\lim_{x\to\pm\infty}\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{x\to\pm\infty}\frac{3x}{x^2+1} = 0.$$

Part (c). Evaluating the general solution at $x = 0$ and $y = 2$, we get

$$2 = \frac{3}{2}\ln\left(0^2+1\right) + C \implies C = 2.$$

Thus, the particular solution is

$$y = \frac{3}{2}\ln\left(x^2+1\right) + 2.$$



Problem 7. The variables x, y and z are connected by the following differential equations.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3 - 2z \qquad (*)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = z$$

(a) Given that $z < \frac{3}{2}$, solve equation $(*)$ to find z in terms of x.

(b) Hence, find y in terms of x.

(c) Use the result in part (b) to show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = a\frac{\mathrm{d}y}{\mathrm{d}x} + b$$

for constants a and b to be determined.

(d) The curve of the solution in part (b) passes through the points $(0, 1)$ and $\left(2, 3 + \mathrm{e}^{-4}\right)$. Sketch this curve, indicating its axial intercept and asymptote (if any).

Solution.

Part (a).

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3 - 2z \implies \frac{1}{3-2z}\frac{\mathrm{d}z}{\mathrm{d}x} = 1 \implies \int \frac{1}{3-2z}\,\mathrm{d}z = \int 1\,\mathrm{d}x$$
$$\implies -\frac{1}{2}\ln(3-2z) = x + C_1 \implies z = \frac{3}{2} - A\mathrm{e}^{-2x}, \quad A = \frac{\mathrm{e}^{-2C_1}}{2}.$$

Thus, the general solution is

$$z = \frac{3}{2} - A\mathrm{e}^{-2x}, \quad A \in \mathbb{R}^+.$$

Part (b).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2} - A\mathrm{e}^{-2x} \implies y = \int\left(\frac{3}{2} - A\mathrm{e}^{-2x}\right)\mathrm{d}x = \frac{3}{2}x + \frac{A}{2}\mathrm{e}^{-2x} + B, \quad B \in \mathbb{R}.$$

Part (c).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2} - A\mathrm{e}^{-2x} \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2A\mathrm{e}^{-2x} = 2\left(\frac{3}{2} - \frac{\mathrm{d}y}{\mathrm{d}x}\right) = -2\frac{\mathrm{d}y}{\mathrm{d}x} + 3.$$

Hence, $a = -2$ and $b = 3$.

Part (d). Evaluating the general solution at $(0, 1)$, we obtain

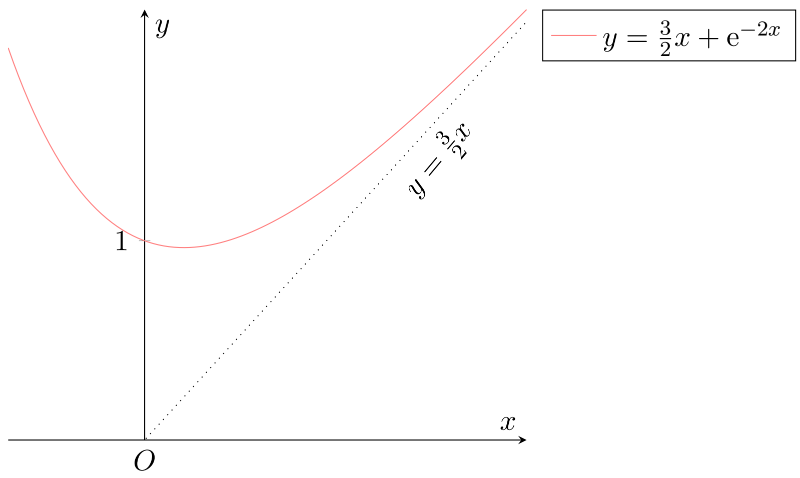
$$1 = \frac{3}{2}(0) + \frac{A}{2}\mathrm{e}^{-2(0)} + B \implies B = 1 - \frac{A}{2}.$$

Evaluating the general solution at $\left(2, 3 + \mathrm{e}^{-4}\right)$, we obtain

$$3 + \mathrm{e}^{-4} = \frac{3}{2}(2) + \frac{A}{2}\mathrm{e}^{-2(2)} + \left(1 - \frac{A}{2}\right) \implies A = 2.$$

The curve thus has equation

$$y = \frac{3}{2}x + \mathrm{e}^{-2x}.$$



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Problem 8. A bottle containing liquid is taken from a refrigerator and placed in a room where the temperature is a constant 20 °C. As the liquid warms up, the rate of increase of its temperature θ °C after time t minutes is proportional to the temperature difference $(20-\theta)$ °C. Initially the temperature of the liquid is 10 °C and the rate of increase of the temperature is 1 °C per minute. By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-t/10}$.

Find the time it takes the liquid to reach a temperature of 15 °C, and state what happens to θ for large values of t. Sketch a graph of θ against t.

Solution. Since $\frac{d\theta}{dt} \propto (20-\theta)$, we have $\frac{d\theta}{dt} = k(20-\theta)$, where k is a constant. We now solve for θ.

$$\frac{d\theta}{dt} = k(20-\theta) \implies \frac{1}{20-\theta}\frac{d\theta}{dt} = k \implies \int \frac{1}{20-\theta}\,d\theta = \int k\,dt$$
$$\implies -\ln(20-\theta) = kt + C_1 \implies \theta = 20 - Ce^{-kt}, \quad C = e^{-C_1}.$$

Evaluating at $\theta = 0$ and $\theta = 10$, we get

$$10 = 20 - Ce^{-0} \implies C = 10.$$

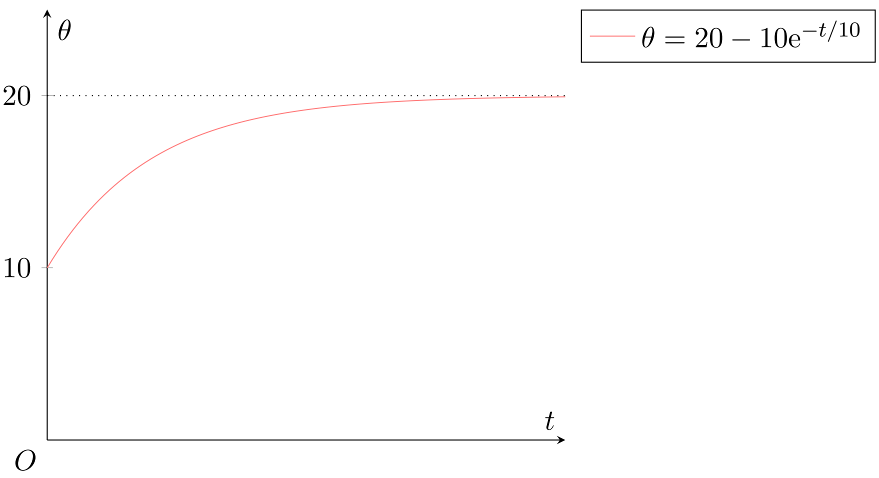
Additionally, since $\frac{d\theta}{dt} = 1$ when $t = 0$, we have

$$1 = k\left[20 - (20 - 10e^{0})\right] = 10k \implies k = \frac{1}{10}.$$

Thus,

$$\theta = 20 - 10e^{-t/10}.$$

Using G.C., when $\theta = 15$, we have $t = 6.93$. Thus, it takes 6.93 minutes for the liquid to reach a temperature of 15°C. As t tends to infinity, θ tends towards 20.



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Problem 9.

(a) Find $\int \frac{1}{100-v^2} \, \mathrm{d}x$.

(b) A stone is dropped from a stationary balloon. It leaves the balloon with zero speed, and t seconds later its speed v metres per second satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.1v^2.$$

(i) Find t in terms of v. Hence, find the exact time the stone takes to reach a speed of 5 metres per second.

(ii) Find the speed of the stone after 1 second.

(iii) What happens to the speed of the stone for large values of t?

Solution.

Part (a).

$$\int \frac{1}{100-v^2} \, \mathrm{d}v = \frac{1}{2(10)} \ln\left(\frac{10+v}{10-v}\right) + C = \frac{1}{20} \ln\left(\frac{10+v}{10-v}\right) + C.$$

Part (b).

Part (b)(i).

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.1v^2 = \frac{100-v^2}{10} \implies \frac{1}{100-v^2}\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{10} \implies \int \frac{1}{100-v^2} \, \mathrm{d}v = \int \frac{1}{10} \, \mathrm{d}t$$
$$\implies \frac{1}{20} \ln\left(\frac{10+v}{10-v}\right) + C_1 = \frac{t}{10} \implies t = \frac{1}{2} \ln\left(\frac{10+v}{10-v}\right) + C, \quad C = 10C_1.$$

Evaluating the solution at $t = 0$ and $v = 0$, we get

$$0 = \frac{1}{2} \ln\left(\frac{10+0}{10+0}\right) + C \implies C = 0.$$

Thus, the general solution is

$$t = \frac{1}{2} \ln\left(\frac{10+v}{10-v}\right).$$

Consider $v = 5$, we have

$$t = \frac{1}{2} \ln\left(\frac{10+5}{10-5}\right) = \frac{1}{2} \ln 3.$$

It thus takes $\frac{1}{2} \ln 3$ seconds for the stone to reach a speed of 5 m/s.

Part (b)(ii). Consider $t = 1$. Using G.C., we get $v = 7.62$. Thus, after 1 second, the stone has a speed of 7.62 m/s.

Part (b)(iii). As $t \to \infty$, we have $\ln\left(\frac{10+v}{10-v}\right) \to \infty \implies \frac{10+v}{10-v} \to \infty$. Thus, $v \to 10$. Hence, for large values of t, the speed of the stone approaches 10 m/s.

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Problem 10. Two scientists are investigating the change of a certain population of an animal species of size n thousand at time t years. It is known that due to its inability to reproduce effectively, the species is unable to replace itself in the long run.

(a) One scientist suggests that n and t are related by the differential equation $\frac{\mathrm{d}^2 n}{\mathrm{d}t^2} = 10 - 6t$. Given that $n = 100$ when $t = 0$, show that the general solution of this differential equation is $n = 5t^2 - t^3 + Ct + 100$, where C is a constant. Sketch the solution curve of the particular solution when $C = 0$, stating the axial intercepts clearly.

(b) The other scientist suggests that n and t are related by the differential equation $\frac{\mathrm{d}n}{\mathrm{d}t} = 3 - 0.02n$. Find n in terms of t, given again that $n = 100$ when $t = 0$. Explain in simple terms what will eventually happen to the population using this model.

Which is a more appropriate model in modelling the population of the animal species?

Solution.

Part (a).

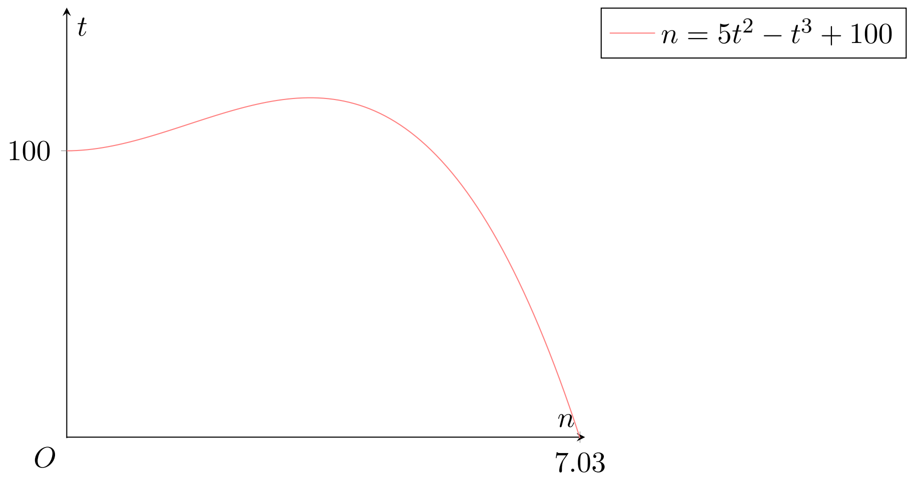
$$\frac{\mathrm{d}^2 n}{\mathrm{d}t^2} = 10 - 6t \implies \frac{\mathrm{d}n}{\mathrm{d}t} = \int (10 - 6t)\,\mathrm{d}\theta = 10t - 3t^2 + C$$
$$\implies n = \int \left(10t - 3t^2 + C\right)\,\mathrm{d}t = 5t^2 - t^3 + Ct + D.$$

Evaluating the solution at $t = 0$ and $n = 100$, we obtain $D = 100$. Thus,

$$n = 5t^2 - t^3 + Ct + 100.$$

Hence, when $C = 0$,

$$n = 5t^2 - t^3 + 100.$$



Part (b).

$$\frac{\mathrm{d}n}{\mathrm{d}t} = 3 - 0.02n = \frac{150 - n}{50} \implies \frac{1}{150 - n}\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{1}{50} \implies \int \frac{1}{150 - n}\,\mathrm{d}n = \int \frac{1}{50}\,\mathrm{d}t$$
$$\implies -\ln(150 - n) = \frac{1}{50}t + C_1 \implies n = 150 - Ce^{-t/50}, \quad C = e^{-C_1}.$$

When $t = 0$ and $n = 100$, we have $C = 50$. Hence,

$$n = 150 - 50e^{-t/50}.$$

As $t \to \infty$, $n \to 150$. Hence, the population will decrease before plateauing at 150 thousand.

The first model is more appropriate, as it account for the fact that the species will eventually go extinct ($n = 0$) due to the fact that they cannot replace itself in the long run.

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Problem 11. A rectangular tank has a horizontal base. Water is flowing into the tank at a constant rate, and flows out at a rate which is proportional to the depth of water in the tank. At time t seconds, the depth of water in the tank is x metres. If the depth is 0.5 m, it remains at this constant value. Show that $\frac{dx}{dt} = -k(2x-1)$, where k is a positive constant. When $t = 0$, the depth of water in the tank is 0.75 m and is decreasing at a rate of 0.01 m s^{-1}. Find the time at which the depth of water is 0.55 m.

Solution. Let V_i m^3/s be the rate at which water is flowing into the tank. Note that $V_i \geq 0$. Let the rate at which water is flowing out of the tank be $V_o x$ m^3/s. Let the base of the container be A m^2. Then

$$\frac{dx}{dt} = \frac{V_i - V_o x}{A}.$$

At $x = 0.5$, the volume of water in the tank is constant. Thus,

$$\left.\frac{dx}{dt}\right|_{x=0.5} = 0 \implies V_i - 0.5V_o = 0 \implies V_o = 2V_i \implies \frac{dx}{dt} = -\frac{V_i(2x-1)}{A}.$$

Letting $k = V_i/A$, we have

$$\frac{dx}{dt} = -k(2x-1).$$

We now solve for t.

$$\frac{dx}{dt} = -k(2x-1) \implies \frac{1}{2x-1}\frac{dx}{dt} = -k \implies \int \frac{1}{2x-1}\,dx = \int -k\,dt$$
$$\implies \frac{\ln(2x-1)}{2} + C_1 = -kt \implies t = -\frac{\ln(2x-1)+C}{2k}, \quad C = 2C_1$$

Evaluating the solution at $t = 0$ and $x = 0.75$, we get

$$0 = -\frac{\ln(2(0.75)-1)+C}{2k} \implies C_2 = \ln 2.$$

Additionally,

$$\left.\frac{dx}{dt}\right|_{t=0} = -0.01 \implies -0.01 = -k[2(0.75)-1] \implies k = 0.02.$$

Thus,

$$t = -\frac{\ln(2x-1)+\ln 2}{2(0.02)} = -25\ln(4x-2).$$

Consider $x = 0.55$. Then

$$t = -25\ln(4(0.55)-2) = 25\ln 5.$$

Thus, when $t = 25\ln 5$, the depth of the water is 0.55 m.

Problem 12. In a model of mortgage repayment, the sum of money owned to the Building Society is denoted by x and the time is denoted by t. Both x and t are taken to be continuous variables. The sum of money owned to the Building Society increases, due to interest, at a rate proportional to the sum of money owed. Money is also repaid at a constant rate r.

When $x = a$, interest and repayment balance. Show that, for $x > 0$, $\frac{dx}{dt} = \frac{r}{a}(x - a)$.
Given that, when $t = 0$, $x = A$, find x in terms of t, r, a and A.
On a single, clearly labelled sketch, show the graph of x against t in the two cases:

(a) $A > a$.

(b) $A < a$.

State the circumstances under which the loan is repaid in a finite time T and show that, in this case, $T = \frac{a}{r} \ln \frac{a}{a-A}$.

Solution. Let the rate at which money is owned to the Building Society be kx. Then

$$\frac{dx}{dt} = kx - r.$$

At $x = a$, interest and repayment balance. Hence,

$$dx/dt|_a = ka - r = 0 \implies k = \frac{r}{a}.$$

Thus,

$$\frac{dx}{dt} = \frac{r}{a}x - r = \frac{r}{a}(x - a).$$

We now solve for x.

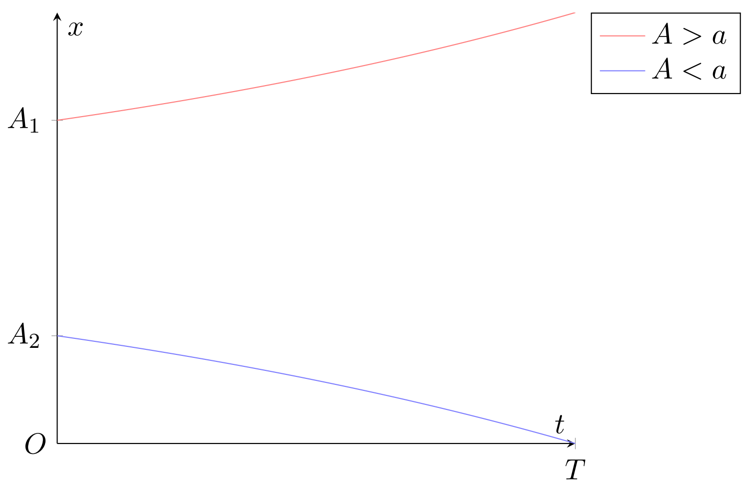
$$\frac{dx}{dt} = \frac{r}{a}(x - a) \implies \frac{1}{x - a}\frac{dx}{dt} = \frac{r}{a} \implies \int \frac{1}{x - a}\,dx = \int \frac{r}{a}\,dt$$
$$\implies \ln|x - a| = \frac{r}{a}t + C_1 \implies x = Ce^{rt/a} + a.$$

When $t = 0$, we have $x = A$. Substituting this into the solution, we obtain

$$A = C + a \implies C = A - a.$$

Thus,

$$x = (A - a)e^{rt/a} + a.$$



For the loan to be repaid in finite time, $A < a$. At time T, the loan has been repaid, i.e. $x = 0$. Thus,

$$(A - a)e^{rt/a} + a = 0 \implies e^{rt/a} = \frac{a}{a - A} \implies \frac{rt}{a} = \ln\left(\frac{a}{a - A}\right) \implies t = \frac{a}{r}\ln\left(\frac{a}{a - A}\right).$$

Self-Practice B12

Problem 1. Show that the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy + 3 = 0$ may be reduced by means of the substitution $y = ux^2$ to $\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{3}{x^4}$. Hence, other otherwise, show that the general solution for y in terms of x is $y = Cx^2 + \frac{1}{x}$, where C is an arbitrary constant.

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Problem 2. Use the substitution $z = y\mathrm{e}^x$ to find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x + 3$. Sketch on one diagram, the curve of a particular solution for which $y \to \infty$ as $x \to -\infty$, labelling the equation of this particular solution.

* * * * *

Problem 3.

(a) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x+2)(y-3),$$

giving your answer in the form $y = f(x)$.

(b) Given that u and t are related by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 16 - 9u^2,$$

and that $u = 1$ when $t = 0$, find t in terms of u, simplifying your answer.

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Problem 4. At each instant of time the rate of increase of money in a savings account is proportional to the amount in the account at that instant. The constant of proportionality does not vary with time. Denote the amount in the account at time t years by $\$x$. When $x = 1000$, the rate of increase is \$50 per year. Obtain a differential equation relating x and t.

(a) Initially, when $t = 0$, the account contained \$900. Find the amount in the account exactly 3 years later.

(b) Find, in years correct to 2 places of decimals, the time when the account contains \$1800.

(c) Comment on whether the model can be regarded as a good model of the situation in the real world.

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Problem 5. Salt is dissolved in a tank filled with 120 litres of water. Salt water containing 20 g of salt per litre is poured in at a rate of 3 litres per minute and the mixture flows out at a constant rate of 3 litres per minute. The contents of the tank are kept well mixed at all times. Let the amount of salt in the tank (in grams) be denoted by S and the time (in minutes) be denoted by t.

(a) Show that $\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{2400-S}{40}$.

(b) Given that 400g of salt was dissolved in the tank initially, find the amount of salt in the tank after 1 hour, giving your answer to the nearest grams.

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Problem 6. In a certain country, the price of a brand-new car of a particular make, manufactured on 1 January 1996, is \$32,000. According to a model of car pricing, the price P of the car (in \$) depreciates at a rate proportional to P when the car is t years old (as from 1 January 1996). Write down a differential equation relating P and t.

By solving this differential equation, show that $P = 32000\mathrm{e}^{-kt}$ where k is a positive constant.

A man purchased a used car of this particular make for \$2000, at the price predicted by the model, on 1 January 2006. Subsequently on 1 January 2007, the man sold the used car for \$800. Determine if the man sold his car below the price predicted by the model.

Assignment B12

Problem 1. The curve $y = f(x)$ passes through the origin and has gradient given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - 4x + 1}{2y - 5}.$$

(a) Find $f(x)$.

(b) By considering $\frac{\mathrm{d}y}{\mathrm{d}x}$, deduce the coordinates of the point on the curve where it is tangent to the x-axis.

(c) Determine the interval of validity for the solution.

Solution.

Part (a).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - 4x + 1}{2y - 5} \implies (2y-5)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x + 1$$
$$\implies \int (2y-5)\,\mathrm{d}y = \int \left(3x^2 - 4x + 1\right)\,\mathrm{d}x \implies y^2 - 5y = x^3 - 2x^2 + x + C_1.$$

Note that $x^3 - 2x^2 + x = x(x-1)^2$. Hence,

$$y^2 - 5y - x(x-1)^2 + C_2 = 0 \implies y = \frac{5 \pm \sqrt{4x(x-1)^2 + C_3}}{2}.$$

Since the curve passes through the origin $(0, 0)$, we have

$$0 = \frac{5 - \sqrt{C_3}}{2} \implies C_3 = 25.$$

Thus,

$$f(x) = \frac{5 - \sqrt{4x(x-1)^2 + 25}}{2}.$$

Note that we reject the positive branch since $f(x) > 0$ in that case.

Part (b). When the curve is tangent to the x-axis, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $y = 0$. Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies 3x^2 - 4x + 1 = 0 \implies x = \frac{1}{3} \text{ or } 1.$$

Also note that

$$y = 0 \implies 4x(x-1)^2 = 0 \implies x = 0 \text{ or } 1.$$

Hence, the required point is $(1, 0)$.

Part (c). Since the square root function is defined only on the non-negative reals, we require

$$4x(x-1)^2 + 25 \geq 0 \implies x \geq -1.24.$$

Thus, the interval of validity is $[-1.24, \infty)$.

* * * * *

Problem 2.

(a) Using the substitution $y = ux$, find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y}{x},$$

where $x > 0$.

(b) Find the particular solution of the differential equation for which $y = -1$ when $x = 1$.

(c) Without sketching the curve of the solution in (b), determine the number of stationary points the solution curve has.

Solution.

Part (a). Note that

$$y = ux \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u.$$

Substituting this into the given differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x} \implies \frac{\mathrm{d}u}{\mathrm{d}x}x + u = \frac{x+ux}{x} \implies \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\implies u = \ln(x) + C \implies y = x\ln x + Cx.$$

Part (b). Evaluating the solution at $x = 1$ and $y = -1$, we get

$$-1 = 1\ln 1 + C(1) \implies C = -1.$$

Thus,

$$y = x\ln x - x.$$

Part (c). Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x} = \frac{x + x\ln x - x}{x} = \ln x.$$

Since $\ln x$ only has one root (at $x = 1$), the solution curve has only 1 stationary point.

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Problem 3. As a tree grows, the rate of increase of its height, h m, with respect to time, t years after planting, is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{10}\sqrt{16 - \frac{1}{2}h}.$$

The tree is planted as a seedling of negligible height, so that $h = 0$ when $t = 0$.

(a) State the maximum height of the tree, according to this model.

(b) Find an expression for t in terms of h, and hence find the time the tree takes to reach half of its maximum height.

Solution.

Part (a). Note that $\frac{\mathrm{d}h}{\mathrm{d}t} \geq 0 \implies h \leq 32$. Thus, the maximum height of the tree is 32 m.

Part (b).

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{10}\sqrt{16 - \frac{1}{2}h} \implies 10\left(16 - \frac{1}{2}h\right)^{-1/2}\frac{\mathrm{d}h}{\mathrm{d}t} = 1$$
$$\implies 10\int\left(16 - \frac{1}{2}h\right)^{-1/2}\,\mathrm{d}h = \int 1\,\mathrm{d}t \implies -10\sqrt{16 - \frac{1}{2}h} + C = t.$$

Since $h = 0$ when $t = 0$, we have

$$-10\sqrt{16} + C = 0 \implies C = 40.$$

Thus,

$$t = 40 - 10\sqrt{16 - \frac{1}{2}h}.$$

When $h = \frac{32}{2} = 16$, we have

$$t = 40 - 10\sqrt{16 - \frac{1}{2}(16)} = 11.7 \text{ (3 s.f.)}.$$

Thus, it takes 11.7 years for the tree to reach half its maximum height.

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Problem 4.

(a) Find $\int \frac{1}{x(1000-x)}\,\mathrm{d}x$.

(b) A communicable disease is spreading within a small community with a population of 1000 people. A scientist found out that the rate at which the disease spreads is proportional to the product of the number of people who are infected with the disease and the number of people who are not infected with the disease. It is known that one person in this community is infected initially and five days later, 12% of the population is infected.

Given that the infected population is x at time t days after the start of the spread of the disease, show that it takes less than 8 days for half the population to contract the disease.

(c) State an assumption made by the scientist.

Solution.

Part (a).

$$\int \frac{1}{x(1000-x)}\,\mathrm{d}x = \int \frac{1}{1000}\left(\frac{1}{x} + \frac{1}{1000-x}\right)\mathrm{d}x$$
$$= \frac{\ln|x| - \ln|1000-x|}{1000} + C = \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right| + C.$$

Part (b). Note that $\frac{\mathrm{d}x}{\mathrm{d}t} \propto x(1000-x) \implies \frac{\mathrm{d}x}{\mathrm{d}t} = kx(1000-x)$ for some $k \in \mathbb{R}^+$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx(1000-x) \implies \frac{1}{x(1000-x)}\frac{\mathrm{d}x}{\mathrm{d}t} = k$$
$$\implies \int \frac{1}{x(1000-x)}\,\mathrm{d}x = \int k\,\mathrm{d}t \implies \frac{1}{1000}\ln\left(\frac{x}{1000-x}\right) + C = kt.$$

Note that when $t = 0$, we have $x = 1$. Thus,

$$\frac{1}{1000}\ln\left(\frac{1}{999}\right) + C = 0 \implies C = \frac{\ln 999}{1000}.$$

When $t = 5$, $x = 120$. Hence,

$$\frac{1}{1000}\ln\left(\frac{120}{880}\right) + \frac{\ln 999}{1000} = 5k \implies k = \frac{1}{5000}\left(\ln\frac{3}{22} + \ln 999\right).$$

Thus,

$$\begin{aligned} t &= \left[\frac{1}{5000}\left(\ln\frac{3}{22} + \ln 999\right)\right]^{-1}\left[\frac{1}{1000}\ln\left(\frac{x}{1000-x}\right) + \frac{\ln 999}{1000}\right] \\ &= \frac{5}{\ln(3/22) + \ln 999}\left[\ln\left(\frac{x}{1000-x}\right) + \ln 999\right] \end{aligned}$$

Hence, when half the population is infected, i.e. $x = 500$, we have $t = 7.03 < 8$. Thus, it takes less than 8 days for half the population to contract the disease.

Part (c). The assumption is that there are no measures taken by the population to limit the spread of the disease (e.g. quarantine).

B13 Linear First Order Differential Equations

Tutorial B13

Problem 1. Solve the following differential equations:

(a) $2\sec x\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1-y^2}$

(b) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y-t^2y}$, given $y = 4$ when $t = 0$

Solution.
Part (a).

$$2\sec x\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1-y^2} \implies \frac{2}{\sqrt{1-y^2}}\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \implies \int \frac{2}{\sqrt{1-y^2}}\,\mathrm{d}y = \int \cos x\,\mathrm{d}x$$
$$\implies 2\arcsin y = \sin x + C_1 \implies y = \sin\left(\frac{1}{2}\sin x + C\right), \quad C = \frac{C_1}{2}.$$

Part (b).

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y-t^2y} \implies y\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{1-t^2} \implies \int y\,\mathrm{d}y = \int \frac{t}{1-t^2}\,\mathrm{d}t$$
$$\implies \frac{1}{2}y^2 = -\frac{1}{2}\ln\left|1-t^2\right| + C_1 \implies y^2 = C_2 - \ln\left|1-t^2\right|, \quad C_2 = 2C_1.$$

Since $y = 4 \geq 0$ when $t = 0$, we have

$$4^2 = C_2 - \ln|1-0| \implies C_2 = 16.$$

Hence,

$$y = \sqrt{16 - \ln|1-t^2|}.$$

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Problem 2. Solve the following differential equations:

(a) $xy' + (2x-3)y = 4x^4$

(b) $(1+x)y' + y = \cos x$, $y(0) = 1$

(c) $(1+t^2)\frac{\mathrm{d}y}{\mathrm{d}t} = 2ty + 2$

(d) $(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{\ln(x+1)} = x^2 + x$, where $x > 0$

Solution.

Part (a). Note that

$$xy' + (2x-3)y = 4x^4 \implies y' + \left(2 - \frac{3}{x}\right) y = 4x^3.$$

Hence, the integrating factor is

$$\text{I. F.} = \exp \int \left(2 - \frac{3}{x}\right) \mathrm{d}x = \exp(2x - \ln 3) = \frac{\mathrm{e}^{2x}}{x^3}.$$

Multiplying through by the integrating factor, we get

$$\frac{\mathrm{e}^{2x}}{x^3} y' + \frac{\mathrm{e}^{2x}}{x^3}\left(2 - \frac{3}{x}\right) y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{2x}}{x^3} y\right) = 4\mathrm{e}^{2x} \implies \frac{\mathrm{e}^{2x}}{x^3} y = \int 4\mathrm{e}^{2x}\,\mathrm{d}x = 2\mathrm{e}^{2x} + C$$
$$\implies y = \frac{x^3}{\mathrm{e}^{2x}}\left(2\mathrm{e}^{2x} + C\right) = 2x^3 + Cx^3\mathrm{e}^{-2x}.$$

Part (b).

$$(1+x)y' + y = \frac{\mathrm{d}}{\mathrm{d}x}[(1+x)y] = \cos x \implies (1+x)y = \int \cos x\,\mathrm{d}x = \sin x + C$$
$$\implies y = \frac{\sin x + C}{x+1}$$

Since $y(0) = 1$,

$$1 = \frac{\sin 0 + C}{0+1} \implies C = 1 \implies y = \frac{\sin x + 1}{x+1}.$$

Part (c). Let $t = \tan\theta$. Observe that

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \sec^2\theta = 1 + t^2.$$

Hence,

$$(1+t^2)\frac{\mathrm{d}y}{\mathrm{d}t} = (1+t^2)\frac{\mathrm{d}y}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1+t^2}{1+t^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}\theta}.$$

Substituting this into the given differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2y\tan\theta + 2 \implies \cos^2\theta\frac{\mathrm{d}y}{\mathrm{d}\theta} - 2y\sin\theta\cos\theta = 2\cos^2\theta \implies \frac{\mathrm{d}}{\mathrm{d}\theta}\left(y\cos^2\theta\right) = 2\cos^2\theta$$
$$\implies y\cos^2\theta = \int 2\cos^2\theta\,\mathrm{d}t = \int (1+\cos 2\theta)\,\mathrm{d}\theta = \theta + \frac{\sin 2\theta}{2} + C = \theta + \sin\theta\cos\theta + C$$
$$\implies y = (\theta + C)\sec^2\theta + \tan\theta = (\arctan t + C)\left(1+t^2\right) + t.$$

Part (d). Note that

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{\ln(x+1)} = x^2 + x \implies \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{(x+1)\ln(x+1)} = x.$$

Hence, the integrating factor is

$$\text{I. F.} = \exp\int \frac{1/(x+1)}{\ln(x+1)}\,\mathrm{d}x = \exp\ln\,\ln(x+1) = \ln(x+1).$$

Multiplying through by the integrating factor, we get

$$\ln(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{(x+1)} = \frac{\mathrm{d}}{\mathrm{d}x}\left(y\ln(x+1)\right) = x\ln(x+1)$$
$$\implies y\ln(x+1) = \int x\ln(x+1)\,\mathrm{d}x.$$

Integrating by parts, we get

$$\begin{aligned} y\ln(x+1) &= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\,\mathrm{d}x \\ &= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \left(x - 1 + \frac{1}{x+1}\right)\mathrm{d}x \\ &= \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{\ln(x+1)}{2} + C. \end{aligned}$$

Thus,

$$y = \frac{x^2}{2} - \frac{x^2}{4\ln(x+1)} + \frac{x}{2\ln(x+1)} - \frac{1}{2} + \frac{C}{\ln(x+1)}.$$

* * * * *

Problem 3. Given a general first order differential equation, $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, if $f(x, y)$ is such that $f(kx, ky) = f(x, y)$, then the equation may be reduced to a separable equation by means of the substitution $y = ux$. Hence, solve the following differential equation: $(x+y)y' = x - y$.

Solution. Note that

$$(x+y)y' = x - y \implies y' = \frac{x-y}{x+y}.$$

Let $f(x, y) = \frac{x-y}{x+y}$. Then

$$f(kx, ky) = \frac{kx - ky}{kx + ky} = \frac{x-y}{x+y} = f(x, y).$$

Hence, the differential equation can be solved with the substitution $y = ux$, whence $y' = u'x + u$. Substituting this into the differential equation, we get

$$\begin{aligned} u'x + u = \frac{x - ux}{x + ux} = \frac{1-u}{1+u} &\implies \left(\frac{1+u}{1-2u-u^2}\right)u' = \frac{1}{x} \\ &\implies \int \frac{1+u}{1-2u-u^2}\,\mathrm{d}u = \int \frac{1+u}{2-(1+u)^2}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x \\ \implies -\frac{1}{2}\ln\left|2-(1+u)^2\right| = -\frac{1}{2}\ln\left|2 - \left(1 + \frac{y}{x}\right)^2\right| &= \ln x + C_1 \implies (x+y)^2 = 2x^2 + C, \end{aligned}$$

where $C = -\mathrm{e}^{-2C_1} \in \mathbb{R}^-$.

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Problem 4. Using the substitution $u = \frac{1}{y}$, solve $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^x y^2$.

Solution. Note that

$$u = \frac{1}{y} \implies y = \frac{1}{u} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{u^2}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Substituting this into the differential equation,

$$\begin{aligned} -\frac{1}{u^2}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{2}{u} = \frac{\mathrm{e}^x}{u^2} \implies \frac{\mathrm{d}u}{\mathrm{d}x} - 2u = -\mathrm{e}^x \implies \mathrm{e}^{-2x}\frac{\mathrm{d}u}{\mathrm{d}x} - 2\mathrm{e}^{-2x}u = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-2x}u\right) = -\mathrm{e}^{-x} \\ \implies \mathrm{e}^{-2x}u = \int -\mathrm{e}^{-x}\,\mathrm{d}x = \mathrm{e}^{-x} + C \implies u = \mathrm{e}^x + C\mathrm{e}^{2x} \implies y = \frac{1}{\mathrm{e}^x + C\mathrm{e}^{2x}}. \end{aligned}$$

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Problem 5. Assuming that $p(x) \neq 0$, state conditions under which the linear equation $y' + p(x)y = f(x)$ is separable. If the equation satisfies these conditions, solve it by separation of variables and by the method of integrating factor.

Solution. The linear equation $y' + p(x)y = f(x)$ is separable if $p(x)$ is a scalar multiple of $f(x)$, i.e. $p(x) = \lambda f(x)$ for some $\lambda \in \mathbb{R}$.

We begin by solving using separation of variables. Note that

$$y' = f(x) - p(x)y = f(x) - \lambda f(x)y = f(x)(1 - \lambda y).$$

Hence,

$$\begin{aligned} \frac{1}{1-\lambda y} y' = f(x) &\implies \int \frac{1}{1-\lambda y} \, \mathrm{d}y = -\frac{\ln|1-\lambda y|}{\lambda} = \int f(x) \, \mathrm{d}x \\ &\implies \ln|1-\lambda y| = -\int \lambda f(x) \, \mathrm{d}x = -\int p(x) \, \mathrm{d}x \\ &\implies y = \frac{1}{\lambda}\left[1 - C_1 \mathrm{e}^{-\int p(x) \, \mathrm{d}x}\right] = \frac{1}{\lambda} + C\mathrm{e}^{-\int p(x) \, \mathrm{d}x}. \end{aligned}$$

Integrating Factor. Note that the integrating factor is $\mathrm{e}^{\int p(x) \, \mathrm{d}x}$. Multiplying through, we get

$$\begin{aligned} &\mathrm{e}^{\int p(x) \, \mathrm{d}x} y' + \mathrm{e}^{\int p(x) \, \mathrm{d}x} p(x) y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int p(x) \, \mathrm{d}x} y\right) = \mathrm{e}^{\int p(x) \, \mathrm{d}x} f(x) = \frac{1}{\lambda} e^{\int p(x) \, \mathrm{d}x} p(x) \\ \implies &\mathrm{e}^{\int p(x) \, \mathrm{d}x} y = \int \frac{1}{\lambda} \mathrm{e}^{\int p(x) \, \mathrm{d}x} p(x) \, \mathrm{d}x = \frac{1}{\lambda} \mathrm{e}^{\int p(x) \, \mathrm{d}x} + C \implies y = \frac{1}{\lambda} + Ce^{-\int p(x) \, \mathrm{d}x}. \end{aligned}$$

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Problem 6. The variables x and y are related by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = y^3$.

(a) State clearly why the integrating factor method cannot be used to solve this equation.

(b) The variables y and z are related by the equation $\frac{1}{y^2} = -2z$. Show that $\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{2z}{x} = 1$.

(c) Find the solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = y^3$, given that $y = 2$ when $x = 1$.

Solution.

Part (a). The differential is non-linear due to the presence of the y^3 term.

Part (b).

$$\frac{1}{y^2} = -2z \implies \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{y^3} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y^3}\left(y^3 - \frac{y}{x}\right) = 1 - \frac{1}{y^2} \cdot \frac{1}{x} = 1 + \frac{2z}{x} \implies \frac{\mathrm{d}z}{\mathrm{d}x} - \frac{2z}{x} = 1.$$

Part (c).

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} - \frac{2z}{x} = 1 &\implies \frac{1}{x^2}\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{2z}{x^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{z}{x^2}\right) = \frac{1}{x^2} \implies \frac{z}{x^2} = \int \frac{1}{x^2} \, \mathrm{d}x = -\frac{1}{x} + C_1 \\ &\implies z = -\frac{1}{2y^2} = -x + C_1 x^2 \implies y^2 = \frac{1}{2x + C_2 x^2}. \end{aligned}$$

Since $y(1) = 2$, we have

$$2^2 = \frac{1}{2 + C_2} \implies C_2 = -\frac{7}{4}.$$

Thus,

$$y^2 = \frac{1}{2x - 7x^2/4} = \frac{4}{8x - 7x^2} \implies y = \frac{2}{\sqrt{8x - 7x^2}}.$$

Note that we reject the negative branch since $y(1) = 2 \geq 0$.

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Problem 7. Show that the substitution $v = \ln y$ transforms the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)(y \ln y)$ into the linear equation $\frac{\mathrm{d}v}{\mathrm{d}x} + P(x) = Q(x)v(x)$. Hence, solve the equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - 4x^2 y + 2y \ln y = 0$.

Solution. Note that

$$v = \ln y \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = y\frac{\mathrm{d}v}{\mathrm{d}x}.$$

Substituting this into the differential equation, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)(y\ln y) \implies y\frac{\mathrm{d}v}{\mathrm{d}x} + P(x)y = Q(x)(yv) \implies \frac{\mathrm{d}v}{\mathrm{d}x} + P(x) = Q(x)v.$$

We now solve the given differential equation. Note that

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4x^2y + 2y\ln y = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} - 4xy = -\frac{2}{x}(y\ln y).$$

Hence,

$$P(x) = -4x, \quad Q(x) = -\frac{2}{x}.$$

Now, from the previous part, we can rewrite the differential equation as

$$\frac{\mathrm{d}v}{\mathrm{d}x} - 4x = -\frac{2v}{x},$$

where $v(x) = \ln y$. Thus,

$$x^2\frac{\mathrm{d}v}{\mathrm{d}x} + 2xv = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2v\right) = 4x^3 \implies x^2v = x^2\ln y = \int 4x^3\,\mathrm{d}x = x^4 + C$$
$$\implies y = \exp\left(x^2 + Cx^{-2}\right).$$

* * * * *

Problem 8. The normal at any point on a certain curve always passes through the point $(2, 3)$. Form a differential equation to express this property. Without solving the differential equation, find the equation of the curve where the stationary points of the family of curves will lie on. Which family of standard curves will have their stationary points lying along a curve with such an equation found earlier?

Solution. Clearly,

$$y - 3 = \frac{-1}{\mathrm{d}y/\mathrm{d}x}(x - 2).$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x-2}{y-3}.$$

For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, whence $x = 2$ and $y \neq 3$.

Also note that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{(y-3) - (x-2)\frac{\mathrm{d}y}{\mathrm{d}x}}{(y-3)^2}.$$

At stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, giving

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y-3}.$$

Hence, when $y > 3$, we have $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, giving a maximum. Likewise, when $y < 3$, we have $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, giving a minimum. This suggests that the required family of standard curves is the family of circles with centre $(2, 3)$.

Problem 9. Obtain the general solution of the differential equation $x\frac{dy}{dx} - y = x^2 + 1$ in the form $y = x^2 + Cx - 1$, where C is an arbitrary constant.

Show that each solution curve of the differential equation has one minimum point.

Find the equation of the curve of which all these minimum points lie.

Sketch some of the family of solution curves including those corresponding to some negative values of C, some positive values of C, and $C = 0$.

Solution.

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2 + 1 \implies \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{y}{x} = 1 + \frac{1}{x^2}$$
$$\implies \frac{y}{x} = \int \left(1 + \frac{1}{x^2}\right) \mathrm{d}x = x - \frac{1}{x} + C \implies y = x^2 + Cx - 1.$$

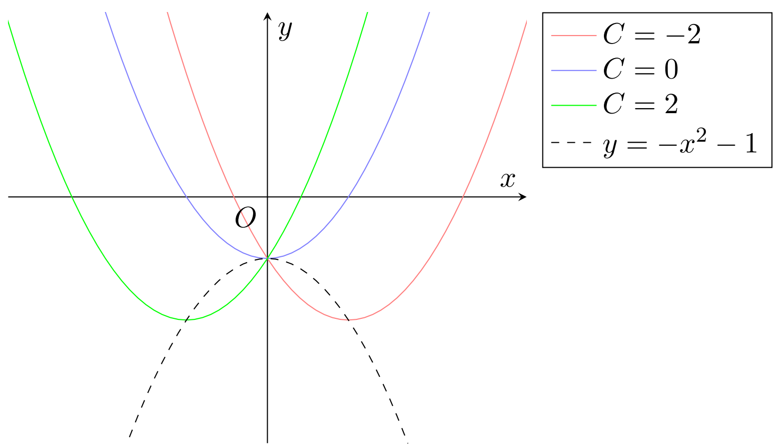
Note that

$$y = x^2 + Cx - 1 = \left(x + \frac{C}{2}\right)^2 - \left(1 + \frac{C^2}{4}\right).$$

Thus, y has a unique minimum point at $\left(-\frac{C}{2}, -\left(1 + \frac{C^2}{4}\right)\right)$.

For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Thus, the minimum points lie on the curve with equation

$$-y = x^2 + 1 \implies y = -x^2 - 1.$$



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Problem 10. Show that the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy - 2x\left(x^2 + 1\right) = 0$$

can be expressed in the form $y = x^2 + Ce^{-x^2}$, where C is an arbitrary constant.

Deduce, with reasons, the number of stationary points of the solution curves of the equation when

(a) $C \leq 1$;

(b) $C > 1$.

Solution. Note that the integrating factor is $\exp\left(\int 2x\,\mathrm{d}x\right) = \mathrm{e}^{x^2}$. Multiplying the integrating factor throughout the differential equation, we get

$$\mathrm{e}^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\mathrm{e}^{x^2}y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}y\right) = 2x\mathrm{e}^{x^2}\left(x^2 + 1\right)$$
$$\implies \mathrm{e}^{x^2}y = \int 2x\mathrm{e}^{x^2}\left(x^2 + 1\right)\mathrm{d}x = \mathrm{e}^{x^2}(x^2 + 1) - \int 2x\mathrm{e}^{x^2}\,\mathrm{d}x = \mathrm{e}^{x^2}(x^2 + 1) - \mathrm{e}^{x^2} + C$$
$$\implies y = \left(x^2 + 1\right) - 1 + C\mathrm{e}^{-x^2} = x^2 + C\mathrm{e}^{-x^2}.$$

For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Hence,

$$2xy - 2x\left(x^2+1\right) = 0 \implies x = 0 \text{ or } y - \left(x^2+1\right) = 0.$$

Consider $y - \left(x^2+1\right) = 0$.

$$y - \left(x^2+1\right) = \left(x^2 + C\mathrm{e}^{-x^2}\right) - \left(x^2+1\right) = C\mathrm{e}^{-x^2} - 1 = 0 \implies x^2 = \ln C.$$

Part (a). When $C < 1$, we have $\ln C < 0$. Hence, there are no solutions to $x^2 = \ln C$, whence there is only 1 stationary point (at $x = 0$).

When $C = 1$, we have $\ln C = 0$, whence the only solution to $x^2 = \ln C$ is $x = 0$. Thus, there is still only 1 stationary point (at $x = 0$).

Part (b). When $C > 1$, we have $\ln C > 0$, whence there are two solutions to $x^2 = \ln C$, namely $x = \pm \ln C$. Thus, there are 3 stationary points (at $x = 0$ and $x = \pm \ln C$).

* * * * *

Problem 11. Using the substitution $y = x^2 \ln t$, where $t > 0$, show that the differential equation

$$2xt\ln t\frac{\mathrm{d}x}{\mathrm{d}t} + (3\ln t + 1)x^2 = \frac{\mathrm{e}^{-2t}}{t} \tag{$*$}$$

can be reduced to a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + P(t)y = \frac{\mathrm{e}^{-2t}}{t^2},$$

where $P(t)$ is some function of t to be determined.

Hence, find x^2 in terms of t.

Sketch, on a single diagram, two solution curves for the differential equation $(*)$, C_1 and C_2, of which only C_1 has stationary point(s). Label the equations of any asymptotes in your diagram.

Solution. Note that

$$y = x^2\ln t \implies \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{x^2}{t} + 2x\ln t\frac{\mathrm{d}x}{\mathrm{d}t} \implies 2xt\ln t\frac{\mathrm{d}x}{\mathrm{d}t} = t\frac{\mathrm{d}y}{\mathrm{d}t} - x^2.$$

Hence,

$$2xt\ln t\frac{\mathrm{d}x}{\mathrm{d}t} + (3\ln t+1)x^2 = \left(t\frac{\mathrm{d}y}{\mathrm{d}t} - x^2\right) + (3\ln t + 1)x^2 = t\frac{\mathrm{d}y}{\mathrm{d}t} + 3x^2\ln t = t\frac{\mathrm{d}y}{\mathrm{d}t} + 3y.$$

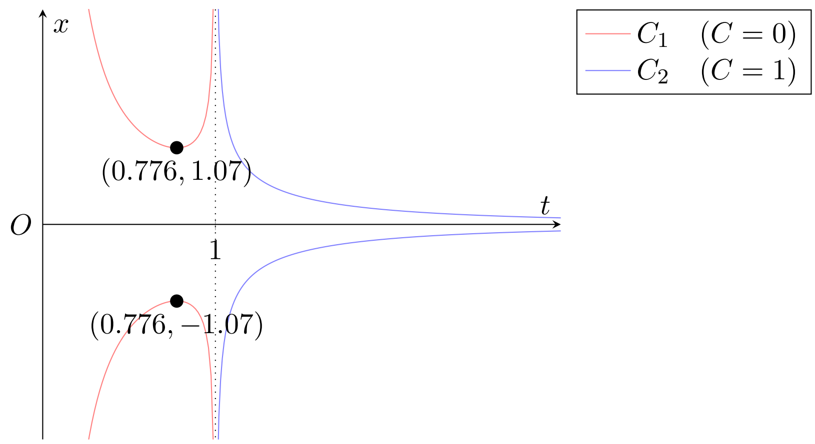
Our differential equation thus becomes

$$t\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = \frac{\mathrm{e}^{-2t}}{t} \implies \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{3y}{t} = \frac{\mathrm{e}^{-2t}}{t^2},$$

whence $P(t) = 3/t$. We now solve the differential equation. Observe that

$$t^3\frac{\mathrm{d}y}{\mathrm{d}t} + 3t^2y = \frac{\mathrm{d}}{\mathrm{d}t}\left(t^3y\right) = t\mathrm{e}^{-2t} \implies t^3y = \int t\mathrm{e}^{-2t}\,\mathrm{d}t = -\frac{1}{2}t\mathrm{e}^{-2t} - \frac{1}{4}\mathrm{e}^{-2t} + C_1$$

$$\implies t^3x^2\ln t = -\frac{(2t+1)\,\mathrm{e}^{-2t}}{4} + \frac{C}{4} \implies x^2 = \frac{C - (2t+1)\,\mathrm{e}^{-2t}}{4t^3\ln t}.$$



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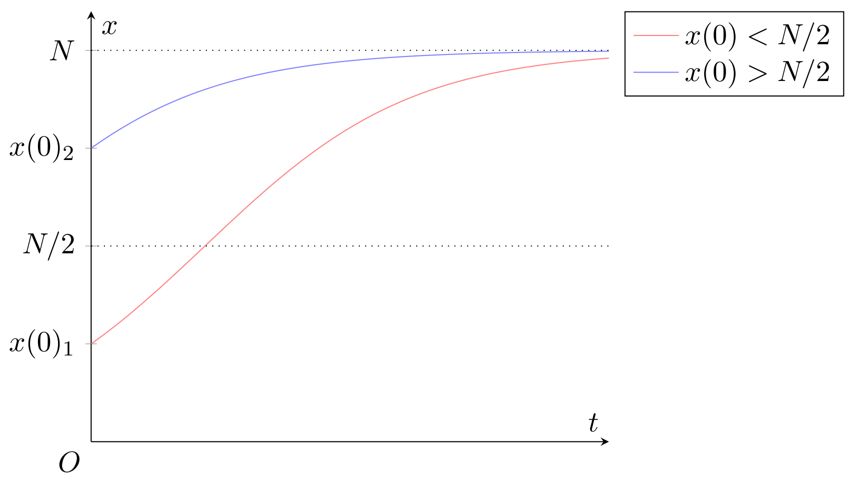
Problem 12. It is suggested that the spread of a highly contagious disease on an isolated island with a population of N may be modelled by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = kx(N-x)$, where k is a positive constant, and $x(t)$ is the number of individuals infected with the disease at time t.

(a) Without solving the differential equation, sketch the graph of $x(t)$ against t for cases when $x(0) < \frac{N}{2}$ and $x(0) > \frac{N}{2}$.

(b) Given that $x(0) = x_0$, solve the differential equation for an explicit expression of $x(t)$.

Solution.

Part (a).



Part (b).

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx(N-x) \implies \frac{1}{x(N-x)}\frac{\mathrm{d}x}{\mathrm{d}t} = k \implies \int \frac{1}{x(N-x)}\,\mathrm{d}x = \int k\,\mathrm{d}t$$

$$\implies \int \frac{1}{N}\left(\frac{1}{x} - \frac{1}{N-x}\right) = \int k\,\mathrm{d}t \implies \frac{\ln x - \ln(N-x)}{N} = kt + C_1$$

$$\implies \ln\frac{x}{N-x} = Nkt + C_2 \implies x = \frac{C_3 N \mathrm{e}^{Nkt}}{1 + C_3 \mathrm{e}^{Nkt}}.$$

At $t = 0$, we have $x = x_0$. Hence,

$$x_0 = \frac{C_3 N e^0}{1 + C_3 e^0} \implies C_3 = \frac{x_0}{N - x_0}.$$

This gives

$$x = \frac{\frac{x_0}{N-x_0} N e^{Nkt}}{1 + \frac{x_0}{N-x_0} e^{Nkt}} = \frac{N x_0 e^{Nkt}}{N - x_0 + x_0 e^{Nkt}}.$$

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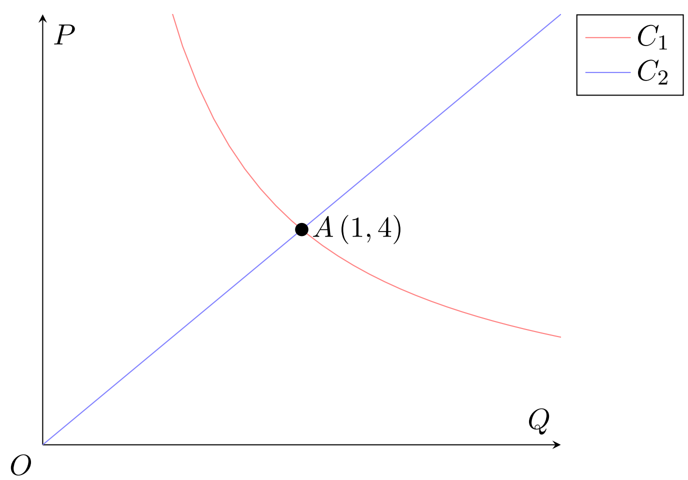
Problem 13. In the diagram below, the curve C_1 and the line C_2 illustrate the relationship between price (P dollars per kg) and quantity (Q tonnes) for consumers and producers respectively.

The curve C_1 shows the quantity of rice that consumers will buy at each price level while the line C_2 shows the quantity of rice that producers will produce at each price level. C_1 and C_2 intersect at point A, which has the coordinates $(1, 4)$.

The quantity of rice that consumers will buy is inversely proportional to the price of the rice. The quantity of rice that producers will produce is directly proportional to the price.



(a) Interpret the coordinates of A in the context of the question.

(b) Solve for the equations of C_1 and C_2, expressing Q in terms of P.

Shortage occurs when the quantity of rice consumers will buy exceeds the quantity of rice producers will produce. It is known that the rate of increase of P after time t months is directly proportional to the quantity of rice in shortage.

(c) Given that the initial price is \$3 and that after 1 month, the price is \$3.65, find P in terms of t and sketch this solution curve, showing the long-term behaviour of P.

Suggest a reason why producers might use $P = aQ + b$, where $a, b \in \mathbb{R}^+$, instead of C_2 to model the relationship between price and quantity of rice produced.

Solution.

Part (a). The coordinates of A represent the equilibrium price and quantity of rice. That is, 1 tonne of rice will be transacted at a price of \$4 per kg.

Part (b). Note that $C_1 : P = \frac{k_1}{Q}$ and $C_2 : P = k_2 Q$ for some constants k_1 and k_2. At $A(1, 4)$, we obtain $k_1 = k_2 = 4$. Thus,

$$C_1 : Q = \frac{4}{P}, \quad C_2 : Q = \frac{P}{4}.$$

Part (c). At a given price $P < 4$, the difference in the amount of rice demanded and produced is given by $\frac{4}{P} - \frac{P}{4} = \frac{16-P^2}{4P}$. Hence, $\frac{\mathrm{d}P}{\mathrm{d}t} = k \cdot \frac{16-P^2}{4P}$.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k \cdot \frac{16 - P^2}{4P} \implies \frac{2P}{16 - P^2} \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{k}{2} \implies \int \frac{2P}{16 - P^2} \,\mathrm{d}P = \int \frac{k}{2} \,\mathrm{d}t$$
$$\implies -\ln\left(16 - P^2\right) = \frac{kt}{2} + C_1 \implies P = \sqrt{16 - C_2 \mathrm{e}^{-kt/2}}.$$

Note that we used the fact that $0 < P < 4$ when solving for P.

At $t = 0$, $P = 3$. Hence,

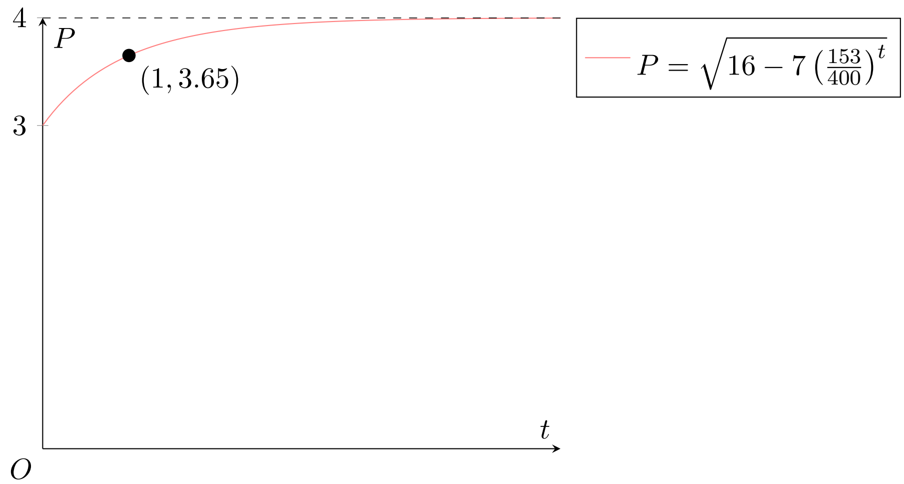
$$3 = \sqrt{16 - C_2 \mathrm{e}^0} \implies C_2 = 7.$$

At $t = 1$, $P = 3.65$. Hence,

$$3.65 = \sqrt{16 - 7\mathrm{e}^{-k(1)/2}} \implies \mathrm{e}^{-k/2} = \frac{153}{400}.$$

Thus,

$$P = \sqrt{16 - 7\left(\frac{153}{400}\right)^t}.$$



The model $P = aQ + b$ accounts for the fixed cost involved in producing rice.

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Problem 14. A rectangular tank contains 100 litres of salt solution at a concentration of 0.01 kg/litre. A salt solution with a concentration of 0.5 kg/litre flows into the tank at the rate of 6 litres/min. The mixture in the tank is kept uniform by stirring the mixture and the mixture flows out at the rate of 4 litres/min. If y kg is the mass of salt in the solution in the tank after t minutes, show that y satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3 - \frac{ky}{100 + mt},$$

where k and m are constants to be determined.

Find the particular solution of the differential equation.

Solution. Note that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.5(6) - 4\left(\frac{y}{100 + (6-4)t}\right) = 3 - \frac{4y}{100 + 2t}.$$

Hence, $k = 4$ and $m = 2$.

We now solve the differential equation. Multiplying throughout by $(100 + 2t)^2$, we get

$$2(100 + 2t)^2 \frac{\mathrm{d}y}{\mathrm{d}t} + 4(100 + 2t)y = \frac{\mathrm{d}}{\mathrm{d}t}\left[(100 + 2t)^2 y\right] = 3(100 + 2t)^2$$

$$\implies (100 + 2t)^2 y = \int 3(100 + 2t)^2 \,\mathrm{d}t = \frac{1}{2}(100 + 2t)^3 + C_1 = 4(50 + t)^3 + C_1$$

$$\implies y = \frac{4(50 + t)^3 + C_1}{(100 + 2t)^2} = \frac{4(50 + t)^3 + C_1}{4(50 + t)^2} = 50 + t + \frac{C}{(50 + t)^2}.$$

At $t = 0$, $y = 100(0.01) = 1$. Hence,

$$1 = 50 + 0 + \frac{C}{(50 + 0)^2} \implies C = -122500.$$

Thus,

$$y = 50 + t - \frac{122500}{(50 + t)^2}.$$

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Problem 15. A first order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n, \quad n \neq 0, 1$$

is called a Bernoulli equation. Show that the substitution $u = y^{1-n}$ reduces the Bernoulli equation into the linear equation $\frac{\mathrm{d}u}{\mathrm{d}x} + (1 - n)p(x)u(x) = (1 - n)q(x)$.

A cardiac pacemaker is designed to provide electrical impulses I amps such that as time t increases, I oscillates with a fixed amplitude of one amp. It is proposed that the following differential equation $\frac{\mathrm{d}I}{\mathrm{d}t} + (\tan t)I = (I \sin t)^2$ can be used to describe how I changes with t.

By using a substitution of the form $u = I^{1-n}$, find I in terms of t.

State one limitation of this model.

Solution. Note that

$$u = y^{1-n} \implies \frac{\mathrm{d}u}{\mathrm{d}x} = (1 - n)y^n \frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{y^n}{1 - n}.$$

Substituting this into the given differential equation, we get

$$\frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{y^n}{1 - n} + p(x)y = q(x)y^n \implies \frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{1}{1 - n} + p(x)u = q(x)$$

$$\implies \frac{\mathrm{d}u}{\mathrm{d}x} + (1 - n)p(x)u = (1 - n)q(x)$$

Let $n = 2$. Then $u = I^{-1}$. We also have $p(x) = \tan t$ and $q(x) = \sin^2 t$.

$$\frac{\mathrm{d}I}{\mathrm{d}t} + (\tan t)I = (I \sin t)^2 \implies \frac{\mathrm{d}u}{\mathrm{d}t} - (\tan t)u = -\sin^2 t$$

$$\implies \cos t \frac{\mathrm{d}u}{\mathrm{d}t} - (\sin t)u = -\cos t \sin^2 t \implies \frac{\mathrm{d}}{\mathrm{d}t}(u \cos t) = -\cos t \sin^2 t$$

$$\implies u \cos t = \int -\cos t \sin^2 \theta \,\mathrm{d}t = -\frac{1}{3}\sin^3 t + C$$

$$\implies u = \frac{-1/3 \cdot \sin^3 t + C}{\cos t} \implies I = \frac{\cos t}{-1/3 \cdot \sin^3 t + C} = \frac{3 \cos t}{3C - \sin^3 t}$$

Consider the stationary points of I. For stationary points, we have $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$. Hence,

$$\frac{\sin t}{\cos t} I = I^2 \sin^2 t \implies I \sin t \left(I \sin t - \frac{1}{\cos t} \right) = 0.$$

Hence, $\sin t = 0$ or $I \sin t - \frac{1}{\cos t} = 0$. Note that $I = \frac{3 \cos t}{3C - \sin^3 t} \neq 0$ since $\cos t \neq 0$. We now consider the latter case.

$$I \sin t - \frac{1}{\cos t} = 0 \implies I \sin t \cos t = 1 \implies I \sin 2t = 2.$$

Since I has an amplitude of 1, we have that $I \in [-1, 1]$. Since $\sin 2t \in [-1, 1]$, we have that $I \sin 2t \in [-1, 1]$. Thus, $I \sin 2t$ can never be 2. Hence, stationary points only occur when $\sin t = 0$, implying $t = k\pi$. Thus, $I(0) = \pm 1$. This gives

$$I(0) = \frac{3 \cos 0}{3C - \sin^3 0} = 1 \implies C = 1,$$

whence

$$I = \frac{3 \cos t}{3 - \sin^3 t}.$$

A limitation of this model is that it does not reflect the fact that the oscillations may gradually get weaker.

Self-Practice B13

Problem 1. Food energy taken in by a man goes partly to maintain the healthy functioning of his body and partly to increase body mass. The total food energy intake of the man per day is assumed to be a constant denoted by I (in joules). The food energy required to maintain the healthy functioning of his body is proportional to his body mass M (in kg). The increase of M with respect to time t (in days) is proportional to the energy not used by his body. If the man does not eat for one day, his body mass will be reduced by 1%.

(a) Show that I, M and t are related by the following differential equation:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{I - aM}{100a},$$

where a is a constant. State an assumption for this model to be valid.

(b) Find the total food energy intake per day, I, of the man in terms of a and M if he wants to maintain a constant body mass.

It is given that the man's initial mass is 10 kg.

(c) Solve the differential equation in part (a), giving M in terms of I, a and t.

(d) Sketch the graph of M against t for the case where $I > 100a$. Interpret the shape of the graph with regard to the man's food energy intake.

(e) If the man's total food energy intake per day is $50a$, find the time taken in days for the man to reduce his body mass from 100 kg to 90 kg.

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Problem 2. Find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x^5\mathrm{e}^{2x}.$$

Sketch the family of solution curves, showing clearly all the essential features sufficiently.

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Problem 3. Let the variables x and y be related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = xy^n,$$

where n is a real number. Find the general solution for y in terms of x for the following cases:

(a) $n = 0$;

(b) $n = 1$;

(c) $n \geq 2$, using the substitution $u = y^{1-n}$.

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Problem 4. *Orthogonal trajectories* are a family of curves that intersect another family of curves perpendicularly.

The electrostatic field created by a single positive charge is a collection of straight lines that radiate away from the charge. Equipotential lines are where the electric potentials are equal on a 2-dimensional surface (**these lines can be curves**). It is given that the equipotential lines are orthogonal trajectories of the electric field lines.

(a) By forming a differential equation satisfied by equipotential lines and solving it, show that the equipotential line of a point charge forms a family of circles with centre at the origin, taking the point charge to be at the origin.

When a point charge is placed at $(0, h_1)$, there is an equipotential line tangential to the x-axis. The collection of these equipotential lines for all $h_1 \in \mathbb{R}$, $h_1 \neq 0$ forms a family of circles denoted by C.

(b) By first writing the Cartesian equation of a circle tangential to the x-axis and with centre $(0, h_1)$, show that the orthogonal trajectories of the family of circles C satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - x^2}{2xy}.$$

Hence, by using the substitution $Y = y^2$ show that the orthogonal trajectories of the family of circles C, form a family of circles that are tangential to the y-axis at the origin.

Assignment B13

Problem 1. Two biological cultures, X and Y, react with each other, and their volumes at time t are x and y respectively, in appropriate units. Their rates of growth are modelled by the simultaneous equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (2-x)y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y^2}{x}$$

When $t=0$, $x=y=1$.

(a) Show that $x = \frac{2y^2}{1+y^2}$.

(b) Find and simplify expressions for y and x in terms of t.

(c) Sketch the graph of y against x for $0 < t < \frac{\pi}{2}$.

Solution.

Part (a). Note that $x, y > 0$ since they represent volume. Also, for $x \in (0,2)$, we have $\frac{\mathrm{d}x}{\mathrm{d}t} = (2-x)y > 0$. When $x=2$, we have $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. Hence, $0 < x \leq 2$. Now observe that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{y^2/x}{(2-x)y} = \frac{y}{x(2-x)} \implies \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x(2-x)}.$$

Integrating both sides with respect to x, we get

$$\implies \int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x(2-x)}\,\mathrm{d}x = \frac{1}{2}\int \left(\frac{1}{x} + \frac{1}{2-x}\right)\mathrm{d}x$$
$$\implies \ln y = \frac{1}{2}\left[\ln x - \ln(2-x)\right] + C_1 \implies y = C_2\sqrt{\frac{x}{2-x}}.$$

At $t=0$, $x=y=1$. Hence,

$$1 = C_2\sqrt{\frac{1}{2-1}} \implies C_2 = 1.$$

Thus,

$$y = \sqrt{\frac{x}{2-x}} \implies x = \frac{2y^2}{1+y^2}.$$

Part (b). Observe that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y^2}{x} = \frac{y^2}{2y^2/(1+y^2)} = \frac{1}{2}(1+y^2) \implies \frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}.$$

Integrating both sides with respect to t, we get

$$\int \frac{1}{1+y^2}\,\mathrm{d}y = \int \frac{1}{2}\,\mathrm{d}t \implies \arctan y = \frac{t}{2} + C \implies y = \tan\left(\frac{t}{2} + C\right).$$

At $t=0$, $y=1$. Hence,

$$1 = \tan C \implies C = \frac{\pi}{4},$$

whence

$$y = \tan\left(\frac{t}{2} + \frac{\pi}{4}\right) = \frac{1 - \cos(t+\pi/2)}{\sin(t+\pi/2)} = \frac{1+\sin t}{\cos t} = \sec t + \tan t.$$

Observe that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (2-x)y = (2-x)\sqrt{\frac{x}{2-x}} = \sqrt{x(2-x)} \implies \frac{1}{\sqrt{x(2-x)}}\frac{\mathrm{d}x}{\mathrm{d}t} = 1.$$

Integrating both sides with respect to t, we get

$$\int \frac{1}{\sqrt{x(2-x)}}\,\mathrm{d}x = \int 1\,\mathrm{d}t \implies 2\arcsin\left(\sqrt{\frac{x}{2}}\right) = t + C_1 \implies x = 2\sin^2\left(\frac{t}{2} + C_2\right).$$

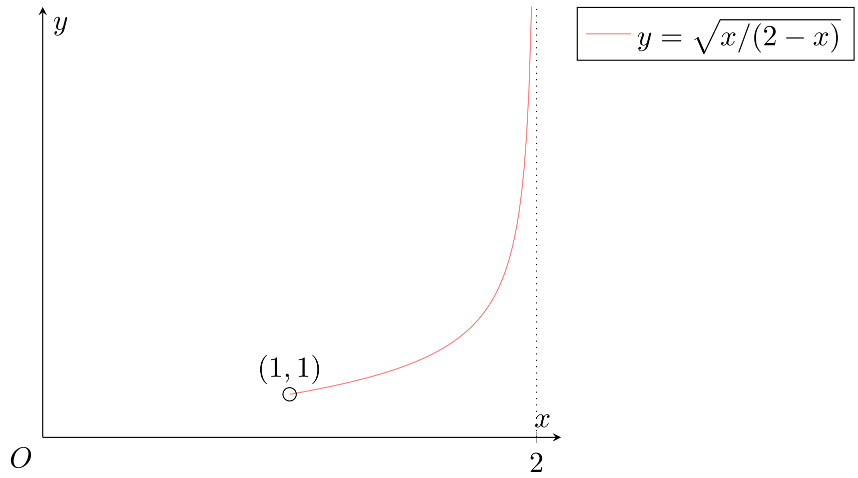
At $t = 0$, $x = 1$. Hence,

$$1 = 2\sin C_2 \implies C_2 = \frac{\pi}{4}.$$

Thus,

$$x = 2\sin^2\left(\frac{1}{2}t + \frac{\pi}{4}\right) = 1 - \cos\left(t + \frac{\pi}{2}\right) = 1 + \sin t.$$

Part (c). Note that $0 < t < \frac{\pi}{2} \implies 1 < x < 2$.



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Problem 2. Find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - 10x = 0.$$

Find the particular solution such that $y \to 0$ as $x \to 0$.

Show, on a single diagram, a sketch of this particular solution and one typical member of the family, F of solution curves for which $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive whenever x is positive.

Show that there is a straight line which passes through the maximum point of every member of F and find its equation.

Solution.

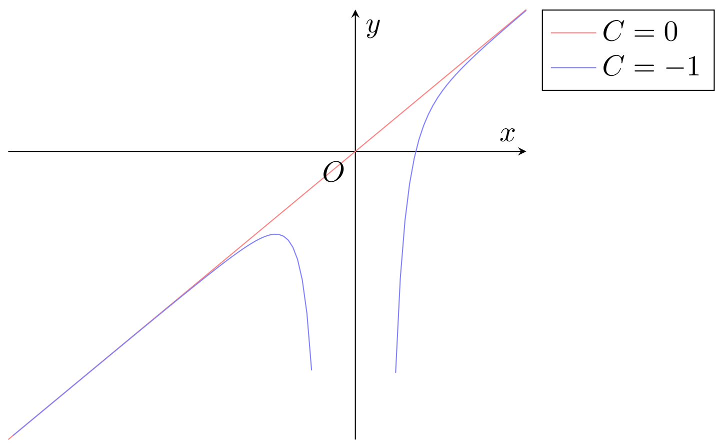
$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - 10x = 0 \implies x^4\frac{\mathrm{d}y}{\mathrm{d}x} + 4x^3y = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^4y\right) = 10x^4$$
$$\implies x^4y = \int 10x^4\,\mathrm{d}x = 2x^5 + C \implies y = 2x + Cx^{-4}$$

As $x \to 0$, $x^{-4} \to \infty$. Hence, C must be 0, whence the particular solution is $y = 2x$.

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 4Cx^{-5} > 0 \implies C < \frac{x^5}{2}.$$

Since $x > 0$, we hence have the constraint $C \leq 0$ for members of F.



Consider the stationary points of members of F. For stationary points, $\frac{dy}{dx} = 0$. Hence,

$$x\frac{dy}{dx} + 4y - 10x = 0 \implies 4y - 10x = 0 \implies y = \frac{5}{2}x.$$

Differentiating the original differential equation with respect to x, we obtain

$$x\frac{dy}{dx} + 4y - 10x = 0 \implies \left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) + 4\frac{dy}{dx} - 10 = 0 \implies \frac{d^2y}{dx^2} = \frac{10}{x}.$$

Note that for members of F, we have that $\frac{dy}{dx} > 0$ for $x > 0$. Hence, there are no stationary points when $x > 0$. That is, any stationary point must occur when $x < 0$ (indeed, there is a stationary point when $x = \sqrt[5]{2C} < 0$). Furthermore, when $x < 0$, $\frac{d^2y}{dx^2} < 0$. Hence, all stationary points must be a maximum point. Thus, $y = \frac{5}{2}x$ passes through the maximum point of every member of F.

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Problem 3.

(a) The variables x and y are related by the differential equation

$$x^2\frac{dy}{dx} - 2xy + y = 0.$$

(i) Find the general solution of this differential equation, expressing y in terms of x.

(ii) Find the particular solution for which $y = -\mathrm{e}$ when $x = 1$. Obtain the coordinates of the turning point of the solution curve of this particular solution and sketch the curve for $x > 0$.

(b) Find the general solution of the differential equation

$$\frac{dy}{dx} + xy = \mathrm{e}^x x^2,$$

expressing y in terms of x.

Solution.

Part (a).

Part (a)(i). Note that

$$x^2\frac{dy}{dx} - 2xy + y = 0 \implies \frac{1}{y}\frac{dy}{dx} = \frac{2}{x} - \frac{1}{x^2}.$$

Integrating with respect to x on both sides, we get

$$\int \frac{1}{y}\,\mathrm{d}y = \int \left(\frac{2}{x} - \frac{1}{x^2}\right)\mathrm{d}x \implies \ln|y| = 2\ln|x| + \frac{1}{x} + C_1 \implies y = C_2 x^2 \mathrm{e}^{1/x}.$$

Part (a)(ii). When $x = 1$, $y = -e$. Hence,

$$-e = C_2\left(1^2\right)\left(\mathrm{e}^1\right) \implies C_2 = -1 \implies y = -x^2 e^{1/x}.$$

For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Hence, $y(2x-1) = 0$, whence $x = \frac{1}{2}$. Note that we reject $y = 0$ since $e^{1/x} \neq 0$ and $x \neq 0$ due to the presence of a $\frac{1}{x}$ term. Hence, y has a stationary point at $(1/2, -\mathrm{e}^2/4)$.

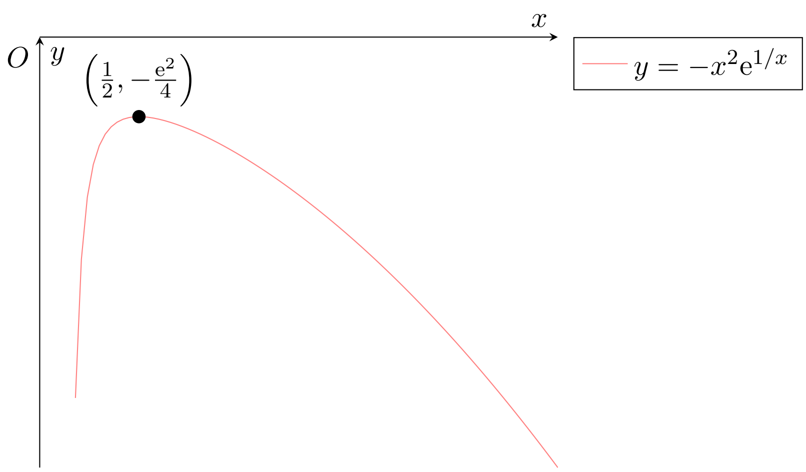
Differentiating the original differential equation with respect to x, we obtain

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y = 0 \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2y}{x^2}.$$

Hence, at $(1/2, -\mathrm{e}^2/4)$, we have

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-e^2/2}{1/4} < 0,$$

whence it is a turning point.



Part (b). Observe that

$$\frac{\mathrm{d}y}{\mathrm{d}x} + xy = \mathrm{e}^x x^2 \implies e^{\frac{1}{2}x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + x\mathrm{e}^{\frac{1}{2}x^2}y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{1}{2}x^2}y\right) = \mathrm{e}^{\frac{1}{2}x^2+x}x^2.$$

Thus,

$$\mathrm{e}^{\frac{1}{2}x^2}y = \int \mathrm{e}^{\frac{1}{2}x^2+x}x^2\,\mathrm{d}x.$$

Suppose $\int e^{\frac{1}{2}x^2+x}x^2\,\mathrm{d}x = P(x)e^{\frac{1}{2}x^2+x} + C$ for some function $P(x)$. Differentiating both sides with respect to x, we obtain

$$x^2e^{\frac{1}{2}x^2+x} = e^{\frac{1}{2}x^2+x}\left[(x+1)P(x) + P'(x)\right],$$

whence

$$x^2 = (x+1)P(x) + P'(x).$$

Thus, $P(x)$ is a polynomial of degree 1. Let $P(x) = ax + b$. For some constants a and b. Then

$$x^2 = ax^2 + (a+b)x + (a+b).$$

Comparing coefficients of x^2, x and constant terms, we have $a = 1$ and $a + b = 0 \implies b = -1$. Thus,

$$\int x^2 e^{\frac{1}{2}x^2+x}\,\mathrm{d}x = (x-1)e^{\frac{1}{2}x^2+x} + C.$$

Hence, we have

$$y = (x-1)e^x + Ce^{-\frac{1}{2}x^2}.$$

B14 Euler Method and Improved Euler Method

Tutorial B14

Problem 1. Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4y - 1, \qquad y(0) = 1.$$

Use the Euler method with step size $\Delta t = 0.1$ to estimate $y(0.5)$.

Explain whether the approximation is an underestimate or an overestimate of the actual value.

Solution. Let $f(y) = 4y - 1$ and $y_0 = 1$. By the Euler method ($y_{n+1} = y_n + \Delta t f(y_n)$),

t	n	y_n
0.0	0	1
0.1	1	1.3
0.2	2	1.72
0.3	3	2.308
0.4	4	3.1312
0.5	5	4.28368

Hence, $y(0.5) \approx 4.28$.

Observe that $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 4\frac{\mathrm{d}y}{\mathrm{d}t} > 0$ for $y > 0$. Hence, y is concave upward. Thus, the approximation is an underestimate.

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Problem 2. A solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$ has $y = 0.5$ at $x = 0$.

(a) Use the Euler method with step size 0.2 to estimate y at $x = 1$. State with a reason whether this value of y is an underestimate or an overestimate.

(b) Find the exact value of y at $x = 1$.

(c) By changing the step size to $\Delta x = 0.1$, comment on the accuracy of the approximations. What are the trade-offs, if any?

Solution.

Part (a). Let $f(x, y) = y - x$, $x_0 = 0$, $y_0 = 0.5$ and $\Delta x = 0.2$. By the Euler method ($y_{n+1} = y_n + \Delta t f(y_n)$),

x	n	y_n
0.0	0	0.5
0.2	1	0.6
0.4	2	0.68
0.6	3	0.736
0.8	4	0.7632
1.0	5	0.75584

Hence, $y(1) \approx 0.756$.
Observe that for $x \in [0, 1]$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y - x < 1 \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x} - 1 < 0.$$

Thus, y is concave downward near $x = 1$, whence the approximation is an overestimate.

Part (b).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y - x \implies \mathrm{e}^{-x}\frac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^{-x}y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x}y\right) = -x\mathrm{e}^{-x}$$
$$\implies \mathrm{e}^{-x}y = \int -x\mathrm{e}^{-x}\,\mathrm{d}x = x\mathrm{e}^{-x} + \mathrm{e}^{-x} + C \implies y = x + 1 + C\mathrm{e}^{x}.$$

At $x = 0$, $y = \frac{1}{2}$. Hence,

$$\frac{1}{2} = 1 + C \implies C = -\frac{1}{2} \implies y = x + 1 - \frac{\mathrm{e}^x}{2}.$$

Evaluating y at $x = 1$ yields,

$$y = 1 + 1 - \frac{e^1}{2} = 2 - \frac{\mathrm{e}}{2}.$$

Part (c). The accuracy of the approximations will improve. However, more calculations will need to be done.

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Problem 3. Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t + y, \qquad y(0) = 1.$$

(a) Use the Euler method with step size $\Delta t = 0.2$ to estimate y at $t = 0.6$. Compare the approximated results with the exact solution.

(b) Use the improved Euler method with step size $\Delta t = 0.2$ to estimate y at $t = 0.6$. Compare the approximated results with the exact solution.

Solution. We begin by finding the exact solution to the differential equation. Observe that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t + y \implies \mathrm{e}^{-t}\frac{\mathrm{d}y}{\mathrm{d}t} - \mathrm{e}^{-t}y = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t}y\right) = t\mathrm{e}^{-t}$$
$$\implies \mathrm{e}^{-t}y = \int t\mathrm{e}^{-t}\,\mathrm{d}t = -t\mathrm{e}^{-t} - \mathrm{e}^{-t} + C \implies y = -t - 1 + C\mathrm{e}^{t}.$$

At $t = 0$, $t = 1$. Hence,

$$1 = -1 + C \implies C = 2 \implies y = -t - 1 + 2\mathrm{e}^{t}.$$

Evaluating at $t = 0.6$, we get

$$y = -0.6 - 1 + 2\mathrm{e}^{0.6} = 2.044.$$

Part (a). Let $f(t, y) = t + y$, $t_0 = 0$, $y_0 = 1$ and $\Delta t = 0.2$. By the Euler method,

t	n	y_n
0.0	0	1
0.2	1	1.2
0.4	2	1.48
0.6	3	1.856

Hence, $y(0.6) \approx 1.856$.

The approximation is not very close to the exact solution, with a percentage error of 9.20%.

Part (b). By the improved Euler method,

$$\begin{aligned} \widetilde{y_1} &= y_0 + \Delta t\, f(t_0, y_0) = 1.2 \\ y_1 &= y_0 + \frac{1}{2}\Delta t\Big[f(t_0, y_0) + f(t_1, \widetilde{y_1})\Big] = 1.24 \end{aligned}$$

$$\begin{aligned} \widetilde{y_2} &= y_1 + \Delta t\, f(t_1, y_1) = 1.528 \\ y_2 &= y_1 + \frac{1}{2}\Delta t\Big[f(t_1, y_1) + f(t_2, \widetilde{y_2})\Big] = 1.5768 \end{aligned}$$

$$\begin{aligned} \widetilde{y_3} &= y_2 + \Delta t\, f(t_2, y_2) = 1.97216 \\ y_3 &= y_2 + \frac{1}{2}\Delta t\Big[f(t_2, y_2) + f(t_3, \widetilde{y_3})\Big] = 2.031696 \end{aligned}$$

Hence, $y(0.6) \approx 2.032$.

The approximation is very close to the exact solution, with a percentage error of 0.602%.

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Problem 4. Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y^2, \qquad y(0) = \frac{1}{2}, \qquad 0 \le t \le 2.$$

(a) Determine an analytic solution for the problem.

(b) Using the improved Euler method with a step size of 0.5, determine an approximate value for $y(2)$ and its error.

Solution.

Part (a).

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y^2 \implies -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}t} = 1 \implies \int -y^2\,\mathrm{d}y = \int \mathrm{d}t$$

$$\implies y^{-1} = t + C \implies y = \frac{1}{t+C}.$$

Note that

$$y(0) = \frac{1}{2} \implies \frac{1}{2} = \frac{1}{C} \implies C = 2.$$

Hence,

$$y = \frac{1}{t+2}.$$

Part (b). Let $f(y) = -y^2$, $t_0 = 0$, $y_0 = \frac{1}{2}$ and $\Delta t = 0.5$. By the improved Euler method,

$$\widetilde{y_1} = y_0 + \Delta t\, f(y_0) = 0.375$$
$$y_1 = y_0 + \frac{1}{2}\Delta t\Big[f(y_0) + f(\widetilde{y_1})\Big] = 0.4023438$$

$$\widetilde{y_2} = y_1 + \Delta t\, f(y_1) = 0.3214035$$
$$y_2 = y_1 + \frac{1}{2}\Delta t\Big[f(y_1) + f(\widetilde{y_2})\Big] = 0.3360486$$

$$\widetilde{y_3} = y_2 + \Delta t\, f(y_2) = 0.2795843$$
$$y_3 = y_2 + \frac{1}{2}\Delta t\Big[f(y_2) + f(\widetilde{y_3})\Big] = 0.2882746$$

$$\widetilde{y_4} = y_3 + \Delta t\, f(y_3) = 0.2467235$$
$$y_4 = y_3 + \frac{1}{2}\Delta t\Big[f(y_3) + f(\widetilde{y_4})\Big] = 0.2522809$$

Hence, $y(2) \approx 0.252$. Thus, the error is

$$\text{Error} = 0.2522809 - \frac{1}{2+2} = 0.0022809$$

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Problem 5. It is given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^y + x$, and that a particular solution curve passes through the point $(0, 1)$.

(a) Use the Euler method with a step size of 0.1 to estimate the value of y at $x = 0.5$.

(b) If the estimate for y at $x = 0.5$ is calculated using the improved Euler method with a step size of 0.1, determine whether this estimate will be greater or less than the value you have calculated in (a). Justify your answer.

Solution.

Part (a). Let $f(x, y) = \mathrm{e}^y + x$, $x_0 = 0$, $y_0 = 1$ and $\Delta x = 0.1$. By the Euler method,

t	n	y_n
0.0	0	1
0.1	1	1.27183
0.2	2	1.63857
0.3	3	2.17334
0.4	4	3.08210
0.5	5	5.30253

Hence, $y(0.5) \approx 5.30$.

Part (b). Observe that for $x > 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^y + x > 0 \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} + 1 > 0.$$

Thus, y is concave upwards, whence the estimates are underestimates. Since the improved Euler method is more accurate than the Euler method, it will be greater than the value calculated in (a).

Problem 6. A differential equation is given by

$$\frac{dy}{dt} = y^2 - 2y + 1, \qquad y(0) = 2, \qquad 0 \le t \le 2.$$

Copy and complete the table showing the use of the improved Euler's method with step size 0.5 to estimate y at $t = 2$.

n	t_n	Euler (y_n)	$\widetilde{y_n}$	Improved Euler (y_n)	Actual y_n
0	0.0	2		2	2
1	0.5	2.5	2.5	2.8125	
2	1.0				
3	1.5				
4	2.0				

Compare and comment on your values obtained using the improved Euler method with the values obtained from the Euler method and the actual solution.

Solution. Let $f(y) = y^2 - 2y + 1$ and $\Delta t = 0.5$.

Euler Method.

$$\begin{aligned} y_2 &= y_1 + \Delta t\, f(y_1) = 3.625 \\ y_3 &= y_2 + \Delta t\, f(y_2) = 7.070313 \\ y_4 &= y_3 + \Delta t\, f(y_3) = 25.49466 \end{aligned}$$

Improved Euler Method.

$$\begin{aligned} \widetilde{y_2} &= y_1 + \Delta t\, f(y_1) = 4.455078 \\ y_2 &= y_1 + \frac{1}{2}\Delta t\big[f(y_1) + f(\widetilde{y_2})\big] = 6.618180 \end{aligned}$$

$$\begin{aligned} \widetilde{y_3} &= y_2 + \Delta t\, f(y_2) = 22.40016 \\ y_3 &= y_2 + \frac{1}{2}\Delta t\big[f(y_2) + f(\widetilde{y_3})\big] = 129.0008 \end{aligned}$$

$$\begin{aligned} \widetilde{y_4} &= y_3 + \Delta t\, f(y_3) = 8321.107 \\ y_4 &= y_3 + \frac{1}{2}\Delta t\big[f(y_3) + f(\widetilde{y_4})\big] = 17310270 \end{aligned}$$

Actual Value.

$$\frac{dy}{dt} = y^2 - 2y + 1 = (y-1)^2 \implies \frac{1}{(y-1)^2}\frac{dy}{dt} = 1 \implies \int \frac{1}{(y-1)^2}\,dy = \int 1\,dt$$

$$\implies -\frac{1}{y-1} = x + C \implies y = 1 - \frac{1}{x+C}.$$

Note that

$$y(0) = 2 \implies 2 = 1 - \frac{1}{C} \implies C = -1.$$

Hence,

$$y = 1 - \frac{1}{x-1}.$$

n	t_n	Euler (y_n)	$\widetilde{y_n}$	Improved Euler (y_n)	Actual y_n
0	0.0	2		2	2
1	0.5	2.5	2.5	2.8125	3
2	1.0	3.625	4.455078	6.618180	-
3	1.5	7.070313	22.40016	129.0008	-1
4	2.0	25.49466	8321.107	17310270	0

The values obtained using the improved Euler method deviate significantly from that obtained using the Euler method and the actual solution. This is because y has a discontinuity at $t = 1$, making both Euler methods inappropriate to use.

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Problem 7. A solution of the differential equation

$$\frac{dy}{dx} = y - x$$

has $y = 2$ at $x = 0$.

(a) Use the Euler method with step size 0.5 to estimate y at $x = 1$. Explain whether you expect this value of y to be an underestimate or overestimate of the true value.

(b) Copy and complete the table showing the use of the improved Euler method with step size 0.5 to estimate y at $x = 1$.

x	y	$y - x$	$\widetilde{y}$	$\Delta y/\Delta x$
0	2	2	3	$(2 + 2.5)/2$
0.5	3.125	2.625	4.438	
1				

(c) Show that the exact value of y at $x = 1$ is $2 + e$.

Solution.

Part (a). Let $f(x, y) = y - x$, $x_0 = 0$, $y_0 = 2$, $\Delta x = 0.5$. By the Euler method,

x	n	y_n
0.0	0	2
0.5	1	3
1.0	2	4.25

Hence, $y(1) \approx 4.25$.

Note that for $x > 0$, we get

$$\frac{dy}{dx} = y - x > \implies \frac{d^2y}{dx^2} = \frac{dy}{dx} - 1 > 0.$$

Thus, y is concave upwards, whence the value of y is an underestimate.

Part (b). From the improved Euler method, one has

$$y_2 = y_1 + \frac{1}{2}\Delta x\Big[f(x_1, y_1) + f(x_2, \widetilde{y_2})\Big].$$

Thus,

$$\frac{y_2 - y_1}{\Delta x} = \frac{\Delta y}{\Delta x} = \frac{1}{2}\Big[f(x_1, y_1) + f(x_2, \widetilde{y_2})\Big] = \frac{1}{2}\left[2.625 + (4.438 - 1)\right] = 3.031.$$

Also,

$$y_2 = y_1 + \Delta x \left(\frac{\Delta y}{\Delta x} \right) = 3.125 + 0.5\,(3.031) = 4.64.$$

x	y	$y - x$	$\widetilde{y}$	$\Delta y / \Delta x$
0	2	2	3	$(2+2.5)/2$
0.5	3.125	2.625	4.438	3.031
1	4.64			

Part (c).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y - x \implies \mathrm{e}^{-x}\frac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^{-x}y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x}y\right) = -x\mathrm{e}^{-x}$$
$$\implies \mathrm{e}^{-x}y = \int \left(-x\mathrm{e}^{-x}\right)\,\mathrm{d}x = x\mathrm{e}^{-x} + \mathrm{e}^{-x} + C \implies y = x + 1 + C\mathrm{e}^{x}.$$

At $x = 0$, $y = 2$, whence

$$2 = 1 + C \implies C = 1 \implies y = x + 1 + \mathrm{e}^{x}.$$

Thus, at $x = 1$,

$$y = 1 + 1 + \mathrm{e}^{1} = 2 + \mathrm{e}.$$

* * * * *

Problem 8. Initially, a tank is fully filled with 100 litres of pure water. There exists a tap at the top of the tank. This tap supplies brine, containing 1 g of salt per litre, into the tank at a rate of 1 litre per minute. There also exists another tap at the bottom of the tank which allows the mixture to flow out at a constant rate of 2 litres per minute. At time T (in minutes), the amount of salt and the volume of the mixture in the tank are denoted by S (in grams) and V (in litres) respectively. Both taps are turned on simultaneously at time $T = 0$. The tap at the bottom of the tank is turned off at time $T = 75$. The mixture in the tank is assumed to be well-stirred and homogenous at all times.

(a) Show that $\frac{\mathrm{d}S}{\mathrm{d}T} = \frac{100-T-2S}{100-T}$, $0 < T < 75$.

(b) By solving the differential equation, show that the amount of salt in the tank after 75 minutes is 18.75 grams.

At the instance when the tap at the bottom is turned off, a crack is accidentally created at the bottom of the tank. According to Torricelli's law, the mixture flows out from the crack at a rate proportional to the square-root of its volume. It can be assumed that the mixture flow obeys Torricelli's law, regardless of its viscosity. Let the amount of salt and the volume of the mixture in the tank be denoted by s (in grams) and v (in litres) respectively, t minutes after the crack has been accidentally created. It has been observed that the volume of the mixture in the tank stays constant at 36 litres after a long period of time.

(c) Show that $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{6-\sqrt{v}}{6}$. Estimate the time taken for the mixture in the tank to rise to 26 litres after the crack has been created, by using

 (i) Euler's Method with two iterations,

 (ii) Simpson's Rule with two strips.

(d) Show that $\frac{\mathrm{d}s}{\mathrm{d}v} = \frac{6\sqrt{v}-s}{6\sqrt{v}-v}$. Use the improved Euler method with one iteration to estimate the amount of salt in the tank at the instant when the mixture in the tank rises to 26 litres after the crack has been created. Given your answer to 4 decimal places.

Solution.

Part (a). The concentration of salt in the tank is given by $\frac{S}{V}$. Let V_i and V_o be the volume of liquid entering and leaving the tank in litres, respectively. Then

$$\frac{\mathrm{d}V}{\mathrm{d}T} = \frac{\mathrm{d}V_i}{\mathrm{d}T} - \frac{\mathrm{d}V_o}{\mathrm{d}T} = 1 - 2 = -1 \implies V = 100 - T,$$

since $V = 100$ initially. Thus,

$$\frac{\mathrm{d}S}{\mathrm{d}T} = 1\left(\frac{\mathrm{d}V_i}{\mathrm{d}T}\right) - \frac{S}{V}\left(\frac{\mathrm{d}V_o}{\mathrm{d}T}\right) = 1 - \frac{2S}{100-T} = \frac{100-T-2S}{100-T}.$$

Part (b).

$$\frac{\mathrm{d}S}{\mathrm{d}T} = 1 - \frac{2S}{100-T} \implies \frac{\mathrm{d}S}{\mathrm{d}T} + \frac{2S}{100-T} = 1$$
$$\implies \frac{1}{(100-T)^2}\frac{\mathrm{d}S}{\mathrm{d}T} + \frac{2S}{(100-T)^3} = \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{S}{(100-T)^2}\right) = \frac{1}{(100-T)^2}$$
$$\implies \frac{S}{(100-T)^2} = \int \frac{\mathrm{d}T}{(100-T)^2} = \frac{1}{100-T} + C \implies S = 100 - T + C(100-T)^2.$$

When $T = 0$, $S = 0$, whence

$$0 = 100 + C\left(100^2\right) \implies C = -\frac{1}{100} \implies S = 100 - T - \frac{(100-T)^2}{100}.$$

When $T = 75$,

$$S = 100 - 75 - \frac{(100-75)^2}{100} = 18.75.$$

Hence, the amount of salt in the tank after 75 minutes is 18.75 grams.

Part (c). Let v_i and v_o be the volume of liquid entering and leaving the tank in litres, respectively. Since the top tap is still open, we have $\frac{\mathrm{d}v_i}{\mathrm{d}t} = 1$. By Torricelli's law, we also have $\frac{\mathrm{d}v_0}{\mathrm{d}t} = k\sqrt{v}$. Thus,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v_i}{\mathrm{d}t} - \frac{\mathrm{d}v_o}{\mathrm{d}t} = 1 - k\sqrt{v}.$$

Since the volume of the tank remains constant at 36 litres eventually, we have

$$1 - k\sqrt{36} = 0 \implies k = \frac{1}{6}.$$

Thus,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 1 - \frac{\sqrt{v}}{6} = \frac{6-\sqrt{v}}{6}.$$

Observe that $\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{6}{6-\sqrt{v}}$.

Part (c)(i). Let $f(v) = \frac{6}{6-\sqrt{v}}$, $v_0 = 25$, $t_0 = 0$ and $\Delta v = 0.5$. By the Euler method,

$$t_1 = t_0 + \Delta v\, f(v_0) = 3, \qquad t_2 = t_1 + \Delta v\, f(v_1) = 6.157.$$

Hence, when $t \approx 6.16$, the mixture in the tank has risen to 26 litres.

Part (c)(ii). Note that $t = \int \frac{6}{6-\sqrt{v}}\,\mathrm{d}v$. Hence, the desired time is given by $\int_{25}^{26} \frac{6}{6-\sqrt{v}}\,\mathrm{d}v$. By Simpson's rule,

$$\int_{25}^{26} \frac{6}{6-\sqrt{v}}\,\mathrm{d}v \approx \frac{1}{3} \cdot \frac{26-25}{2}\Big[f(25)+4f(25.5)+f(26)\Big] = 6.319.$$

Hence, when $t \approx 6.32$, the mixture in the tank has risen to 26 litres.

Part (d). Observe that

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 1\left(\frac{\mathrm{d}v_i}{\mathrm{d}t}\right) - \frac{s}{v}\left(\frac{\mathrm{d}v_o}{\mathrm{d}t}\right) = 1 - \frac{s}{v}\frac{\sqrt{v}}{6} = \frac{6\sqrt{v}-s}{6\sqrt{v}}.$$

By the chain rule,

$$\frac{\mathrm{d}s}{\mathrm{d}v} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{6\sqrt{v}-s}{6\sqrt{v}} \cdot \frac{6}{6-\sqrt{v}} = \frac{6\sqrt{v}-s}{6\sqrt{v}-v}.$$

Let $f(s,v) = \frac{6\sqrt{v}-s}{6\sqrt{v}-v}$, $v_0 = 25$, $s_0 = 18.75$ and $\Delta v = 1$. By the improved Euler method,

$$\begin{aligned}\widetilde{s_1} &= s_0 + \Delta v\, f(s_0, v_0) = 21\\ s_1 &= s_0 + \frac{1}{2}\Delta v\Big[f(s_0,v_0) + f(\widetilde{s_1}, v_1)\Big] = 20.9192 \text{ (4 d.p.)}\end{aligned}$$

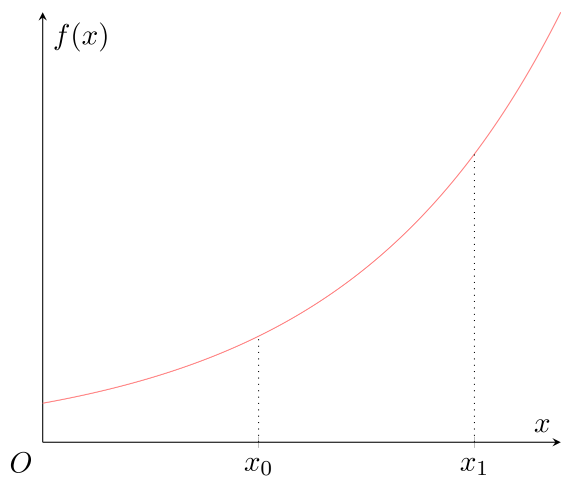
Hence, there is approximately 20.9192 grams of salt in the tank.

* * * * *

Problem 9. A solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ has $y = y_0$ at $x = x_0$. It is required to estimate the value of y at $x = x_1$ using a numerical method with one step.

(a) Write down expressions for the value of y at $x = x_1$ obtained by using the Euler method and by using the improved Euler method.

(b) The graph of f is as below. Copy the graph and use it to illustrate the errors in the two estimates of y obtained by using the methods of part (a). State clearly whether the errors correspond to overestimates or underestimates.



(c) Given that $x_0 = 0$ and $f(x) = a + bx + cx^2$, where a, b and c are constants, find the error in using the improved Euler method with a single step of size h.

Solution.

Part (a). Let $\Delta x = x_1 - x_0$.

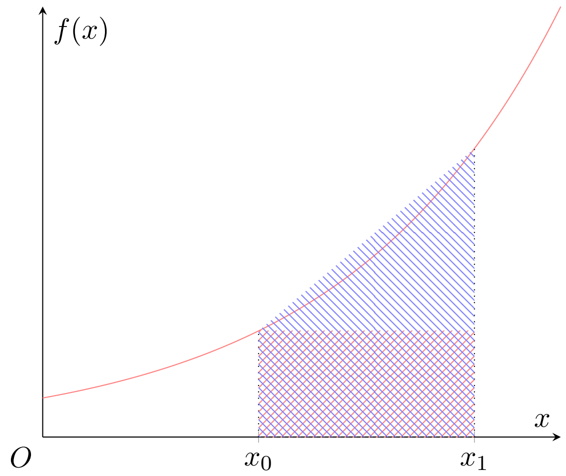
Euler Method.

$$y_1 = y_0 + \Delta x\, f(x_0)$$

Improved Euler Method.

$$y_1 = y_0 + \frac{1}{2}\Delta x\Big[f(x_0) + f(x_1)\Big]$$

Part (b).



By the fundamental theorem of calculus, the area under the graph of $f(x)$ between x_0 and x_1 is precisely $y_1 - y_0$. That is,

$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}x = y_1 - y_0 = \Delta y.$$

Hence, the better the approximation of the integral, the better the approximation of Δy and thus y_1.

The Euler method gives the approximation $\Delta y = \Delta x\, f(x_0)$. This is represented by the area of the red-shaded rectangle with base Δx and height $f(x_0)$.

The improved Euler method gives the approximation $\Delta y = \Delta x\, \frac{1}{2}\big[f(x_0) + f(x_1)\big]$. This is represented by the area of the blue-shaded trapezium with base Δx and heights $f(x_0)$ and $f(x_1)$.

Thus, the improved Euler method gives a better approximation for the integral of $f(x)$ than the Euler method. Thus, the error of the estimate given by the improved Euler method is smaller than that of the Euler method.

The Euler method underestimates the integral, hence y_1 is also underestimated. Similarly, the improved Euler method overestimates the integral, hence y_1 is also overestimated.

Part (c). We have $f(x) = a + bx + cx^2$, $x_0 = 0$ and $\Delta x = h$. Let $y_0 = d$.

Improved Euler Method.

$$y_1 = d + \frac{1}{2}h\Big[(a) + \big(a + bh + ch^3\big)\Big] = d + ah + \frac{1}{2}bh^2 + \frac{1}{2}ch^3.$$

Actual Value.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a + bx + cx^2 \implies y = \int \left(a + bx + cx^2\right)\,\mathrm{d}x = ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3 + C.$$

When $x = 0$, $y = d$. Hence, $C = d$, thus $y = d + ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3$. At $x = h$,

$$y_1 = d + ah + \frac{1}{2}bh^2 + \frac{1}{3}ch^3$$

Error.

$$\text{Error} = \left(d + ah + \frac{1}{2}bh^2 + \frac{1}{3}ch^3\right) - \left(d + ah + \frac{1}{2}bh^2 + \frac{1}{2}ch^3\right) = \frac{1}{6}ch^3.$$

* * * * *

Problem 10. The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 \tan x = 1,$$

where $y = 1$ when $x = 1$, is to be solved numerically.

(a) Carry out two steps of Euler's method with step length 0.1 to estimate the value of y when $x = 1.2$, giving your answer to 4 decimal places.

(b) The method in part (a) is now replaced by the improved Euler method. The estimate obtained is 2.0156, given to 4 decimal places. State, with a reason, whether this estimate and the one found in part (a) are likely to be overestimates or underestimates of the actual value of y when $x = 1.2$.

(c) Explain why it would be inappropriate to continue this process in part (a) to estimate the value of y when $x = 1.6$.

Solution.

Part (a). Let $f(x, y) = \mathrm{d}y/\mathrm{d}x = 1+y^2 \tan x$, $x_0 = 1$, $y_0 = 1$, $\Delta x = 0.1$ and $x_n = x_0+n\Delta x$. By the Euler method,

$$y_1 = y_0 + \Delta x\, f(x_0, y_0) = 1.2557408, \qquad y_2 = y_1 + \Delta x\, f(x_1, y_1) = 1.6655608.$$

Hence, $y(1.2) \approx 1.6656$ (4 d.p.).

Part (b). Observe that on the interval $x \in I = [1, 1.2]$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2 \tan x > 0.$$

Since y is continuous on I, and $y(1) = 1$, we also have $y > 0$ on I. Thus,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} \tan x + y^2 \sec^2 x > 0,$$

whence y is concave upwards. Thus, the estimates are likely to be underestimates.

Part (c). $f(x, y)$ has a vertical asymptote at $x = \pi/2 \in (1.5, 1.6)$. Thus, the Euler method will fail.

Self-Practice B14

Problem 1. A solution of the differential equation $\frac{dx}{dt} + x \cot t = \csc t$ has $x = 0$ when $t = 1$.

(a) Use the improved Euler method with a step size of 0.5 to estimate x at $t = 2$.

(b) Solve the differential equation to find the exact value of x when $t = 2$.

(c) By considering the gradients of the curve at $t = 1.5$, $t = 2$ and $t = 2.5$, comment on the accuracy of the Euler method to estimate x at $t = 2$ and $t = 3$.

* * * * *

Problem 2. For the differential equation $\frac{dy}{dx} + \frac{y}{x} = 3x$, consider the solution curve passing through the point $(1, 2)$.

(a) Compute the Euler approximation to $y(1.1)$ using step-size $h = 0.1$.

(b) State, giving a reason, if you would expect the estimate in part (a) to be an underestimate or an overestimate of the true value.

(c) Find the general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = 3x$.

(d) Hence, find the solution curve that passes through the point $(1, 2)$ and calculate the percentage error of your estimate in (a).

* * * * *

Problem 3. A differential equation is of the form

$$\frac{dy}{dx} = \frac{y}{x} f\left(\frac{y}{x}\right).$$

(a) By using the substitution $z = y/x$, show that

$$\ln|x| = \int \frac{1}{z\left[f(z) - 1\right]}\, dz.$$

(b) A particular solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x} \ln\left(\frac{x}{y}\right)$$

has $y = 1$ when $x = 1$. Find y in terms of x.

(c) Copy and complete the table below, using the improved Euler method with step size 0.5 to estimate y at $x = 2$.

x	y	dy/dx	$\widetilde{y}$	$\Delta y/\Delta x$
1	1	0	1	0.13516
1.5	1.06758			
2				

(d) Use a graphical method to explain whether the estimated value of y found in part (c) is an under-estimate or over-estimate.

Problem 4. The variables P and t are related by the "modified" logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{10}P\left(1 - \frac{P}{10}\right)(P-1).$$

The differential equation is used to model the size, P (in thousands), of species of wolves in time, t (in years) in a given habitat.

(a) Biologists observe that if the population of wolves is "too small", adults run the risk of being unable to find a mate, resulting in a decrease to the population.

(i) Explain how the model accounts for this observation.

(ii) State the maximum population that the resources in the habitat can support.

(iii) Find the equilibrium solutions.

(iv) Sketch the possible solution curves for P as a function of t.

(b) You are given that the population is now 2000.

(i) Copy and complete the table showing the use of the improved Euler method with step size 0.5 to estimate the population in a year's time.

t	P	$\mathrm{d}P/\mathrm{d}t$	$\widetilde{P}$	$\Delta P/\Delta t$
0	2	0.16	2.08	0.16896
0.5	2.08448			
1				

(ii) How could the accuracy of the numerical method in part (b)(i) be improved?

(c) Suppose the wolves are being hunted at a fixed rate E (in thousands per year). Write down the new model for the population.

Assignment B14

Problem 1.

(a) Explain why the Euler method will fail for the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\cos\sqrt{x}, \qquad y(0) = 0,$$

where $y = y(x)$ satisfies that differential equation and is not a constant.

(b) Suppose the initial condition for the problem in part (a) is now $y(0) = 10$. Use the improved Euler method with a step size of 0.1 to find, to three decimal places, an estimate for $y(0.1)$.

(c) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}y(2-x), \qquad y > 0, \quad y(0) = 10,$$

expressing y in terms of x, and simplifying your answer as far as possible.

(d) Explain why the solution found in part (c) will give a reasonable estimate for $y(0.1)$ in part (b).

Solution.

Part (a). By the Euler method,

$$y_1 = y_0 + \Delta x(y_0\cos\sqrt{x_0}) = 0 + \Delta x(0\cos 0) = 0.$$

It follows that $y_n = 0$ for all $n \in \mathbb{N}$, whence y is the zero function. However, because y is not a constant function, y cannot be the zero function, a contradiction. Hence, the Euler method fails.

Part (b). Let $\Delta x = 0.1$, $y_0 = 10$ and $x_n = n\Delta x$.

$$\begin{aligned}\widetilde{y_1} &= y_0 + \Delta x(y_0\cos\sqrt{x_0}) = 11\\ y_1 &= y_0 + \frac{1}{2}\Delta x\left[y_0\cos\sqrt{x_0} + \widetilde{y_1}\cos\sqrt{x_1}\right] = 11.023 \text{ (3 d.p.)}\end{aligned}$$

Hence, $y(0.1) \approx 11.023$.

Part (c).

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}y(2-x) \implies \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(2-x) \implies \int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{2}(2-x)\,\mathrm{d}x\\ &\implies \ln y = \frac{1}{2}\left[2x - \frac{1}{2}x^2\right] + C_1 = x - \frac{1}{4}x^2 + C_1 \implies y = C\exp\left(x - \frac{1}{4}x^2\right).\end{aligned}$$

Since $y(0) = 10$, we have $C = 10$. Thus,

$$y = 10\exp\left(x - \frac{1}{4}x^2\right).$$

Part (d). For small x, we have that $\cos\sqrt{x} \approx 1 - \frac{1}{2}(\sqrt{x})^2 = \frac{1}{2}(2-x)$. Thus,

$$y\cos\sqrt{x} \approx \frac{1}{2}y(2-x),$$

whence the two differential equations and thus their solutions are approximately equal. Since $x = 0.1$ is small, the solution found in part (c) will give a reasonable estimate for $y(0.1)$ in part (b).

* * * * *

Problem 2. Rewriting the given differential equation, we obtain

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{7x}{v} - 24.$$

Let $f(x, v) = -\frac{7x}{v} - 24$, $\Delta x = 1$, $v_0 = 121$, and $x_n = n\Delta x$.

Solution.

Part (a). By the Euler method,

$$v_1 = v_0 + \Delta x f(x_0, v_0) = 97.$$

Thus, $y(1) \approx 97$.

Part (b). By the improved Euler method,

$$\begin{aligned}\widetilde{v_1} &= v_0 + \Delta x f(x_0, v_0) = 97\\ v_1 &= v_0 + \frac{1}{2}\Delta x \left[f(x_0, v_0) + f(x_1, \widetilde{v_1})\right] = 96.964\end{aligned}$$

Thus, $y(1) \approx 96.964$.

The gradient of v at $x = 0$ is $f(x_0, v_0) = -24$, which is very close to the gradient of v at $x = 1$, $f(x_1, \widetilde{v_1}) = -24.072$. Since the gradient of v is approximately constant for $0 \le x \le 1$, we have that v is approximately a linear function on that interval.

Observe that for $0 \le x \le 1$, $x/v \approx 0$ since $x \in [0, 1]$, while $v \ge 96$. Thus, $\mathrm{d}v/\mathrm{d}x \approx -24$, whence $v = -24x + C$. Since $v = 121$ when $x = 0$, we have $v \approx -24x + 121$.

* * * * *

Problem 3. The function $y = y(x)$ satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{5}(\tan x + x^3 y).$$

The value of $y(h)$ is to be found, where h is a small positive number, and $y(0) = 0$.

(a) Use one step of the improved Euler method to find an alternative approximation to $y(h)$ in terms of h.

(b) It can be shown that $y = y(x)$ satisfies

$$y(h) = e^{0.05h^4} \int_0^h \frac{\tan x}{5} \mathrm{e}^{-0.05x^4} \, \mathrm{d}x.$$

Assume that h is small and hence find another approximation to $y(h)$ in terms of h.

(c) Discuss the relative merits of these two methods employed to obtain these approximations.

Solution.

Part (a). Let $f(x,y) = \frac{1}{5}(\tan x + x^3 y)$, $\Delta x = h$ and $y_0 = 0$. By the improved Euler method,

$$\widetilde{y_1} = y_0 + \Delta x f(x_0, y_0) = 0$$

$$y_1 = y_0 + \frac{1}{2}\Delta x \left[f(x_0, y_0) + f(x_1, \widetilde{y_1})\right] = 0 + \frac{1}{2}h\left[0 + \frac{1}{5}(\tan h + 0)\right] = \frac{h \tan h}{10}.$$

Hence,

$$y(h) \approx \frac{h \tan h}{10}.$$

Part (b). Since h is small, we have that $\mathrm{e}^{0.05h^4} \approx 1$. Furthermore, since we are integrating over the interval $x \in [0, h]$, the integrand $\frac{\tan x}{5}\mathrm{e}^{-0.05x^4}$ can likewise be approximated by $\frac{x}{5}$. Our integral hence transforms to

$$y(h) = \mathrm{e}^{0.05h^4} \int_0^h \frac{\tan x}{5} \mathrm{e}^{-0.05x^4}\,\mathrm{d}x \approx \int_0^h \frac{x}{5}\,\mathrm{d}x = \frac{h^2}{10}.$$

Part (c). The improved Euler method involves more steps, while the approximation in (b) is more direct.

B15 Modelling Populations with First Order Differential Equations

Assignment B15

Problem 1. In response to a massive ecosystem-wide destruction by goats on the island of Isabela in Ecuador, Project Isabela was started on the first day of 1997 to eliminate all goats on the island. Goat elimination was done by hunting at a constant rate. Suppose that the goat population, P (in thousands), can be modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P}{4}\left(1 - \frac{P}{150}\right) - H,$$

where t is measured in months and H is measured in thousands.

(a) State, in context, the significance of the term H.

(b) Find the greatest integer value of H for which it is still possible for some goats to survive in the long run.

(c) Based on the answer from part (b), discuss the long-term behaviour of the goat population for different initial populations.

The hunters involved in Project Isabela finally managed to eliminate all the goats on the island of Isabela on the first day of 2006.

(d) State an inequality that must be satisfied by H.

(e) Given that the initial goat population was 100 thousand, find the value of H, correct to 3 decimal places.

Solution.

Part (a). H represents the number of goats killed (in thousands) per month.

Part (b). Consider the equilibrium points of the differential equation.

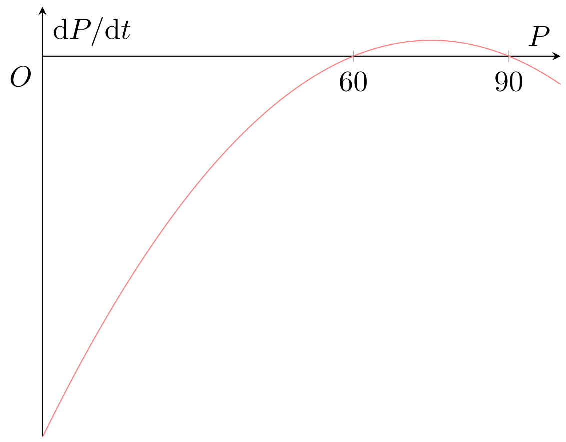
$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P}{4}\left(1 - \frac{P}{150}\right) - H = -\frac{1}{600}\left(P^2 - 150P + 600H\right) = 0.$$

By the quadratic formula,

$$P = 75 \pm 5\sqrt{225 - 24H}.$$

For it to be possible for goats to survive in the long term, there must be at least one equilibrium point. That is, $\sqrt{225 - 24H} \geq 0 \implies H \leq 9.375$. Thus, the maximum integer value of H is 9.

Part (c). When $H = 9$, the equilibrium points are $P = 75 \pm 5\sqrt{225 - 24(9)} = 60$ or 90. Let the initial population be P_0.



When $P_0 = 0$, there are no goats initially. Hence, the population will remain at 0.

When $0 < P_0 < 60$, $\mathrm{d}P/\mathrm{d}t < 0$. Hence, the population of goats will decrease towards 0.

When $P_0 = 60$, $\mathrm{d}P/\mathrm{d}t = 0$. Hence, the population of goats will remain at 60 thousand.

When $60 < P_0 < 90$, $\mathrm{d}P/\mathrm{d}t > 0$. Hence, the population of goats will increase towards 90 thousand.

When $P_0 = 90$, $\mathrm{d}P/\mathrm{d}t = 0$. Hence, the population of goats will remain at 90 thousand.

When $P_0 > 90$, $\mathrm{d}P/\mathrm{d}t < 0$. Hence, the population of goats will decrease towards 90 thousand.

Part (d). H must satisfy the inequality $H > 9.375$.

Part (e). Note that $t = 120$, $P(0) = 100$ and $P(108) = 0$. Now observe that

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{1}{600}\left(P^2 - 150P + 600H\right) = -\frac{1}{600}\left[(P-75)^2 + \left(600H - 75^2\right)\right]$$
$$\implies \frac{1}{(P-75)^2 + (600H - 75^2)}\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{1}{600}.$$

Integrating both sides with respect to t,

$$\int \frac{1}{(P-75)^2 + (600H-75^2)}\,\mathrm{d}P = \int -\frac{1}{600}\,\mathrm{d}t$$
$$\implies \frac{1}{\sqrt{600H-75^2}}\arctan\left(\frac{P-75}{\sqrt{600H-75^2}}\right) = -\frac{1}{600}t + C.$$

Let $X = \frac{1}{\sqrt{600H-75^2}}$. This simplifies the above result to

$$X\arctan((P-75)X) = -\frac{1}{600}t + C.$$

When $t = 0$, $P = 100$. Hence,

$$C = X\arctan(25X)$$

When $t = 108$, $P = 0$. Hence,

$$X\arctan(-75X) = -\frac{108}{600} + X\arctan(25X),$$

which has the solution $X = 0.073145$. Note that we reject $X = -0.073145$ since $X \geq 0$. We thus have

$$H = \frac{1}{600}\left(\frac{1}{0.073145^2} + 75^2\right) = 9.377 \text{ (3 d.p.)}.$$

B16 Second Order Differential Equations

Tutorial B16

Problem 1. Find the general solution of $3\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} - 7x = 0$.

Solution. Consider the characteristic equation of the DE:

$$3m^2 + 4m - 7 = (3m+7)(m-1) = 0.$$

We hence have $m = -7/3$ or $m = 1$, whence

$$x = A\mathrm{e}^{-\frac{7}{3}t} + B\mathrm{e}^{t}.$$

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Problem 2. Solve the following homogeneous second-order linear differential equations.

(a) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 0$, given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -4$ when $x = 0$.

(b) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0$, given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

(c) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \sqrt{3}\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 0$, given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -4$ when $x = 0$.

Solution.

Part (a). Consider the characteristic equation of the DE:

$$m^2 + 4m + 3 = (m+1)(m+3) = 0.$$

We hence have $m = -1$ or $m = -3$, whence

$$y = A\mathrm{e}^{-x} + B\mathrm{e}^{-3x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -A\mathrm{e}^{-x} - 3B\mathrm{e}^{-3x}.$$

Using the given conditions, we obtain the system

$$\begin{cases} A + B = 0 \\ -A - 3B = -4 \end{cases},$$

which has solution $A = -2$ and $B = 2$. Thus,

$$y = -2\mathrm{e}^{-x} + 2\mathrm{e}^{-3x}.$$

Part (b). Consider the characteristic equation of the DE:

$$m^2 + 6m + 9 = (m+3)^2 = 0.$$

We have a repeated root $m = -3$, whence

$$y = (A + Bx)\mathrm{e}^{-3x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -3(A + Bx)\mathrm{e}^{-3x} + B\mathrm{e}^{-3x}.$$

Using the given conditions, we obtain the system

$$\begin{cases} A & = 1 \\ -3A + B = 1 \end{cases},$$

which has solution $A = 1$ and $B = 4$. Thus,

$$y = (1 + 4x)\mathrm{e}^{-3x}.$$

Part (c). Consider the characteristic equation of the DE:

$$m^2 + \sqrt{3}m + 3 = 0.$$

Solving, we get

$$m = \frac{-\sqrt{3}}{2} \pm \frac{3}{2}\mathrm{i},$$

whence

$$y = \mathrm{e}^{-\frac{\sqrt{3}}{2}x}\left(A\cos\frac{3}{2}x + B\sin\frac{3}{2}x\right).$$

Differentiating, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-\frac{\sqrt{3}}{2}x}\left[\left(-\frac{\sqrt{3}}{2}A + \frac{3}{2}B\right)\cos\frac{3}{2}x + \left(-\frac{\sqrt{3}}{2}B - \frac{3}{2}A\right)\sin\frac{3}{2}x\right].$$

Using the given conditions, we obtain the system

$$\begin{cases} A & = 0 \\ -\frac{\sqrt{3}}{2}A + \frac{3}{2}B = -4 \end{cases},$$

whence $A = 0$ and $B = -8/3$. Thus,

$$y = -\frac{8}{3}\mathrm{e}^{-\frac{\sqrt{3}}{2}x}\sin\frac{3}{2}x.$$

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Problem 3. Find the general solution of

(a) $2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = 10x^2 + 1$,

(b) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 22\mathrm{e}^{4x}$,

(c) $\frac{\mathrm{d}^2s}{\mathrm{d}t^2} - 2\frac{\mathrm{d}s}{\mathrm{d}t} + s = 4\mathrm{e}^t$,

(d) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 16x = 3\cos 4t$.

Solution.

Part (a). Consider the characteristic equation of the DE:

$$2m^2 - 3m - 5 = (2m - 5)(m + 1) = 0.$$

The roots are $m = 5/2$ and $m = -1$, whence the complementary function is

$$y_c = A\mathrm{e}^{\frac{5}{2}x} + B\mathrm{e}^{-x}.$$

For the particular solution, we try

$$y_p = Cx^2 + Dx + E.$$

Note that

$$y_p' = 2Cx + D \quad \text{and} \quad y_p'' = 2C.$$

Substituting this into the DE,

$$2(2C) + -3(2Cx + D) - 5\left(Cx^2 + Dx + E\right) = 10x^2 + 1.$$

Comparing coefficients, we get the system

$$\begin{cases} -5C & = 10 \\ -6C - 5D & = 0 \\ 4C - 3D - 5E & = 1 \end{cases},$$

which has solution $C = -2$, $D = 12/5$ and $E = -81/25$. The general solution is thus

$$y = y_c + y_p = A\mathrm{e}^{\frac{5}{2}x} + B\mathrm{e}^{-x} - 2x^2 + \frac{12}{5}x - \frac{81}{25}.$$

Part (b). Consider the characteristic equation of the DE:

$$m^2 - 2m + 3 = 0 \implies m = 1 \pm \sqrt{2}\mathrm{i}.$$

The complementary function is hence

$$y_c = \mathrm{e}^x \left(A\cos\sqrt{2}x + B\sin\sqrt{2}x\right).$$

For the particular solution, we try

$$y_p = C\mathrm{e}^{4x}.$$

Note that

$$y_p' = 4C\mathrm{e}^{4x} \quad \text{and} \quad y_p'' = 16C\mathrm{e}^{4x}.$$

Substituting this into the DE,

$$16C\mathrm{e}^{4x} - 2\left(4C\mathrm{e}^{4x}\right) + 3C\mathrm{e}^{4x}22\mathrm{e}^{4x} \implies C = 2.$$

The general solution is thus

$$y = y_c + y_p = \mathrm{e}^x \left(A\cos\sqrt{2}x + B\sin\sqrt{2}x\right) + 2\mathrm{e}^{4x}.$$

Part (c). Consider the characteristic equation of the DE:

$$m^2 - 2m + 1 = (m-1)^2 = 0.$$

The only root is $m = 1$, whence the complementary function is

$$s_c = (A + Bt)\mathrm{e}^t.$$

For the particular solution, we try

$$s_p = Ct^2\mathrm{e}^t.$$

Note that

$$s_p' = C\mathrm{e}^t\left(t^2 + 2t\right) \quad \text{and} \quad s_p'' = C\mathrm{e}^t\left(t^2 + 4t + 2\right).$$

Substituting this into the DE,

$$Ce^t\left(t^2+4t+2\right)-2Ce^t\left(t^2+2t\right)+Ce^t\left(t^2\right)=4e^t \implies C=2.$$

The general solution is thus

$$s=s_c+s_p=\left(A+Bt+2t^2\right)e^t.$$

Part (d). Consider the characteristic equation of the DE:

$$m^2+16m=0 \implies m=\pm 4i.$$

The complementary function is hence

$$x_c=A\cos 4t+B\sin 4t.$$

For the particular solution, we try

$$x_p=t\left(C\cos 4t+D\sin 4t\right).$$

Note that

$$x_p'=4t\left(-C\sin 4t+D\cos 4t\right)+\left(C\cos 4t+D\sin 4t\right)$$

and

$$x_p''=16t\left(-C\cos 4t-D\sin 4t\right)+8\left(-C\sin 4t+D\cos 4t\right).$$

Substituting this into the DE,

$$\begin{aligned}16t\left(-C\cos 4t-D\sin 4t\right)&+8\left(-C\sin 4t+D\cos 4t\right)\\&+16t\left(C\cos 4t+D\sin 4t\right)=3\cos 4\theta.\end{aligned}$$

Simplifying, we get

$$-8C\sin 4t+8D\cos 4t=3\cos 4t,$$

whence $C=0$ and $D=3/8$. Thus, the general solution is

$$x=A\cos 4t+B\sin 4t+\frac{3}{8}t\sin 4t.$$

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Problem 4.

(a) Find the general solution of the differential equation $\frac{d^2y}{dx^2}-4y=10e^{3x}$.

(b) Hence, find the solution for which $y=-2$ and $\frac{dy}{dx}=-6$ when $x=0$.

Solution.

Part (a). Observe that the characteristic equation of the DE is $m^2-4=0$, whence the roots are $m=\pm 2$. Hence, the complementary function is

$$y_c=Ae^{2x}+Be^{-2x}.$$

For the particular solution, we try $y_p=Ce^{3x}$. Note that

$$y_p'=3Ce^{3x} \quad \text{and} \quad y_p''=9Ce^{3x}.$$

Substituting this into the DE, get

$$9Ce^{3x} - 4Ce^{3x} = 10e^{3x} \implies C = 2,$$

whence the general solution is

$$y = y_c + y_p = Ae^{2x} + Be^{-2x} + 2e^{3x}.$$

Part (b). Note that

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x} + 6e^{3x}.$$

The given conditions thus give the system

$$\begin{cases} A + B + 2 = -2 \\ 2A - 2B + 6 = -6 \end{cases},$$

whence $A = -5$ and $B = 1$. Hence,

$$y = -5e^{2x} + e^{-2x} + 2e^{3x}.$$

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Problem 5.

(a) Find the general solution of the differential equation $\frac{d^2y}{dx^2} = \sin x$. Find the particular solution that passes through the points $(0, \sqrt{2})$ and $\left(\frac{\pi}{4}, -\sqrt{2}\right)$.

(b) Find the general solution of the differential equation

(i) $\frac{d^2y}{dx^2} = 16 - 9x^2$,

(ii) $\left(9 - x^2\right)^2 \frac{d^2y}{dx^2} - x = 0$,

giving your answer in the form $y = f(x)$.

Solution.

Part (a). Integrating the DE with respect to x,

$$\frac{dy}{dx} = \int \sin x \, dx = -\cos x + A.$$

Integrating once more,

$$y = \int (-\cos x + A) \, dx = -\sin x + Ax + B.$$

At $(0, \sqrt{2})$, we have $B = \sqrt{2}$. At $\left(\frac{\pi}{4}, -\sqrt{2}\right)$, we have

$$-\frac{\sqrt{2}}{2} + A\left(\frac{\pi}{4}\right) + B = -\sqrt{2} \implies A = -\frac{6\sqrt{2}}{\pi}.$$

Thus, the particular solution is

$$y = -\sin x - \frac{6\sqrt{2}}{\pi}x + \sqrt{2}.$$

Part (b).

Part (b)(i). Integrating with respect to x,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \int \left(16 - 9x^2\right)\,\mathrm{d}x = 16x - 3x^2 + A.$$

Integrating once more,

$$y = \int \left(16x - 3x^2 + A\right)\,\mathrm{d}x = 8x^2 - \frac{3}{4}x^4 + Ax + B.$$

Part (b)(ii). Rewriting, we get

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{x}{\left(9 - x^2\right)^2}.$$

Integrating with respect to x,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \int \frac{x}{\left(9 - x^2\right)^2}\,\mathrm{d}x.$$

Using the substitution $x = 3\sin\theta$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \int \frac{3\sin\theta}{81\cos^4\theta} 3\cos\theta\,\mathrm{d}\theta = \frac{1}{9}\int \tan\theta\sec^2\theta\,\mathrm{d}\theta$$
$$= \frac{1}{9}\left(\frac{\tan^2\theta}{2}\right) + C = \frac{1}{18}\left(\frac{(x/3)^2}{1 - (x/3)^2}\right) + C = \frac{1}{18}\left(\frac{x^2}{9 - x^2}\right) + C.$$

Integrating once more,

$$y = \int \left[\frac{1}{18}\left(\frac{x^2}{9 - x^2}\right) + C\right]\mathrm{d}x = \int \left[\frac{1}{18}\left(\frac{9}{9 - x^2} - 1\right) + C\right]\mathrm{d}x$$
$$= \frac{1}{18}\left[\frac{3}{2}\ln\left|\frac{3 + x}{3 - x}\right| - x\right] + Cx + D = \frac{1}{12}\ln\left|\frac{3 + x}{3 - x}\right| + Ex + D,$$

where $E = -1/18 + C$.

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Problem 6.

(a) Find the particular solution of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 16x = 0$, given that $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -8$ when $t = 0$.

(b) By writing the particular solution as $R\cos(4t + \alpha)$, find the first positive value of t for which x is maximum.

Solution.

Part (a). Note that the characteristic equation of the DE is $m^2 + 16 = 0$, whence the roots are $m = \pm 4\mathrm{i}$. Hence,

$$x = A\cos 4t + B\sin 4\theta.$$

Differentiating with respect to t, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4A\sin 4t + 4B\cos 4t.$$

When $x = 3$ and $t = 0$, we have $A = 3$. When $\frac{\mathrm{d}x}{\mathrm{d}t} = -8$ and $t = 0$, we have $B = -2$. Thus,

$$x = 3\cos 4t - 2\sin 4t.$$

Part (b). We have

$$x = 3\cos 4t - 2\sin 4t = \sqrt{3^2 + 2^2}\cos\left(4t - \arctan\frac{-2}{3}\right) = \sqrt{13}\cos(4t + 0.58800).$$

x attains a maximum whenever $\cos(4t + 0.58800) = 1$. Thus,

$$4t + 0.58800 = 2\pi n \implies t = \frac{2\pi n - 0.58800}{4},$$

where n is an integer. The first positive value of t is hence

$$t = \frac{2\pi - 0.58800}{4} = 1.42 \text{ (3 s.f.)},$$

which occurs when $n = 1$.

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Problem 7. Using the substitution $x = \mathrm{e}^u$, find the general solution of

(a) $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$,

(b) $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$.

Solution. Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}.$$

Since $u = \ln x$, we have $\mathrm{d}u/\mathrm{d}x = 1/x$ and $\mathrm{d}^2u/\mathrm{d}x^2 = -1/x^2$. Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}u} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - \frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}u}.$$

Part (a). Substituting the above expressions into the DE, we have

$$x^2\left(\frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - \frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}u}\right) + 2x\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}u}\right) - 2y = 0.$$

Simplifying, we get

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} + \frac{\mathrm{d}y}{\mathrm{d}u} - 2y = 0.$$

The characteristic equation $m^2 + m - 2 = (m+2)(m-1) = 0$ has roots $m = -2$ and $m = 1$. Thus,

$$y = A\mathrm{e}^{-2u} + B\mathrm{e}^{u} = Ax^{-2} + Bx.$$

Part (b). Substituting the above expressions into the DE, we have

$$x^2\left(\frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - \frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}u}\right) - 5x\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}u}\right) - 6y = 0.$$

Simplifying, we get

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - 6\frac{\mathrm{d}y}{\mathrm{d}u} - 6y = 0.$$

The characteristic equation $m^2 - 6m - 6 = 0$ has roots $m = 3 \pm \sqrt{15}$. Thus,

$$y = A\mathrm{e}^{(3+\sqrt{15})u} + B\mathrm{e}^{(3-\sqrt{15})u} = Ax^{3+\sqrt{15}} + Bx^{3-\sqrt{15}}.$$

Problem 8. Show, by means of the substitution $y = x^{-4}z$, that the differential equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(4x^2 + 8x\right) \frac{\mathrm{d}y}{\mathrm{d}x} + \left(3x^2 + 16x + 12\right) y = 0$$

can be reduced to the form

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + a \frac{\mathrm{d}z}{\mathrm{d}x} + bz = 0,$$

where a and b are constants to be determined. Hence, find the general solution of the above differential equation.

Solution. Note that $z = x^4 y$. Differentiating with respect to x,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = x^4 \frac{\mathrm{d}y}{\mathrm{d}x} + 4yx^3.$$

Differentiating with respect to x again,

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = x^4 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 8x^3 \frac{\mathrm{d}y}{\mathrm{d}x} + 12yx^2.$$

Consider the DE in question. Multiplying through by x^2,

$$x^4 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(4x^4 + 8x^3\right) \frac{\mathrm{d}y}{\mathrm{d}x} + \left(3x^4 + 16x^3 + 12x^2\right) y = 0.$$

Now observe that we can split the LHS as

$$\left(x^4 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 8x^3 \frac{\mathrm{d}y}{\mathrm{d}x} + 12yx^2\right) + 4\left(x^4 \frac{\mathrm{d}y}{\mathrm{d}x} + 4yx^3\right) + 3x^4 y.$$

Thus,

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + 4 \frac{\mathrm{d}z}{\mathrm{d}x} + 3z = 0.$$

Hence, $a = 4$ and $b = 3$.

Note that the characteristic equation of this new DE is $m^2 + 4m + 3 = (m+3)(m+1) = 0$. Thus, the roots are $m = -3$ and $m = -1$, whence

$$z = A\mathrm{e}^{-3x} + B\mathrm{e}^{-x} \implies y = x^{-4}\left(A\mathrm{e}^{-3x} + B\mathrm{e}^{-x}\right).$$

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Problem 9. By letting $x = \sqrt{t}$, show that the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(2x - \frac{1}{x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} + 24x^2 = 0$$

where $x > 0$, may be transformed to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a \frac{\mathrm{d}y}{\mathrm{d}t} + b = 0,$$

where a and b are constants to be determined. Hence, find the general solution of y in terms of x.

Solution. Note that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$$
and
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}x^2}.$$
Since $t = x^2$, we have $\mathrm{d}t/\mathrm{d}x = 2x$ and $\mathrm{d}^2u/\mathrm{d}x^2 = 2$. Thus,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\frac{\mathrm{d}y}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4x^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}u}.$$
Substituting this into the given DE,
$$\left(4x^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}u}\right) + \left(2x - \frac{1}{x}\right)\left(2x\frac{\mathrm{d}y}{\mathrm{d}t}\right) + 24x^2 = 0.$$
Simplifying, we get
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 6 = 0,$$
whence $a = 1$ and $b = 6$.

Rewriting,
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} = -6.$$
Integrating with respect to t, we get
$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = -6t + C.$$
Multiplying through by e^t yields
$$\mathrm{e}^t\frac{\mathrm{d}y}{\mathrm{d}t} + \mathrm{e}^t y = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t y\right) = \mathrm{e}^t\left(-6t + C\right).$$
Integrating with respect to t,
$$\mathrm{e}^t y = \int \mathrm{e}^t\left(-6t + C\right)\,\mathrm{d}t = -6\left(t\mathrm{e}^t - \mathrm{e}^t\right) + C\mathrm{e}^t = \mathrm{e}^t\left(-6t + A\right) + B.$$
Thus,
$$y = -6t + A + B\mathrm{e}^{-t} = -6x^2 + A + B\mathrm{e}^{-x^2}.$$

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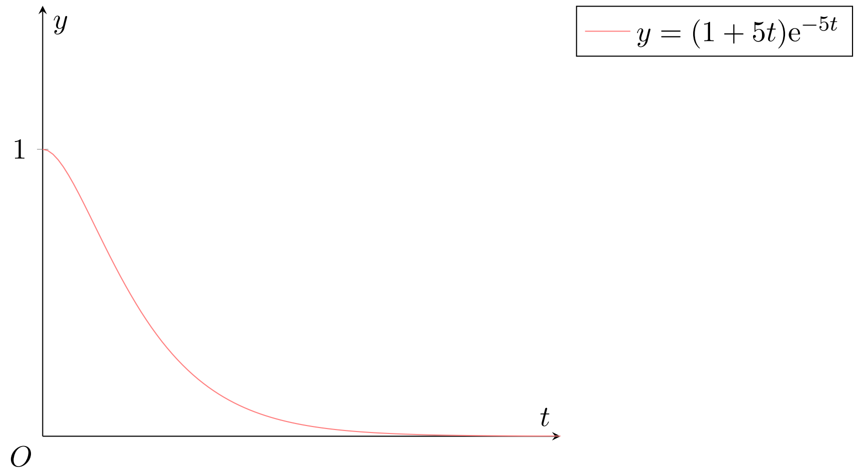
Problem 10. A damped vibrating spring system is described by the differential equation
$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -ky - \lambda\frac{\mathrm{d}y}{\mathrm{d}t},$$
where m, k and λ are positive constants. The variable y represents the displacement of the object from equilibrium position in centimetres, and t is time measured in seconds. Given that $m = 1$, $k = 25$ and $\lambda = 10$, and the object was initially released from rest at $y = 1$, find the equation of motion and sketch its graph. Briefly explain if this motion is suitable to be used to close a door.

Solution. We have
$$\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=0.$$
The characteristic equation $r^2+10r+25=(r+5)^2=0$ has a single root $r=-5$. Thus,
$$y=(A+Bt)\,e^{-5t}.$$
Since $y=1$ when $t=0$, we get $A=1$. Hence,
$$y=(1+Bt)\,e^{-5t} \implies \frac{dy}{dt}=Be^{-5t}-5\,(1+Bt)\,e^{-5t}.$$
Since the object is initially at rest, $\frac{dy}{dt}=0$ when $t=0$. This gives $B=5$. Thus,
$$y=(1+5t)\,e^{-5t}.$$



Since the object does not oscillate (y does not change sign) and y approaches $y=0$ quite quickly, the motion is suitable to be used to close a door.

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Problem 11. The motion of the tip of a tuning fork can be modelled by the differential equation
$$m\frac{d^2x}{dt^2}+k\frac{dx}{dt}+m\omega^2x=0,$$
where x is the displacement of the tip from its equilibrium position at time t and m, k and ω are positive constants. It is known that k is so small that k^2 can be ignored as k models the slight damping due to the resistance of the air. It is given that the tip of the fork is initially in its equilibrium position and moving with speed v in the positive x-direction.

(a) Solve the differential equation.

The amplitude of a vibration is the maximum displacement of the tip from its equilibrium position and one period of a vibration is the time interval between the occurrences of two consecutive amplitudes.

(b) Comment on the period of the vibrations over time and show that the amplitude of successive vibrations follows a geometric progression.

(c) Given that k is no longer small and $k^2>4m^2\omega^2$, describe the behaviour of x as time progresses and sketch a possible graph of x against t. Justify your answer.

Solution.

Part (a). The characteristic equation of the DE is given by $mr^2 + kr + m\omega^2 = 0$. Let the roots be r_1 and r_2. We have

$$r_{1,2} = \frac{-k \pm \sqrt{k^2 - 4m^2\omega^2}}{2m}.$$

Since k^2 can be ignored,

$$r_{1,2} = \frac{-k \pm \sqrt{-4m^2\omega^2}}{2m} = \frac{-k \pm 2m\omega\mathrm{i}}{2m} = -\frac{k}{2m} \pm \omega\mathrm{i}.$$

The general solution is thus given by

$$x = \mathrm{e}^{-\frac{k}{2m}t}\left(A\cos\omega t + B\sin\omega t\right).$$

Since the object is initially at equilibrium, we have $x = 0$ at $t = 0$. There is hence no contribution from the cosine term, i.e. $A = 0$. Thus,

$$x = B\mathrm{e}^{-\frac{k}{2m}t}\sin\omega t.$$

Differentiating with respect to t,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = B\mathrm{e}^{-\frac{k}{2m}t}\left(\omega\cos\omega t - \frac{k}{2m}\sin\omega t\right).$$

Since the object was initially released with speed $v > 0$, we have $\mathrm{d}x/\mathrm{d}t = v$ at $t = 0$. Hence,

$$B\omega = v \implies B = \frac{v}{\omega}.$$

We hence obtain the solution

$$x = \frac{v}{\omega}\mathrm{e}^{-\frac{k}{2m}t}\sin\omega t.$$

Part (b). Let x_n be the (signed) amplitude of the nth vibration, and let t_n be the corresponding time, where $n \in \mathbb{N}$.

To find t_n, we consider the stationary points of x:

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=t_n} = \frac{v}{\omega}\mathrm{e}^{-\frac{k}{2m}t_n}\left(\omega\cos\omega t_n - \frac{k}{2m}\sin\omega t_n\right) = 0 \implies \tan\omega t_n = \frac{2m\omega}{k}.$$

Since tangent has period π,

$$t_n = \frac{1}{\omega}\left(\arctan\frac{2m\omega}{k} + \pi n\right).$$

Quite clearly, x has a constant period $2\pi/\omega$.

We now find x_n. Evaluating x at t_n,

$$x_n = \frac{v}{\omega}\exp\left(-\frac{k}{2m\omega}\left[\arctan\frac{2m\omega}{k} + \pi n\right]\right)\sin\left(\arctan\frac{2m\omega}{k} + \pi n\right).$$

Note that $\sin(X + \pi n) = (-1)^n \sin X$. Hence,

$$x_n = \left[-\exp\left(-\frac{k\pi}{2m\omega}\right)\right]^n \underbrace{\left[\frac{v}{\omega}\exp\left(-\frac{k}{2m\omega}\arctan\frac{2m\omega}{k}\right)\sin\left(\arctan\frac{2m\omega}{k}\right)\right]}_{\text{constant}}.$$

Hence,

$$\frac{|x_{n+1}|}{|x_n|} = \mathrm{e}^{-\frac{k\pi}{2m\omega}},$$

whence the amplitudes $\{|x_n|\}$ are in geometric progression with common ratio $\mathrm{e}^{-\frac{k\pi}{2m\omega}}$.

Part (c). Recall that the roots of the characteristic equation are given by

$$r_{1,2} = \frac{-k \pm \sqrt{k^2 - 4m^2\omega^2}}{2m}.$$

If $k^2 > 4m^2\omega^2$, then the roots are real and distinct, whence x has general solution

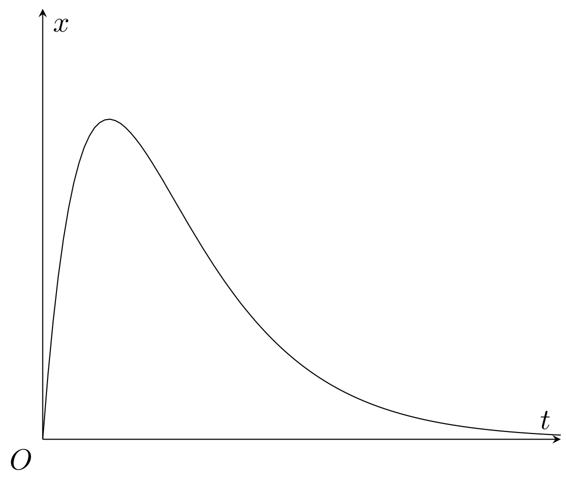
$$x = A\mathrm{e}^{r_1 t} + B\mathrm{e}^{r_2 t}.$$

Since $x = 0$ at $t = 0$, we obtain $A + B = 0$. Thus,

$$x = A\left(\mathrm{e}^{r_1 t} - \mathrm{e}^{r_2 t}\right).$$

Note that both roots are negative (since $\sqrt{k^2 - X^2} < \sqrt{k^2} = k$). Hence, as t tends to infinity, $\mathrm{e}^{r_1 t} - \mathrm{e}^{r_2 t}$ (and by extension x) tends to 0.

A possible graph of x is



Self-Practice B16

Problem 1. Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 24\mathrm{e}^{-2x}.$$

Show that when $x \to \infty$, the solution can be expressed as a single trigonometric expression.

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Problem 2.

(a) By using the substitution $z = xy$, show that the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2-4x)\frac{\mathrm{d}y}{\mathrm{d}x} + 4y(x-1) = 0$$

can be simplified into the differential equation

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 4\frac{\mathrm{d}z}{\mathrm{d}x} + 4z = 0.$$

Hence, find the general solution for y in terms of x.

(b) Using a machine, a particle is accelerated from rest such that at a time t seconds after the machine is turned on, its displacement s from its initial starting point is modelled by the following differential equation:

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} - 4\frac{\mathrm{d}s}{\mathrm{d}t} + 4s = \cos t.$$

Find s in terms of t. Hence, find the amount of time required for the particle's speed to exceed the speed of sound ($340\ \mathrm{ms}^{-1}$), giving your answer to the nearest hundredth of a second.

* * * * *

Problem 3. Given that $\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 9t + y$, $\frac{\mathrm{d}y}{\mathrm{d}t} = y - 4x + 9$, by eliminating the variable y, show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 9t.$$

Find the general solution of x in terms of t and hence obtain the general solution of y in terms of t. Find the ratio $x : y$ when $t \to -\infty$.

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Problem 4. A teacher gave his class the following differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + 16x^3y = 8x^3 \qquad (1)$$

and asked them to find the solution. One of the students, Adrian, who had come across (1) before recalled that the solution is $y = \cos^2(x^2)$.

(a) Show that $y = \cos^2(x^2)$ satisfies (1).

Another student, Bobby, decided to use the substitution $t = x^2$ to solve (1).

(b) Show that by using the substitution, (1) can be transformed into the equation

$$\frac{d^2y}{dt^2} + ay = b,$$

where a and b are constants to be determined.

(c) Find the general solution of (1).

(d) Show that Adrian's solution can be obtained from Bobby's solution be choosing suitable values of the arbitrary constants in the solution.

The teacher found out later that there was a typing error in (1). The differential equation should be

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 16x^3y = 8x^3. \tag{2}$$

Deduce the general solution of (2).

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Problem 5. Use the substitution $x = \sec\theta$, where $0 < \theta < \pi/2$, to show that the differential equation

$$\left(x^3 - x\right)\frac{d^2y}{dx^2} + \left(2x^2 - 1\right)\frac{dy}{dx} + \frac{ky}{x} = \frac{2}{x^3},$$

where k is a positive integer, can be reduced to

$$\frac{d^2y}{d\theta^2} + ky = 2\cos^2\theta.$$

Hence, obtain the general solution for the differential equation in x and y for $k \neq 4$ in the form

$$y = A\cos\sqrt{k}\,(\operatorname{arcsec} x) + B\sin\sqrt{k}\,(\operatorname{arcsec} k) + f(x),$$

where A and B are arbitrary constants and $f(x)$ is a function of x to be determined.

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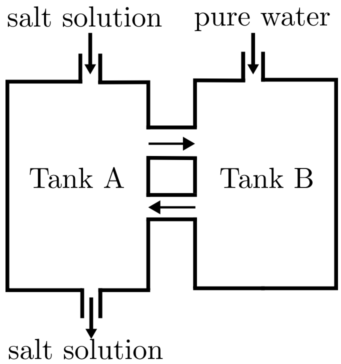
Problem 6. Two 50-litre tanks, Tank A and Tank B (as shown in the diagram below), containing salt solution are connected by two horizontal pipes. Both tanks have inlets and outlets where salt solution flows in and out of the tanks. The rates of flow in the inlets, outlets and pipes are managed in such a way that both tanks will be full at all times.

Tank A is receiving salt solution at a concentration of 1 gram per litre at a rate of 1 litre per minute and Tank B is receiving pure water at the same rate. Salt solution is flowing out from Tank A from the bottom outlet at a rate of 2 litres per minute. Salt solution flows from Tank A to Tank B through one of the horizontal pipes at a rate of 2 litres per minute and flows in the reverse direction through the other horizontal pipe at a rate of 3 litres per minute.



(a) Suppose that at time t minutes, the amount of salt in Tank A and Tank B are x grams and y grams respectively. Show that

$$\frac{dx}{dt} = -\frac{2}{25}x + \frac{3}{50}y + 1 \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{25}x - \frac{3}{50}y.$$

(b) Prove that

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = r,$$

where p, q and r are constants to be determined.

(c) Find the general solution of the differential equation in (b).

(d) By expressing x in terms of t, find the ratio of the amount of salt in Tank A to the amount of salt in Tank B in the long run.

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Problem 7 (🌶). A model for the vibrations of a wine glass when struck by an external force is

$$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + \omega^2 x = 0,$$

where λ is a constant due to the external force, ω is a constant of the wine glass, $2\omega > \lambda > 0$, and x is the deformation of the glass.

(a) Find the general solution of the model in the form

$$x = e^{At}\left(c_1 \cos Bt + c_2 \sin Bt\right),$$

where A and B are expressions to be determined in terms of λ and ω, and c_1 and c_2 are arbitrary constants.

Suppose that the wine glass vibrates at 440 Hz when struck, that is, the period of the oscillation is $1/440$ second.

(b) Show that $\sqrt{4\omega^2 - \lambda^2} = 1760\pi$.

If it takes about 2 second for the sound to die away, and this happens when the original vibrations have reduced to one hundredth of their initial amplitude,

(c) show that $\lambda = \ln 100$ and hence find ω, correct to three significant figures.

A pure tone at 440 Hz is produced at D decibels and aimed at the glass, forcing it to vibrate at its natural frequency. The glass will shatter if the amplitude of the pure tone is approximately 1. The vibrations are now modelled by

$$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + \omega^2 x = \frac{10^{D/10-8}}{3}\cos(880\pi t).$$

(d) Determine how loud the sound show be, i.e. how large D should be, in order to shatter the glass.

Part X

H3 Mathematics

Mathematical Proofs and Reasoning

An Introduction to the Mathematical Vernacular

Problem 1. Prove that the sum of even and even is even.

Proof. Let a and b be even. By definition, there exists $a', b' \in \mathbb{Z}$ such that $a = 2a'$ and $b = 2b'$. Thus,

$$a + b = 2a' + 2b' = 2\left(a' + b'\right) = 2c,$$

where $c = a' + b'$. Since c is an integer, by definition, $a + b$ is even. Hence, the sum of even and even is even. □

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Problem 2. Prove that the sum of even and odd is odd.

Proof. Let a be even and let b be odd. By definition, there exists $a', b' \in \mathbb{Z}$ such that $a = 2a'$ and $b = 2b' + 1$. Thus,

$$a + b = 2a' + \left(2b' + 1\right) = 2\left(a' + b'\right) + 1 = 2c + 1,$$

where $c = a' + b'$. Since c is an integer, by definition, $a + b$ is odd. Hence, the sum of even and odd is odd. □

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Problem 3. Prove that the sum of odd and odd is even.

Proof. Let a and b be odd. By definition, there exists $a', b' \in \mathbb{Z}$ such that $a = 2a' + 1$ and $b = 2b' + 1$. Thus,

$$a + b = 2\left(a' + 1\right) + 2\left(b' + 1\right) = 2\left(a' + b' + 2\right) = 2c,$$

where $c = a' + b' + 2$. Since c is an integer, by definition, $a + b$ is even. Hence, the sum of odd and odd is even. □

An Introduction to Proofs

Problem 1. Let m and N be positive integers. Prove that $\sqrt[m]{N}$ is either an integer or an irrational.

Proof. Let $x = \sqrt[m]{N}$. Let A be the nearest integer to x.

Consider $(x-A)^n$. By the binomial theorem,

$$(x-A)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (-A)^{n-k}.$$

Since $x^m = N \in \mathbb{Z}$, the above n-degree polynomial reduces to an $m-1$ degree polynomial with integer coefficients, i.e.

$$(x-A)^n = \sum_{k=0}^{m-1} c_k x^k, \tag{1}$$

where $\{c_k\}$ are integers.

Now, suppose $x \in \mathbb{Q}$. Then we can write $x = p/q$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. Substituting this into (1), we get

$$(x-A)^n = \sum_{k=0}^{m-1} c_k \left(\frac{p}{q}\right)^k = \sum_{k=0}^{m-1} \frac{c_k p^k}{q^k}.$$

By combining all terms into a single fraction, we can write

$$(x-A)^n = \frac{l}{p^{m-1}},$$

where l is an integer. Thus, the only possible values that $(x-A)^n$ can take on are

$$\ldots, \frac{-2}{p^{m-1}}, \frac{-1}{p^{m-1}}, 0, \frac{1}{p^{m-1}}, \frac{2}{p^{m-1}}, \ldots.$$

Observe that $1/p^{m-1}$ is constant with respect to n, i.e. p and m do not depend on n. Since $|x-A| < 1$, for arbitrarily large n, we can make $(x-A)^n$ as close to 0 as we wish. In other words, we can always find an n large enough such that

$$|(x-A)^n| < \frac{1}{p^{m-1}}.$$

Thus, $(x-A)^n$ must be 0, whence $x = A \in \mathbb{Z}$. Hence, if x is rational, it must necessarily be an integer. This completes the proof. □

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Problem 2. Prove that π is irrational.

Proof. Seeking a contradiction, suppose $\pi = p/q$, where $p, q \in \mathbb{Z}$ with $q \neq 0$.

Consider the function $\sin x$. It is well known that $\sin x$

- is non-negative for all $x \in [0, \pi]$, with equality only when $x = 0$ or $x = \pi$; and
- attains a maximum at $\pi/2$.

Now consider the $2n$th degree polynomial $f(x) = x^n(p - qx)^n$. Clearly, $f(x)$ is non-negative on $[0, \pi]$ and has roots only at $x = 0$ and $x = \pi$. Additionally, $f(x)$ attains a maximum of $(\pi/2)^{2n}$ at $x = \pi/2$. Thus, $f(x)$ also satisfies the above two properties.

Consider now the integral

$$I = \int_0^\pi f(x) \sin x \, \mathrm{d}x.$$

Since both $f(x)$ and $\sin x$ are non-negative on $[0, \pi]$, it follows that I must also be non-negative on $[0, \pi]$. Additionally, since $f(x) \sin x \not\equiv 0$ on $[0, \pi]$, we have the strict lower bound

$$0 < I.$$

We can also bound I from above:

$$I = \int_0^\pi f(x) \sin x \, \mathrm{d}x \leq \int_0^\pi \left(\frac{\pi}{2}\right)^{2n} \mathrm{d}x = \frac{\pi^{2n+1}}{2^{2n}} \leq p^{2n+1}.$$

Putting both inequalities together,

$$0 < I \leq p^{2n+1}. \tag{1}$$

We now evaluate I. Repeatedly integrating by parts, we get

$$I = \sum_{k=0}^{2n+1} \left[f^{(k)}(x) \sin^{(-k-1)}(x) \right]_0^\pi = \sum_{k=0}^{2n+1} \left[f^{(k)}(\pi) \sin^{(-k-1)}(\pi) - f^{(k)}(0) \sin^{(-k-1)}(0) \right].$$

Note that the sum ends at $k = 2n + 1$ since $f^{(k)} = 0$ for $k \geq 2n + 2$. Also observe that

$$\sin^{(-k-1)}(x) = \begin{cases} -\cos x, & k \equiv 0 \pmod 4 \\ -\sin x, & k \equiv 1 \pmod 4 \\ \cos x, & k \equiv 2 \pmod 4 \\ \sin x, & k \equiv 3 \pmod 4 \end{cases}.$$

The odd k terms hence vanish. We are thus left with

$$I = \sum_{k=0}^{n} (-1)^{k+1} \left[f^{(2k)}(\pi) + f^{(2k)}(0) \right].$$

We now consider $f^{(2k)}(x)$. Firstly, notice that $f(x) = f(\pi - x)$. Hence, by differentiating this repeatedly, we get $f^{(2k)}(0) = f^{(2k)}(\pi)$, so

$$I = 2 \sum_{k=0}^{n} (-1)^{k+1} f^{(2k)}(0).$$

Now, observe that when expanded, $f(x)$ is of the form

$$f(x) = \sum_{i=n}^{2n} a_i x^i,$$

where $\{a_i\}$ are integers. Repeatedly differentiating this yields

$$f^{(k)}(x) = x^n bppx - q^n = \sum_{i=n}^{2n} a_i (i)(i-1) \ldots (i-k+1) x^{i-k}.$$

Thus,

$$f^{(k)}(0) = \begin{cases} 0, & 0 \le k < n \\ a_k k!, & n \le k \le 2n \end{cases}.$$

Thus, $f^{(k)}(0)$ is divisible by $k!$ and by extension $n!$ too (since $n \le k$). Hence, I is divisible by $n!$, i.e. $I = Cn!$ for some integer C. From Inequality (1), we have

$$0 < Cn! < p^{2n+1}.$$

However, $n!$ grows much faster than p^{2n+1}. Thus, for sufficiently large n, the inequality does not hold, a contradiction. Hence, π must be irrational. □

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Problem 3. Prove that e is irrational.

Proof. By definition,

$$\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Seeking a contradiction, suppose e is rational, i.e. $\mathrm{e} = a/b$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. Define x as

$$x = b!\left(\mathrm{e} - \sum_{n=0}^{b} \frac{1}{n!}\right). \tag{1}$$

Replacing e with a/b, we get

$$x = b!\left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}\right) = a(b-1)! - \sum_{n=0}^{b} \frac{b!}{n!}.$$

Since $b!/n!$ is an integer for $0 \le n \le b$, it follows that x is also an integer.

Using the definition of e, we can rewrite (1) as

$$x = b!\left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{b} \frac{1}{n!}\right) = \sum_{n=b+1}^{\infty} \frac{b!}{n!}.$$

It follows that $x > 0$. Now, observe that

$$\begin{aligned} x &= \sum_{n=b+1}^{\infty} \frac{b!}{n!} \\ &= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \dots \\ &< \frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \dots \\ &= \frac{1}{b+1}\left(\frac{1}{1 - \frac{1}{b+1}}\right) = \frac{1}{b} \le 1. \end{aligned}$$

Hence, $0 < x < 1$ but $x \in \mathbb{Z}$, a contradiction. Thus, e must be irrational. □

Problem 4. Let a and b be relatively prime. Prove that $\log_a b$ is irrational.

Proof. Seeking a contradiction, suppose $\log_a b$ is rational. Then $\log_a b = m/n$, where $m, n \in \mathbb{Z}$ and $n \neq 0$. Note that $m, n > 0$ since $a, b > 1$. Clearly, we have

$$b = a^{m/n} \implies b^n = a^m.$$

This implies that the integer $k = b^n = a^m$ has two distinct prime factorizations, which is a clear contradiction of the Fundamental Theorem of Algebra. Hence, $\log_a b$ must be irrational. $\square$

Problem Set 1

Problem 1. Determine whether each of the following statements is true or false. Give a direct proof if it is true, and give a counter-example if it is false.

(a) The set of prime numbers is closed under addition.

(b) The set of positive rational numbers is closed under division.

Solution. (a) is false (since 3 and 5 are prime but their sum, 8, is not), while (b) is true.

Proof of (b). Let a/b and c/d be positive rational numbers, i.e. a, b, c and d are positive integers. Then

$$\frac{a/b}{c/d} = \frac{ad}{bc}.$$

Since both ad and bc are positive integers, it follows that ad/bc is a positive rational number. Hence, the set of positive rational numbers is closed under division. □

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Problem 2. Let a, b and c be non-zero integers. Use the definition of divisibility and write down a direct proof for each of the following statements. (Indicate every step clearly).

(a) If a divides b, then ac divides bc.

(b) If a divides b and b divides a, then $a = \pm b$.

Solution.

Proof of (a). Since a divides b, we have

$$b = ka$$

for some integer k. Multiplying this equation through by c,

$$bc = k(ac).$$

Hence, by the definition of divisibility, ac divides bc. □

Proof of (b). Since a divides b, we have

$$b = k_1 a$$

for some integer k_1. Similarly, since b divides a, we have

$$a = k_2 b$$

for some integer k_2. Substituting this into the first equation,

$$b = k_1 k_2 b \implies k_1 k_2 = 1.$$

Since k_1 and k_2 are integers, we either have $k_1 = k_2 = 1$ or $k_1 = k_2 = -1$. Thus, $a = b$ or $a = -b$, i.e. $a = \pm b$. □

Problem 3. Show that 3 divides $n(n+1)(2n+1)$ for any integer n.

Proof. Observe that

$$n(n+1)(2n+1) = 6\sum_{k=1}^{n} k^2 = 3\left(2\sum_{k=1}^{n} k^2\right).$$

Since $2\sum_{k=1}^{n} k^2$ is an integer, $n(n+1)(2n+1)$ is a multiple of 3. □

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Problem 4. Prove that for all integers a, if the remainder is NOT 2 when a is divided by 4, then $4 \mid a^3 + 23a$.

Solution. Observe that

$$a^3 + 23a = (a-1)a(a+1) + 24a \equiv (a-1)a(a+1) \pmod 4.$$

Case 1. If $a \equiv 0 \pmod 4$, i.e. a is a multiple of 4, then $a^3 + 23a$ is trivially a multiple of 4.

Case 2. If $a \equiv 1, 3 \pmod 4$, i.e. a is odd, then both $a-1$ and $a+1$ are even and contribute at least one factor of 2 each to $(a-1)a(a+1)$. Hence, $a^3 + 23a$ is divisible by $2^2 = 4$.

Case 3. If $a \equiv 2 \pmod 4$, then

$$a^3 + 23a \equiv (a-1)a(a+1) \equiv (1)(2)(3) \equiv 2 \not\equiv 0 \pmod 4.$$

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Problem 5. For any integer $n > 1$, let the standard factored form of n be given by

$$n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}.$$

Prove that n is a perfect square if and only if $k_1, k_2, \dots, k_r$ are all even integers.

Solution.

Proof. We begin by proving the backwards case. Suppose $k_1, k_2, \dots, k_r$ are all even integers. We can write $k_i = 2k_i'$ for all $1 \le i \le r$. Then

$$n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r} = p_1^{2k_1'} p_2^{2k_2'} \dots p_r^{2k_r'} = \left(p_1^{k_1'} p_2^{k_2'} \dots p_r^{k_r'}\right)^2.$$

Since $p_1^{k_1'} p_2^{k_2'} \dots p_r^{k_r'}$ is an integer, n is a perfect square.

We now prove the forwards case. Since n is a perfect square, we have $n = m^2$ for some positive integer m. Let the prime factorization of m be given by

$$m = q_1^{k_1'} q_2^{k_2'} \dots q_r^{k_r'},$$

where q_i are primes and k_1' are non-negative integers. Then

$$n = \left(q_1^{k_1'} q_2^{k_2'} \dots q_r^{k_r'}\right)^2 = q_1^{2k_1'} q_2^{2k_2'} \dots q_r^{2k_r'}.$$

Note that this is exactly the prime factorization of n. Also notice that all the exponents are multiples of 2 and are hence even. □

Problem 6. For all integers a and b, prove that $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$.

Solution.

Proof. The backwards case is trivial. We hence only consider the forwards case. We prove this claim using the contrapositive. Suppose $3 \nmid a$ and $3 \nmid b$. Then

$$a \equiv n_1 \pmod 3, \quad b \equiv n_2 \pmod 3,$$

where n_1 and n_2 are integers with $0 < n_1, n_2 < 2$. Without loss of generality, suppose $n_1 \leq n_2$.

Applying standard properties of modular arithmetic, we obtain

$$ab \equiv n_1 n_2 \pmod 3.$$

Case 1. Suppose $n_1 = n_2 = 1$. Then $ab \equiv 1 \not\equiv 0 \pmod 3$.
Case 2. Suppose $n_1 = 1$, $n_2 = 2$. Then $ab \equiv 2 \not\equiv 0 \pmod 3$.
Case 3. Suppose $n_1 = n_2 = 2$. Then $ab \equiv 4 \equiv 1 \not\equiv 0 \pmod 3$.

In any case, $ab \not\equiv 0 \pmod 3$, i.e. 3 does not divide ab. By the contrapositive, it follows that 3 divides ab if 3 divides a or b. □

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Problem 7. Let a, b and n be integers with $n > 1$. Suppose $a \equiv b \pmod n$. Prove the following:

- $ka \equiv kb \pmod{kn}$ for any positive integer k.
- If m is a common divisor of a, b and n, and $1 < m < n$, then

$$\frac{a}{m} \equiv \frac{b}{m} \pmod{\frac{n}{m}}.$$

Solution.

Proof of (a). Since $a \equiv b \pmod n$, we have $a = cn + b$ for some integer c. Multiplying this through by k, we have $ak = c(nk) + bk$. Hence, $ak \equiv bk \mod nk$. □

Proof of (b). Since $a \equiv b \pmod n$, we have $a = cn + b$ for some integer c. Since m is a common divisor of a, b and n, we have $a = ma'$, $b = mb'$ and $n = mn'$ for integers a', b' and m'. Dividing through by m, we get

$$\frac{a}{m} = \frac{cn}{m} + \frac{b}{n} \implies a' = cn' + b'.$$

Hence, $a' \equiv b' \pmod{n'}$, i.e.

$$\frac{a}{m} \equiv \frac{b}{m} \pmod{\frac{n}{m}}.$$

□

Part XI

Examinations

JC1 Weighted Assessment 1

JC1 Weighted Assessment 1 - H2 Mathematics 9758

Problem 1. The curve C has parametric equations

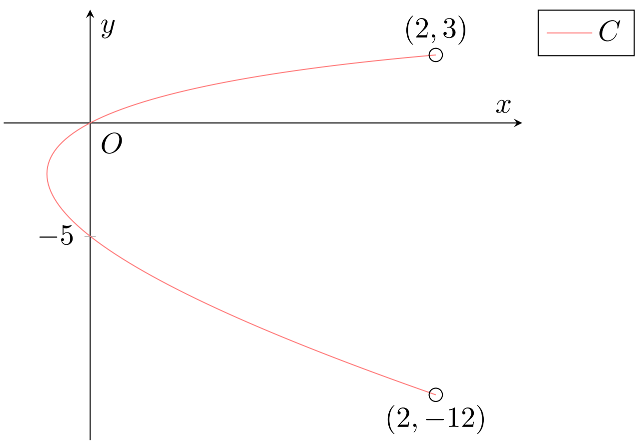
$$x = t^2 + t,\ y = 4t - t^2,\ -2 < t < 1.$$

(a) Sketch C, indicating the coordinates of the end-points and the axial intercepts (if any) of this curve.

(b) Find the coordinates of the point(s) of intersection between C and the line $8y - 12x = 5$.

Solution.

Part (a).



Part (b).

$$28y - 12x = 8(4t - t^2) - 12(t^2 + t) = 5 \implies -20t^2 + 20t - 5 = 0$$
$$\implies t^2 - t + \frac{1}{4} = 0 \implies \left(t - \frac{1}{2}\right)^2 = 0 \implies t = \frac{1}{2}.$$

When $t = 1/2$, we have that $x = 3/4$ and $y = 7/4$. Thus, C and the line $8y - 12x = 5$ intersect at $(3/4, 7/4)$.

* * * * *

Problem 2.

(a) Without using a calculator, solve $\frac{4}{3+2x-x^2} \leq 1$.

(b) Hence, solve $\frac{4}{3+2|x|-x^2} \leq 1$.

Solution.

Part (a).

$$\frac{4}{3+2x-x^2} \leq 1 \implies \frac{4}{x^2-2x-3} \geq -1$$
$$\implies \frac{4}{(x-3)(x+1)} + 1 = \frac{4+(x-3)(x+1)}{(x-3)(x+1)} = \frac{(x-1)^2}{(x-3)(x+1)} \geq 0$$

We thus have that $x = 1$ is a solution. In the case when $(x-1)^2 > 0$,

$$\frac{1}{(x-3)(x+1)} \geq 0 \implies (x-3)(x+1) \geq 0$$

whence $x < 1$ or $x > 3$. Putting everything together, we have

$$x < -1 \text{ or } x = 1 \text{ or } x > 3.$$

Part (b).

$$\frac{4}{3+2|x|-x^2} \leq 1 \implies \frac{4}{3+2|x|-|x|^2} \leq 1.$$

From part (a), we have that $|x| < -1$, $|x| = 1$ or $|x| > 3$.

Case 1: $|x| < -1$. Since $|x| \geq 0$ this case yields no solutions.

Case 2: $|x| = 1$. We have $x = 1$ or $x = -1$.

Case 3: $|x| > 3$. We have $x > 3$ or $x < -3$.

Thus,

$$x < -3 \text{ or } x = -1 \text{ or } x = 1 \text{ or } x > 3.$$

* * * * *

Problem 3. The curve C_1 has equation

$$y = \frac{2x^2+2x-2}{x-1}.$$

(a) Sketch the graph of C_1, stating the equations of any asymptotes and the coordinates of any axial intercepts and/or turning points.

The curve C_2 has equation

$$\frac{(x-a)^2}{1^2} + \frac{(y-6)^2}{b^2} = 1$$

where $b > 0$. It is given that C_1 and C_2 have no points in common for all $a \in \mathbb{R}$.

(b) By adding an appropriate curve in part (a), state the range of values of b, explaining your answer.

(c) The function f is defined by

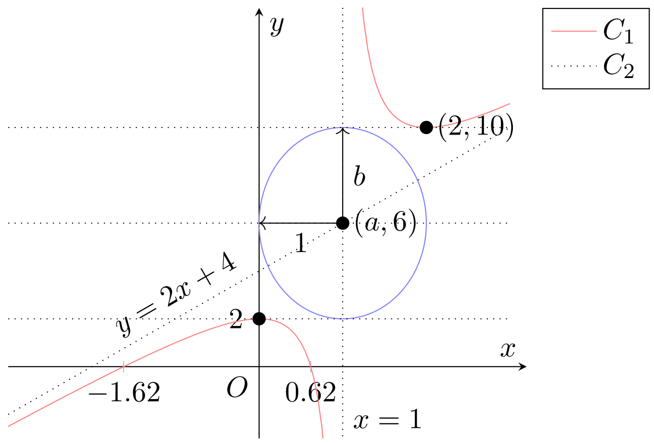
$$f(x) = \frac{2x^2+2x-2}{x-1}, \; x < 1.$$

(i) By using the graph in part (a) or otherwise, explain why the inverse function f^{-1} does not exist.

(ii) The domain of f is restricted to $[c, 1)$ such that c is the least value for which the inverse function f^{-1} exists. State the value of c and define f^{-1} clearly.

Solution.

Part (a).



Part (b). Observe that C_2 describes an ellipse with vertical radius b and horizontal radius 1. Furthermore, the ellipse is centred at $(a, 6)$. Since C_1 and C_2 have no points in common for all $a \in \mathbb{R}$, the maximum y-value of the ellipse corresponds to the y-value of the minimum point $(2, 10)$ of C_1. Similarly, the minimum y-value of the ellipse corresponds to the y-value of the maximum point $(0, 2)$ of C_1. Thus, $2 < y < 10$, whence $b < \min\{|6-2|, |6-10|\} = 4$. Thus, $0 < b < 4$.

Part (c).

Part (c)(i). Observe that $f(-1.62) = f(0.618) = 0$. Hence, there exist two different values of x in D_f that have the same image under f. Thus, f is not one-one. Hence, f^{-1} does not exist.

Part (c)(ii). Clearly, $c = 0$. We now find f^{-1}.

$$f(x) = \frac{2x^2 + 2x - 2}{x - 1} \implies xf(x) - f(x) = 2x^2 + 2x - 2$$

$$\implies 2x^2 + \big[2 - f(x)\big]x + \big[f(x) - 2\big] = 0 \implies x = \frac{f(x) - 2 \pm \sqrt{f(x)^2 - 12f(x) + 20}}{4}.$$

Replacing $x \mapsto f^{-1}(x)$, we get

$$f^{-1}(x) = \frac{x - 2 \pm \sqrt{x^2 - 12x + 20}}{4}.$$

Note that $D_f = R_{f^{-1}} = [0, 1)$. We thus take the positive root. Also note that $R_f = D_{f^{-1}} = (-\infty, 2]$. Hence,

$$f^{-1} : x \mapsto \frac{x - 2 + \sqrt{x^2 - 12x + 20}}{4}, \ x \in \mathbb{R},\ x \leq 2.$$

JC1 Weighted Assessment 1 - H2 Further Mathematics 9649

Problem 1. Show that

$$\sum_{r=1}^{(n-1)^2+3} (3^{2r} - n + 1) = \frac{a}{b}\left(729 \cdot 9^{(n-1)^2} - 1\right) - c(n-1)^3 - d(n-1)$$

where a, b, c and d are constants to be determined.

Solution.

$$\begin{aligned}\sum_{r=1}^{(n-1)^2+3} (3^{2r} - n + 1) &= \sum_{r=1}^{(n-1)^2+3} 9^r - \sum_{r=1}^{(n-1)^2+3} (n-1)\\ &= \frac{9\left(9^{(n-1)^2+3} - 1\right)}{9-1} - (n-1)\left[(n-1)^2+3\right]\\ &= \frac{9}{8}\left(729 \cdot 9^{(n-1)^2} - 1\right) - (n-1)^3 - 3(n-1).\end{aligned}$$

* * * * *

Problem 2. Do not use a calculator in answering this question.
The sequence of positive numbers, u_n, satisfies the recurrence relation:

$$u_{n+1} = \sqrt{2u_n + 3}, \qquad n = 1, 2, 3, \ldots.$$

(a) If the sequence converges to m, find the value of m.

(b) By using a graphical approach, explain why $m < u_{n_1} < u_n$ when $u_n > u_m$. Hence, determine the behaviour of the sequence when $u_1 > m$.

Solution.

Part (a). Observe that

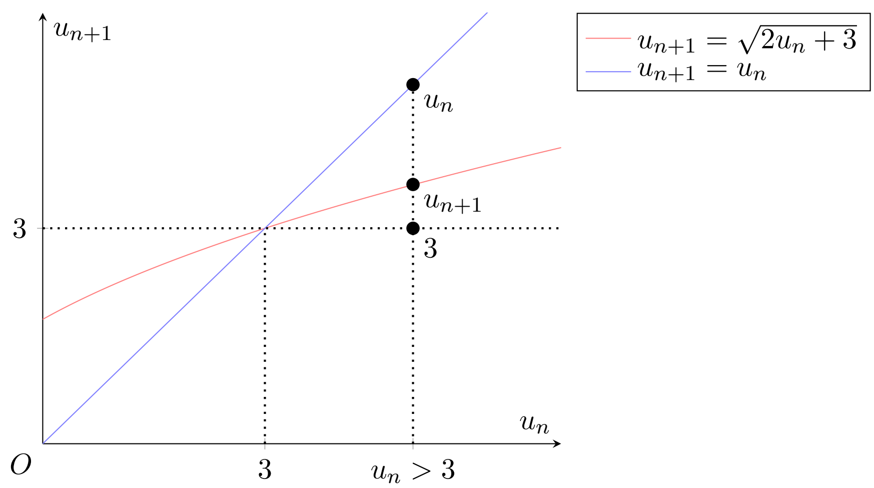
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{2u_{n-1}+3} = \sqrt{2\lim_{n\to\infty} u_{n-1} + 3} = \sqrt{2\lim_{n\to\infty} u_n + 3}.$$

Since the sequence converges to m, we have $\lim_{n\to\infty} u_n = m$. Thus,

$$m = \sqrt{2m+3} \implies m^2 = 2m+3 \implies m^2 - 2m - 3 = (m-3)(m+1) = 0.$$

Thus, $m = 3$ or $m = -1$. Since u_n is always positive, we take $m = 3$.

Part (b).



From the graph, if $u_n > 3$, then $3 < u_{n+1} < u_n$. Hence, the sequence decreases and converges to 3.

* * * * *

Problem 3. Two expedition teams are to climb a vertical distance of 8100 m from the foot to the peak of a mountain. Team A plans to cover a vertical distance of 400 m on the first day. On each subsequent day, the vertical distance covered is 5 m less than the vertical distance covered in the previous day. Team B plans to cover a vertical distance of 800 m on the first day. On each subsequent day, the vertical distance covered is 90% of the vertical distance covered in the previous day.

(a) Find the number of days required for Team A to reach the peak.

(b) Explain why Team B will never be able to reach the peak.

(c) At the end of the 15th day, Team B decided to modify their plan, such that on each subsequent day, the vertical distance covered is 95% of the vertical distance covered in the previous day. Which team will be the first to reach the peak of the mountain? Justify your answer.

Solution.

Part (a). The vertical distance Team A plans to cover in a day can be described as a sequence in arithmetic progression with first term 400 and common difference -5. In order to reach the peak, the total vertical distance covered by Team A has to be greater than 8100 m. Hence,

$$\frac{n}{2}\left(2(400) + (n-1)(-5)\right) \geq 8100.$$

Using G.C., $n \geq 24$. Hence, Team A requires 24 days to reach the peak.

Part (b). The vertical distance Team B plans to cover in the nth day can be described by the sequence U_n in geometric progression with first term 800 and common ratio $r = 0.9$. Let S_n^U be the nth partial sum of U_n. Since $|r| < 1$, the sum to infinity of exists and is equal to

$$S_\infty^U = \frac{800}{1-0.9} = 8000.$$

Hence, Team B will never be able to surpass 8 km in height. Thus, they will not reach the peak no matter how long they take.

Part (c). The new vertical distance covered by Team B after Day 15 can be described by the sequence V_n in geometric progression with first term U_{15} and common ratio $r = 0.95$. Let S_n^V be the nth partial sum of V_n. Then,

$$S_n^V = \frac{U_{15} \cdot 0.95\left[1-(0.95)^n\right]}{1-0.95}.$$

Note that

$$S_n^U = \frac{800\left[1-(0.9)^n\right]}{1-0.9}.$$

Hence, after Day 15, Team B has to climb another $8000 - S_{15}^U = 1747.13$ metres. Since $U_{15} = 183.01$, we have the inequality

$$\frac{183.01 \cdot 0.95\left[1-(0.95)^n\right]}{1-0.95} \geq 1747.13.$$

Using G.C., $n \geq 14$. Hence, Team B will need at least $15 + 14 = 29$ days to reach the peak. Thus, team A will reach the peak first.

Problem 4. The function f is given by $f(x) = x^2 - 3x + 2 - \mathrm{e}^{-x}$. It is known from graphical work that this equation has 2 roots $x = \alpha$ and $x = \beta$, where $\alpha < \beta$.

(a) Show that $f(x) = 0$ has at least one root in the interval $[0, 1]$.

It is known that there is exactly one root in $[0, 1]$ where $x = \alpha$.

(b) Starting with $x_0 = 0.5$, use an iterative method based on the form

$$x_{n+1} = p\left(x_n^2 + q - \mathrm{e}^{-x_n}\right)$$

where p and q are real numbers to be determined, to find the value of α correct to 3 decimal places. You should demonstrate that your answer has the required accuracy.

It is known that the other root $x = \beta$ lies in the interval $[2, 3]$.

(c) With the aid of a clearly labelled diagram, explain why the method in (b) will fail to obtain any reasonable approximation to β, where x_0 is chosen such that $x_0 \in [2, 3]$, $x_0 \neq \beta$.

To obtain an approximation to β, another approach is used.

(d) Use linear interpolation once in the interval $[2, 3]$ to find a first approximation to β, giving your answer to 2 decimal places. Explain whether this approximate is an overestimate or underestimate.

(e) With your answer in (d) as the initial approximate, use the Newton-Raphson method to obtain β correct to 3 decimal places. Your process should terminate when you have two successive iterates that are equal when rounded to 3 decimal places.

Solution.

Part (a). Observe that $f(0) = 1 > 0$ and $f(1) = -e^{-1} < 0$. Since f is continuous and $f(0)f(1) < 0$, there must be at least one root to $f(x) = 0$ in the interval $[0, 1]$.

Part (b). Let $f(x) = 0$. Then,

$$x^2 - 3x + 2 - \mathrm{e}^{-x} = 0 \implies x^2 + 2 - \mathrm{e}^{-x} = 3x \implies x = \frac{1}{3}\left(x^2 + 2 - \mathrm{e}^{-x}\right).$$

Hence, we should use an iterative method based on the form

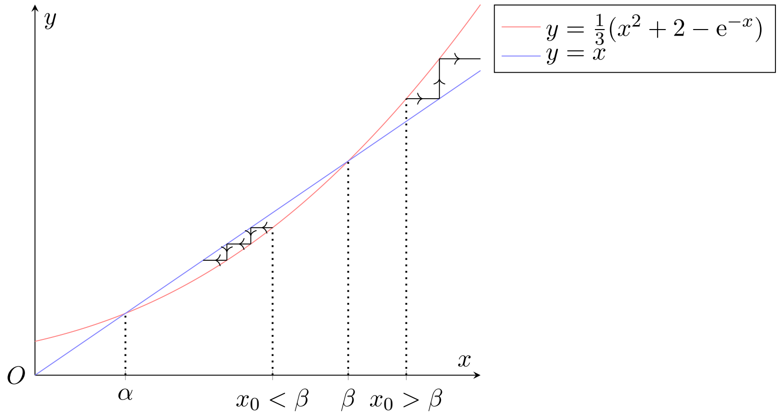
$$x_{n+1} = \frac{1}{3}\left(x_n^2 + 2 - \mathrm{e}^{-x_n}\right).$$

Starting with $x_0 = 0.5$,

n	x_n	n	x_n
0	0.5	6	0.60494
1	0.54782	7	0.60662
2	0.57396	8	0.60759
3	0.58871	9	0.60817
4	0.59718	10	0.60851
5	0.60208	11	0.60870

Since $f(0.6085) = 0.000606 > 0$ and $f(0.6095) = -0.000632 < 0$, we have that $\alpha \in (0.6085, 0.6095)$. Thus, $\alpha = 0.609$ (3 d.p.).

Part (c).



From the diagram, we see that whether we chose $x_0 < \beta$ or $x_0 > \beta$, the approximates move away from the root β. In fact, if we choose $x_0 < \beta$, the approximates converge to the root α instead.

Part (d). Using linear interpolation on the interval $[2, 3]$,

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = 2.06 \text{ (2 d.p.)}.$$

Thus, $\beta = 2.06$ (2 d.p.).

Observe that $f(2.06) = -0.039 < 0$ and $f(3) = 1.950 > 0$. Hence, $\beta \in (2.06, 3)$. Thus, $\beta = 2.06$ is an underestimate.

Part (e). Observe that $f'(x) = 2xx - 3 + e^{-x}$. Using the Newton-Raphson method with $x_1 = 2.06$,

n	x_n
1	2.06
2	2.11118
3	2.10935
4	2.10935

Hence, $\beta = 2.109$ (3 d.p.).

JC1 Weighted Assessment 2

JC1 Weighted Assessment 2 - H2 Mathematics 9758

Problem 1. Differentiate $\arccos\left(\sqrt{1-4x}\right)$ with respect to x, simplifying your answer.

Solution.

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos\left(\sqrt{1-4x}\right) = -\frac{1}{\sqrt{1-(1-4x)}}\left(\frac{-4}{2\sqrt{1-4x}}\right) = \frac{2}{\sqrt{4x}\sqrt{1-4x}} = \frac{1}{\sqrt{x-4x^2}}.$$

* * * * *

Problem 2. It is given that x and y satisfy the equation $xy^2 = \ln\left(x^2\mathrm{e}^y\right) - \frac{2\mathrm{e}}{x}$.

(a) Verify that $(\mathrm{e}, 0)$ satisfies the equation.

(b) Hence, show that at $y = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{\mathrm{e}}$, where k is a constant to be determined.

Solution.

Part (a). Substituting $x = \mathrm{e}$ and $y = 0$ into the given equation,

$$\text{LHS} = \mathrm{e}\cdot 0^2 = 0, \quad \text{RHS} = \ln\left(\mathrm{e}^2\cdot e^0\right) - \frac{2\mathrm{e}}{\mathrm{e}} = 2 - 2 = 0.$$

Since the LHS is equal to the RHS, $(\mathrm{e}, 0)$ satisfies the equation.

Part (b). From the given equation, we have

$$xy^2 = 2\ln x + y - \frac{2\mathrm{e}}{x}.$$

Implicitly differentiating yields

$$x\left(2y\,\frac{\mathrm{d}y}{\mathrm{d}x}\right) + y^2 = \frac{2}{x} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2\mathrm{e}}{x^2}.$$

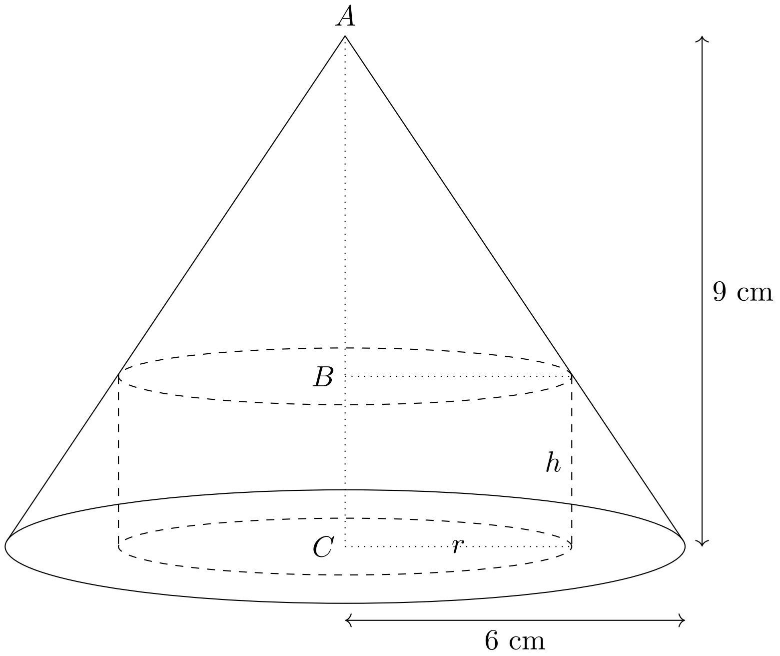
Substituting $x = e$ and $y = 0$ into the above equation gives

$$0 = \frac{2}{\mathrm{e}} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2\mathrm{e}}{\mathrm{e}^2} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4}{\mathrm{e}}.$$

Thus, $k = -4$.

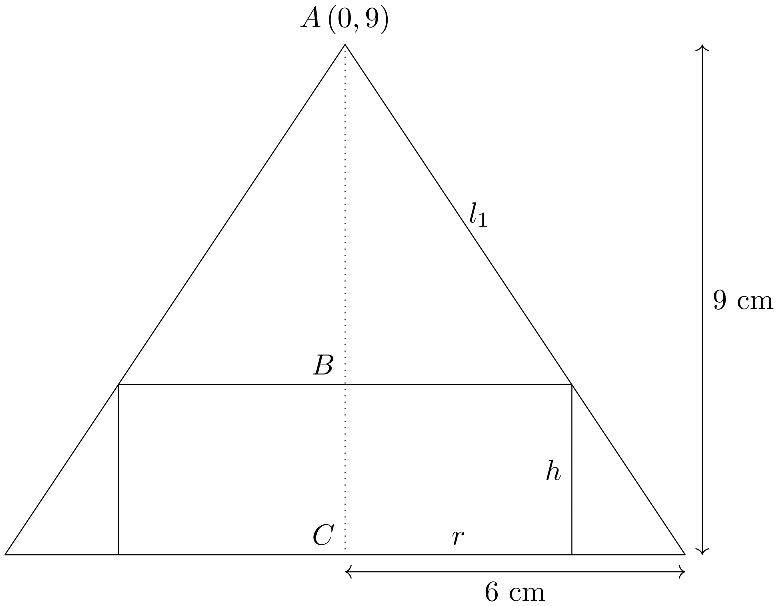
* * * * *

Problem 3. A toy manufacturer wants to make a toy in the shape of a right circular cone with a cylinder drilled out, as shown in the diagram below. The cylinder is inscribed in the cone. The circumference of the top of the cylinder is in contact with the inner surface of the cone and the base of the cylinder is level with the base of the cone. The base radius of the cylinder is r cm and the base radius of the cone is 6 cm. The height of the cylinder, BC, is h cm and the height of the cone, AC is 9 cm.



Using differentiation, find the minimum volume of the toy, V cm^3, in terms of π.

Solution.



Consider the diagram above. Let C be the origin. Note that l_1 has gradient $-\frac{9}{6} = -\frac{3}{2}$. Hence, l_1 has equation

$$l_1 : y = 9 - \frac{3}{2}x.$$

When $x = r$, we have $y = 9 - \frac{3}{2}r$. Thus, the height of the cylinder is $\left(9 - \frac{3}{2}r\right)$ cm. Let the volume of the cylinder be V_1 cm^3.

$$V_1 = \pi r^2 h = \pi r^2 \left(9 - \frac{3}{2}r\right) = 9\pi r^2 - \frac{3}{2}\pi r^3.$$

For stationary points, $\frac{\mathrm{d}V_1}{\mathrm{d}r} = 0$.

$$\frac{\mathrm{d}V_1}{\mathrm{d}r} = 0 \implies 18\pi r - \frac{9}{2}\pi r^2 = 0 \implies \frac{9}{2}\pi r(4 - r) = 0.$$

Hence, V_1 has a stationary point when $r = 4$. Note that we reject $r = 0$ since $r > 0$.

r	4^-	4	4^+
$\frac{dV_1}{dr}$	+ve	0	−ve

By the first derivative test, V_1 attains a maximum when $r = 4$. Hence,

$$\min V = \text{Volume of cone} - \max V_1 = \left[\frac{1}{3}\pi\left(6^2\right)(9)\right] - \left[9\pi\left(4^2\right) - \frac{3}{2}\pi\left(4^3\right)\right] = 60\pi.$$

Thus, the minimum volume of the toy is 60π cm^3.

* * * * *

Problem 4. A curve C has parametric equations

$$x = 2\theta + \sin 2\theta,\ y = \cos 2\theta,\ 0 \leq \theta \leq \pi.$$

(a) Find $\frac{dy}{dx}$, expressing your answer in terms of only a single trigonometric function.

(b) Hence, find the coordinates of point Q, on C, whose tangent is parallel to the y-axis.

Solution.

Part (a). Note that $\frac{dx}{d\theta} = 2 + 2\cos 2\theta$ while $\frac{dy}{d\theta} = -2\sin 2\theta$. Hence,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin 2\theta}{2 + 2\cos 2\theta} = -\frac{\sin 2\theta}{1 + \cos 2\theta}$$
$$= -\frac{2\sin\theta\cos\theta}{1 + (2\cos^2\theta - 1)} = -\frac{2\sin\theta\cos\theta}{2\cos^2\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta.$$

Part (b). Since the tangent at Q is parallel to the y-axis, the derivative $dy/dx = -\tan\theta$ must be undefined there. Hence, $\cos\theta = 0 \implies \theta = \pi/2$. Substituting $\theta = \pi/2$ into the given parametric equations, we obtain $x = \pi$ and $y = -1$, whence $Q(\pi, -1)$.

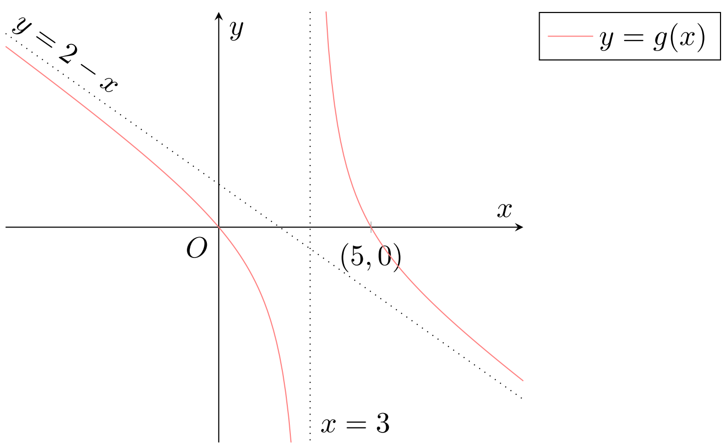
* * * * *

Problem 5.

(a) A function is defined as $f(x) = a(2 - x)^2 - b$, where a and b are positive constants such that $a < 1$ and $b > 4$.

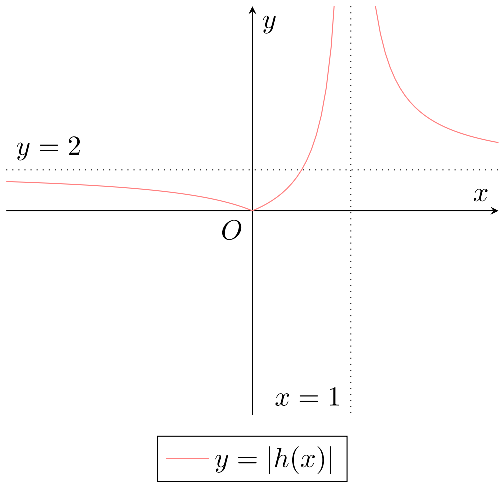
State a sequence of transformations that will transform the curve with equation $y = x^2$ on to the curve with equation $y = f(x)$.

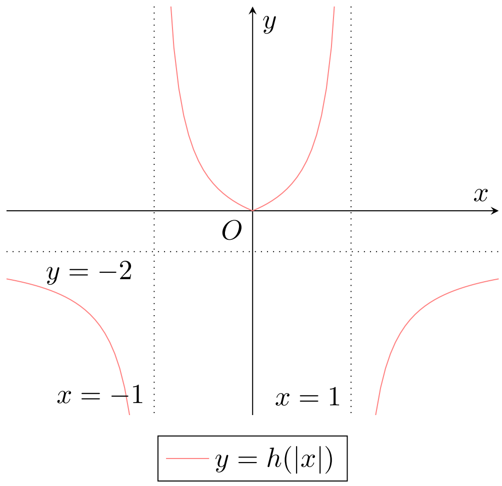
(b) The diagram shows the graph of $y = g(x)$. The lines $x = 3$ and $y = 2 - x$ are asymptotes to the curve, and the graph passes through the points $(0, 0)$ and $(5, 0)$.



Sketch the graph of $y = \frac{1}{g(x)}$, indicating clearly the coordinates of any axial intercepts (where applicable) and the equations of any asymptotes.

(c) Given the graphs of $y = |h(x)|$ and $y = h(|x|)$ below, sketch the two possible graphs of $y = h(x)$.



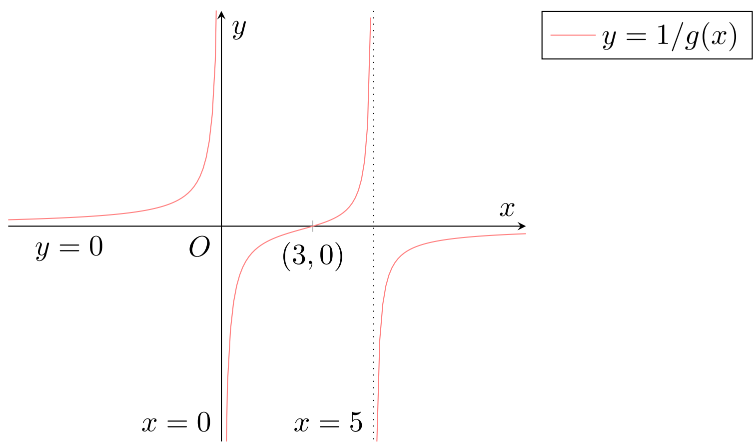


Solution.

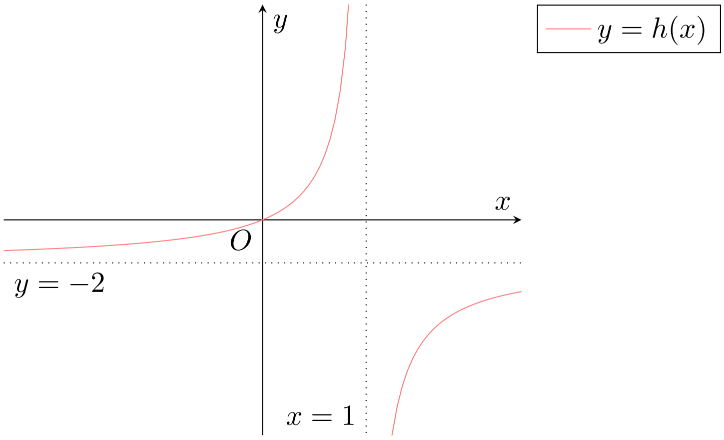
Part (a).

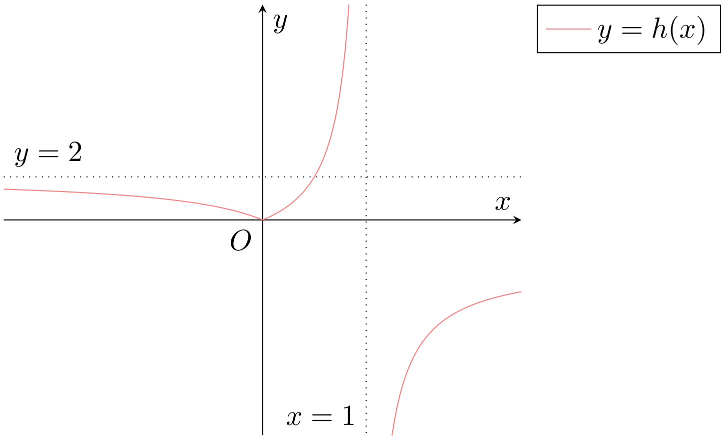
1. Translate the graph 2 units in the positive x-direction.
2. Scale the graph by a factor of a parallel to the y-axis.
3. Translate the graph b units in the negative y-direction.

Part (b).



Part (c).





JC1 Weighted Assessment 2 - H2 Further Mathematics 9649

Problem 1. Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. The point C is such that $\overrightarrow{OC} = m\overrightarrow{OA}$ where m is a constant. The point D lies on AB produced such that B divides AD in the ratio $1 : 2$.

(a) Express the area of triangle ADC in the form $k\left|\mathbf{a} \times \mathbf{b}\right|$, where k is an expression in terms of m. Show your working clearly.

(b) If $\overrightarrow{AC}$ is a unit vector, give a geometrical interpretation of the value of $\left|\mathbf{b} \times \overrightarrow{AC}\right|$ and find the possible values of m in terms of $|\mathbf{a}|$.

Solution.

Part (a).

$$\overrightarrow{OC} = m\overrightarrow{OA} = m\mathbf{a} \implies \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = m\mathbf{a} - \mathbf{a} = (m-1)\mathbf{a}.$$

By the Ratio Theorem,

$$\overrightarrow{OB} = \frac{\overrightarrow{OD} + 2 \cdot \overrightarrow{OA}}{1+2}.$$

Hence,

$$\overrightarrow{OD} = 3\overrightarrow{OB} - 2\overrightarrow{OA} = 3\mathbf{b} - 2\mathbf{a} \implies \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 3\mathbf{b} - 3\mathbf{a}.$$

Thus,

$$\begin{aligned}\text{Area}\,\triangle ADC &= \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{AD}\right| = \frac{1}{2}\left|(m-1)\mathbf{a} \times (3\mathbf{b} - 3\mathbf{a})\right| = \frac{3}{2}\left|m-1\right|\left|\mathbf{a} \times (\mathbf{b} - \mathbf{a})\right| \\ &= \frac{3}{2}\left|m-1\right|\left|\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{a}\right| = \frac{3}{2}\left|m-1\right|\left|\mathbf{a} \times \mathbf{b}\right|\end{aligned}$$

whence $k = \frac{3}{2}\left|m-1\right|$.

Part (b). Since $\overrightarrow{AC}$ is parallel to $\mathbf{a}$, if $\overrightarrow{AC}$ is a unit vector, then $\overrightarrow{AC} = \hat{\mathbf{a}}$. Hence, $\left|\mathbf{b} \times \overrightarrow{AC}\right| = \left|\mathbf{b} \times \hat{\mathbf{a}}\right|$ is the shortest distance from B to the line OA.

Since $\overrightarrow{AC}$ is a unit vector, we have

$$\left|\overrightarrow{AC}\right| = \left|(m-1)\mathbf{a}\right| = 1 \implies \left|m-1\right| = \frac{1}{|\mathbf{a}|} \implies m = 1 \pm \frac{1}{|\mathbf{a}|}.$$

* * * * *

Problem 2. Marine biologist experts calculated that when the concentration of chemical X in a sea inlet reaches 6 milligrams per litre (mg/l), the level of pollution endangers marine life. A factory wishes to release waste containing chemical X into the inlet. It claimed that the discharge will not endanger the marine life, and they provided the local authority with the following information:

- There is no presence of chemical X in the sea inlet at present.
- The plain is to discharge chemical X on a weekly basis into the sea inlet. The discharge, which will be done at the beginning of each week, will result in an increase in concentration of 2.3 mg/l of chemical X in the inlet.

- The tidal streams will remove 7% of chemical X from the inlet at the end of every day.

(a) Form a recurrence relation for the concentration level of chemical X, u_n, at the beginning of week n. Hence, find the concentration at the beginning of week n.

(b) Should the local authority allow the factory to go ahead with the discharge if they are concerned with the marine life at the sea inlet? Justify your answer.

Solution.

Part (a). We have

$$u_n = 0.93^7 u_{n-1} + 2.3, \quad u_0 = 0.$$

Let k be the constant such that $u_n + k = 0.93^7(u_{n-1} + k)$. Then $k = \frac{2.3}{0.93^7 - 1}$. Hence,

$$u_n - \frac{2.3}{1 - 0.93^7} = 0.93^7 \left(u_{n-1} - \frac{2.3}{1 - 0.93^7} \right) = 0.93^{7n} \left(u_0 - \frac{2.3}{1 - 0.93^7} \right) = -\frac{2.3 \cdot 0.93^{7n}}{1 - 0.93^7}$$

$$\implies u_n = \frac{2.3}{1 - 0.93^7} - \frac{2.3 \cdot 0.93^{7n}}{1 - 0.93^7}.$$

Part (b).

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \left(\frac{2.3}{1 - 0.93^7} - \frac{2.3 \cdot 0.93^{7n}}{1 - 0.93^7} \right) = \frac{2.3}{1 - 0.93^7} = 5.77 \text{ (3 s.f.)}.$$

Since $5.77 < 6$, if the local authority's only concern is marine life, they should allow the factory to go ahead with the discharge.

* * * * *

Problem 3. Referred to the origin O, the position vector of the point A is $3\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ and the Cartesian equation of the line l_1 is $x - 1 = 2 - y = 2z + 6$.

(a) Find the position vector of the foot of perpendicular from A to l_1.

Line l_2 has the vector equation $\mathbf{r} = \langle -1, 6, -1 \rangle + \mu \langle -6, 6, -3 \rangle$, where $\mu \in \mathbb{R}$.

(b) Find the shortest distance between l_1 and l_2.

(c) Given that l_2 is the reflection of l_1 about the line l_3, find the vector equation of the line l_3.

Solution.

Part (a). Note that l_1 has vector equation

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Let F be the foot of perpendicular from A to l_1. Since F is on l_1, $\overrightarrow{OF} = \langle 1, 2, -3 \rangle + \lambda \langle 2, -2, 1 \rangle$ for some $\lambda \in \mathbb{R}$. Thus,

$$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Note also that $\overrightarrow{AF}$ is perpendicular to l_1. Hence,

$$\left[\begin{pmatrix}-2\\4\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}=0\implies -9+9\lambda=0\implies\lambda=1.$$

Thus,

$$\overrightarrow{OF}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\0\\-2\end{pmatrix}.$$

Part (b). Note that $\langle -6,\,6,\,-3\rangle \parallel \langle 2,\,-2,\,1\rangle$. Hence, l_2 is parallel to l_1. Hence, the shortest distance between l_1 and l_2 is the perpendicular distance from a point on l_1 to l_2, which is

$$\frac{|[\langle 1,\,2,\,-3\rangle-\langle -1,\,6,\,-1\rangle]\times\langle 2,\,-2,\,1\rangle|}{|\langle 2,\,-2,\,1\rangle|}=\frac{1}{\sqrt{9}}\left|\begin{pmatrix}2\\-4\\-2\end{pmatrix}\times\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|$$
$$=\frac{2}{3}\left|\begin{pmatrix}-1\\2\\1\end{pmatrix}\times\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|=\frac{2}{3}\left|\begin{pmatrix}4\\3\\-2\end{pmatrix}\right|=\frac{2}{3}\sqrt{29}\text{ units.}$$

Part (c). Observe that l_3 passes through the midpoint of $\langle 1,\,2,\,-3\rangle$ and $\langle -1,\,6,\,-1\rangle$, which evaluates to

$$\frac{1}{2}\left[\begin{pmatrix}1\\2\\-3\end{pmatrix}+\begin{pmatrix}-1\\6\\-1\end{pmatrix}\right]=\begin{pmatrix}0\\4\\-2\end{pmatrix}.$$

l_3 is also parallel to both l_1 and l_2. Hence,

$$l_3:\mathbf{r}=\begin{pmatrix}0\\4\\-2\end{pmatrix}+\nu\begin{pmatrix}2\\-2\\1\end{pmatrix},\ \nu\in\mathbb{R}.$$

* * * * *

Problem 4. A first order recurrence relation is given as

$$u_{n+1}\left[u_n+\left(\frac{1}{2}\right)^n\right]+u_n\left[\left(\frac{1}{2}\right)^{n+1}-10\right]=10\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^{2n+1}-16$$

where $u_1=1$.

(a) Using the substitution $u_n=\frac{v_n}{v_{n-1}}-\left(\frac{1}{2}\right)^n$ where $v_{n-1}\neq 0$, show that the recurrence relation can be expressed as a second order recurrence relation of the form $v_{n+1}+av_n+16v_{n-1}=0$, where a is a constant to be found.

(b) By first solving the second order recurrence relation in (a), find an expression for u_n in terms of n.

(c) Describe what happens to the value of u_n for large values of n.

Solution.

Part (a). Substituting in v_n for u_n into the LHS of the recurrence relation, we get

$$\begin{aligned}
&u_{n+1}\left[u_n+\left(\frac{1}{2}\right)^n\right]+u_n\left[\left(\frac{1}{2}\right)^{n+1}-10\right]\\
&=\left[\frac{v_{n+1}}{v_n}-\left(\frac{1}{2}\right)^{n+1}\right]\left[\frac{v_n}{v_{n-1}}\right]+\left[\frac{v_n}{v_{n-1}}-\left(\frac{1}{2}\right)^n\right]\left[\left(\frac{1}{2}\right)^{n+1}-10\right]\\
&=\frac{v_{n+1}}{v_{n-1}}-10\left(\frac{v_n}{v_{n-1}}\right)-\left(\frac{1}{2}\right)^{2n+1}+10\left(\frac{1}{2}\right)^n.
\end{aligned}$$

Cancelling terms from the RHS, we get

$$\frac{v_{n+1}}{v_{n-1}}-10\left(\frac{v_n}{v_{n-1}}\right)=-16 \implies v_{n+1}-10v_n+16v_{n-1}=0.$$

Hence, $a=-10$.

Part (b). Consider the characteristic equation of v_n.

$$x^2-10x+16=(x-2)(x-8)=0.$$

Hence, 2 and 8 are the roots of the characteristic equation. Thus,

$$v_n=A\cdot 2^n+B\cdot 8^n.$$

Consider u_1.

$$u_1=\frac{v_1}{v_0}-\frac{1}{2}=1 \implies \frac{2A+8B}{A+B}=\frac{3}{2} \implies \frac{4A+16B}{A+B}=3 \implies A=-13B.$$

Now observe that

$$\begin{aligned}
\frac{v_n}{v_{n-1}}&=\frac{A\cdot 2^n+B\cdot 8^n}{A\cdot 2^{n-1}+B\cdot 8^{n-1}}=\frac{-13\cdot 2^n+8^n}{-13\cdot 2^{n-1}+8^{n-1}}=8\left(\frac{-13\cdot 2^n+8^n}{-52\cdot 2^n+8^n}\right)\\
&=8\left(1+\frac{39\cdot 2^n}{-52\cdot 2^n+8^n}\right)=8-\frac{312\cdot 2^n}{52\cdot 2^n-8^n}.
\end{aligned}$$

Thus,

$$u_n=\frac{v_n}{v_{n-1}}-\left(\frac{1}{2}\right)^n=8-\frac{312\cdot 2^n}{52\cdot 2^n-8^n}-\left(\frac{1}{2}\right)^n.$$

Part (c).

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left[8-\frac{312\cdot 2^n}{52\cdot 2^n-8^n}-\left(\frac{1}{2}\right)^n\right]=8.$$

Thus, u_n converges to 8 for large values of n.

JC1 Promotional Examination

JC1 Promotional Examination - H2 Mathematics 9758

Problem 1. A dietitian in a hospital is to arrange a special diet meal composed of Food A, Food B and Food C. The diet is to include exactly 7800 units of calcium, 80 units of iron and 7.5 units of vitamin A. The number of units of calcium, iron and vitamin A for each gram of the foods is indicated in the table.

	Units Per Gram		
	Food A	Food B	Food C
Calcium	15	20	24
Iron	0.2	0.15	0.28
Vitamin A	0.015	0.02	0.02

Find the total weight of the foods, in grams, of this special diet.

Solution. Let a, b, and c represent the weight of Food A, B, and C respectively, in grams. We have the following system of equations:

$$\begin{cases} 15a + 20b + 24c = 7800 \\ 0.2a + 0.15b + 0.28c = 80 \\ 0.015a + 0.02b + 0.02c = 7.5 \end{cases}$$

Using G.C., $a = 160$, $b = 180$ and $c = 75$. Hence, the total weight of the foods is $160 + 180 + 75 = 415$ grams.

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Problem 2. By expressing $\frac{3x^2+2x-12}{x-1} - (x+2)$ as a single simplified fraction, solve the inequality

$$\frac{3x^2+2x-12}{x-1} \geq x+2,$$

without using a calculator.

Solution.

$$\frac{3x^2+2x-12}{x-1} - (x+2) = \frac{3x^2+2x-12-(x+2)(x-1)}{x-1}$$
$$= \frac{3x^2+2x-12-(x^2+x-2)}{x-1} = \frac{2x^2+x-10}{x-1} = \frac{(x-2)(2x+5)}{x-1}.$$

Consider the inequality.

$$\frac{3x^2+2x-12}{x-1} \geq x+2 \implies \frac{3x^2+2x-12}{x-1} - (x+2) \geq 0$$
$$\implies \frac{(x-2)(2x+5)}{x-1} \geq 0 \implies (x-2)(2x+5)(x-1) \geq 0$$

Hence, $-5/2 \leq x < 1$ or $x \geq 2$.

Problem 3.

(a) Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, evaluate $\sum_{r=-n}^{n}(r+1)(r+3)$ in terms of n.

(b) Using standard series from the List of Formulae (MF27), find the range of values of x for which the series $\sum_{r=1}^{\infty} \frac{(-1)^{r+1}x^r}{r2^r}$ converges. State the sum to infinity in terms of x.

Solution.

Part (a).

$$\sum_{r=-n}^{n}(r+1)(r+3) = \sum_{r=0}^{2n}(r-n+1)(r-n+3) = \sum_{r=0}^{2n}\left[r^2 + r(4-2n) + (n^2-4n+3)\right]$$
$$= \frac{(2n)(2n+1)(4n+1)}{6} + (4-2n)\frac{(2n)(2n+1)}{2} + (n^2-4n+3)(2n+1)$$
$$= (2n+1)\left[\frac{n(4n+1)}{3} + (4-2n)(n) + (n^2-4n+3)\right] = \frac{2n+1}{3}\left(n^2+n+9\right).$$

Part (b).

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}x^r}{r2^r} = \sum_{r=1}^{\infty}\frac{(-1)^{r+1}(x/2)^r}{r} = \ln\left(1+\frac{x}{2}\right).$$

Range of convergence: $-1 < x/2 \le 1 \implies -2 < x \le 2$.

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Problem 4. The curve C has parametric equations

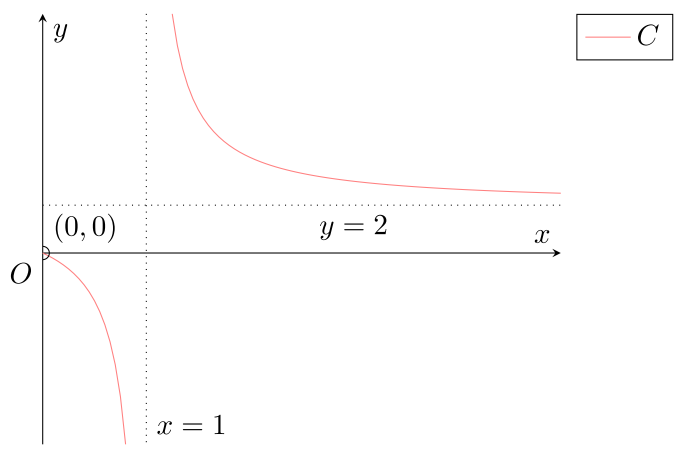
$$x = -\frac{2}{t-1}, \quad y = \frac{4}{t+1}, \quad t < 1, \quad t \neq \pm 1.$$

(a) Sketch a clearly labelled diagram of C, indicating any axial intercepts and asymptotes (if any) of this curve.

(b) Find also its Cartesian equation, stating any restrictions where applicable.

Solution.

Part (a).



Part (b). Note that $x = -\frac{2}{t-1} \implies t = -\frac{2}{x} + 1$. Hence,

$$y = \frac{4}{(-2/x+1)+1} = \frac{4x}{-2+2x} = \frac{2x}{x-1}.$$

Thus,

$$y = \frac{2x}{x-1}, \quad x \neq 1, x > 0.$$

* * * * *

Problem 5.

(a) Find, using an algebraic method, the exact roots of the equation $\left|3x^2 + 5x - 8\right| = 4 - x$.

(b) On the same axes, sketch the curves with equations $y = \left|3x^2 + 5x - 8\right|$ and $y = 4 - x$. Hence, solve exactly the inequality $\left|3x^2 + 5x - 8\right| < 4 - x$.

Solution.

Part (a). *Case 1*: $3x^2 + 5x - 8 = 4 - x$.

$$3x^2 + 5x - 8 = 4 - x \implies 3x^2 + 6x - 12 = 0 \implies x^2 + 2x - 4 = 0.$$

By the quadratic formula, we get

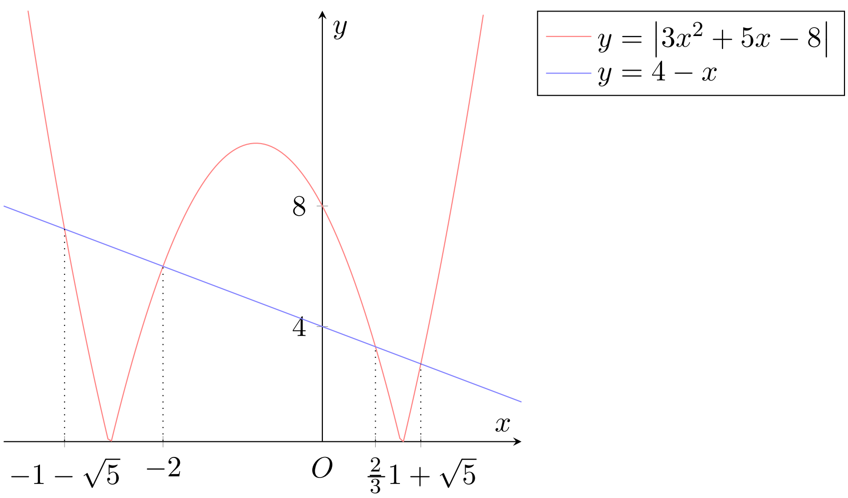
$$x = -1 \pm \sqrt{5}.$$

Case 2: $3x^2 + 5x - 8 = -(4 - x)$.

$$3x^2 + 5x - 8 = -4 + x \implies 3x^2 + 4x - 4 = (3x - 2)(x + 2) = 0 \implies x = \frac{2}{3} \text{ or } -2.$$

Hence, the roots are $x = -1 \pm \sqrt{5}$, $2/3$ and -2.

Part (b).



From the graph, $-1 - \sqrt{5} < x < -2$ or $\frac{2}{3} < x < -1 + \sqrt{5}$.

* * * * *

Problem 6.

(a) The transformations A, B and C are given as follows:

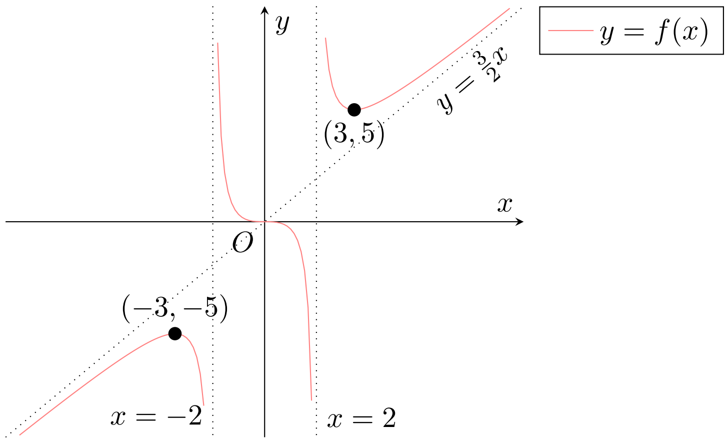
A: A reflection in the x-axis.

B: A translation of 1 unit in the positive y-direction.

C: A translation of 2 units in the negative x-direction.

A curve undergoes in succession, the transformations A, B and C, and the equation of the resulting curve is $y = \frac{2x+1}{2x+2}$. Determine the equation of the curve before the transformations, expressing your answer as a single fraction.

(b) The diagram shows the curve $y = f(x)$. The lines $x = -2$, $x = 2$ and $y = \frac{3}{2}x$ are asymptotes to the curve. The curve has turning points at $(-3, -5)$ and $(3, 5)$. It also has a stationary point of inflexion at the origin O.



(i) State the range of values of x for which the graph is concave downwards.

(ii) Sketch the graph of $y = \frac{1}{f(x)}$.

(iii) Sketch the graph of $y = f'(x)$.

Solution.

Part (a). Observe that

$$A : y \mapsto -y \implies A^{-1} : y \mapsto -y$$
$$B : y \mapsto y - 1 \implies B^{-1} : y \mapsto y + 1$$
$$C : x \mapsto x + 2 \implies C^{-1} : x \mapsto x - 2$$

Hence,

$$y = \frac{2x+1}{2x+2} \xmapsto{C^{-1}} y = \frac{2(x-2)+1}{2(x-2)+2} = \frac{2x-3}{2x-2} \xmapsto{B^{-1}} y + 1 = \frac{2x-3}{2x-2} \xmapsto{A^{-1}} -y + 1 = \frac{2x-3}{2x-2}$$

Thus,

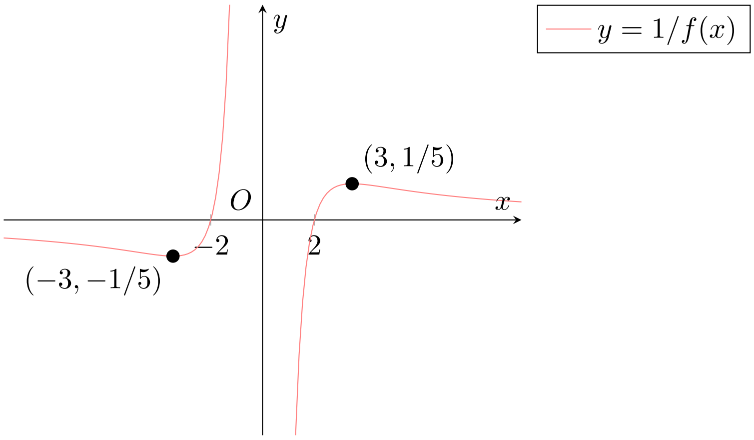
$$y = 1 - \frac{2x-3}{2x-2} = \frac{1}{2x-2}.$$

Part (b).

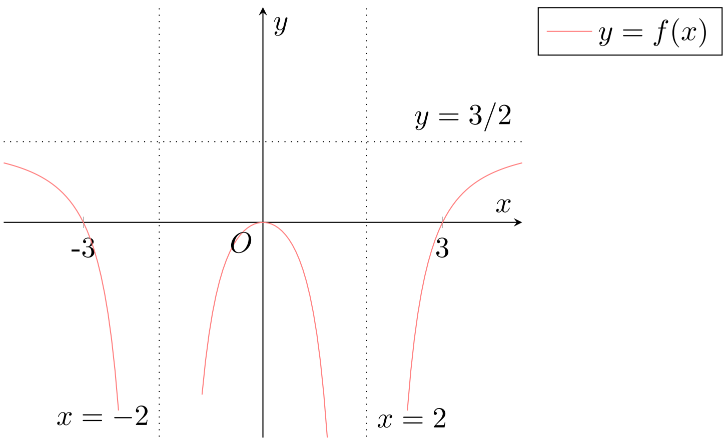
Part (b)(i). From the graph, we clearly have

$$x < -2 \text{ or } 0 < x < 2.$$

Part (b)(ii).



Part (b)(iii).



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Problem 7. A curve has parametric equations

$$x = 3u^2, \quad y = 6u.$$

(a) Find the equations of the normal to the curve at the point $P\left(3p^2, 6p\right)$, where p is a non-zero constant.

(b) The normal meets the x-axis at Q and the y-axis at R. Find the coordinates of Q and of R.

(c) Find two possible expressions for the area bounded by the normal and the axes in terms of p, stating the range of values of p in each case.

(d) Given that p is positive and increasing at a rate of 2 units/s, find the rate of change of the area of the triangle in terms of p.

Solution.

Part (a). Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}u}{\mathrm{d}x/\mathrm{d}u} = \frac{6}{6u} = \frac{1}{u}.$$

At $u = p$, the gradient of the normal is hence $-\frac{1}{1/p} = -p$. Thus, the equation of the normal at P is

$$y - 6p = -p\left(x - 3p^2\right).$$

Part (b). At Q, $y = 0$. Hence, $x = 6 + 3p^2$, whence $Q(6 + 3p^2, 0)$. At R, $x = 0$. Hence, $y = 6p + 3p^2$, whence $R(0, 6p + 3p^2)$.

Part (c). When $p > 0$, the area of the triangle is $\frac{1}{2}(6p+3p^3)(6+3p^2)$ units^2. When $p < 0$, the area of the triangle is $-\frac{1}{2}(6p + 3p^3)(6 + 3p^2)$ units^2.

Part (d). Let the area of the triangle be A unit^2. Since $p > 0$, we have

$$A = \frac{1}{2}(6p + 3p^3)(6 + 3p^2).$$

Thus,

$$\frac{\mathrm{d}A}{\mathrm{d}p} = \frac{1}{2}\left[(6p + 3p^3)(6p) + (6 + 3p^2)(6 + 9p^2)\right] = \frac{9}{2}\left(5p^4 + 12p^2 + 4\right).$$

Hence, the rate of change of area of the triangle is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}p} \cdot \frac{\mathrm{d}p}{\mathrm{d}t} = 9\left(5p^4 + 12p^2 + 4\right) \text{ units}^2\text{/s}.$$

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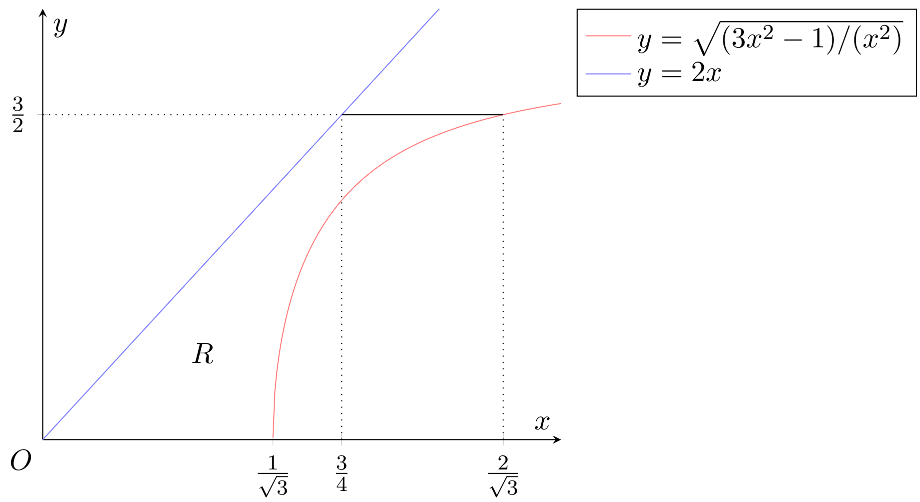
Problem 8. The shaded region R is bounded by the lines $y = 2x$, $y = \frac{3}{2}$, the x-axis and the curve $y = \sqrt{\frac{3x^2-1}{x^2}}$.



(a) By using the substitution $x = \frac{1}{\sqrt{3}} \sec\theta$, find the exact value of $\int_{1/\sqrt{3}}^{2/\sqrt{3}} \sqrt{\frac{3x^2-1}{x^2}} \, \mathrm{d}x$.

(b) Hence, find the exact area of the shaded region R.

(c) Find the volume of the solid generated when R is rotated through 2π radians about the x-axis, giving your answer in 3 decimal places.

Solution.

Part (a). Note that

$$x = \frac{1}{\sqrt{3}} \sec\theta \implies \mathrm{d}x = \frac{1}{\sqrt{3}} \sec\theta \tan\theta \, \mathrm{d}\theta,$$

with the bounds of integration going from $\theta = 0$ to $\pi/3$.

$$\int_{1/\sqrt{3}}^{2/\sqrt{3}} \sqrt{\frac{3x^2 - 1}{x^2}} \, \mathrm{d}x = \int_0^{\pi/3} \sqrt{\frac{\tan^2\theta}{\frac{1}{3}\sec^2\theta}} \frac{1}{\sqrt{3}} \sec\theta \tan\theta \, \mathrm{d}\theta = \int_0^{\pi/3} \tan^2\theta \, \mathrm{d}\theta$$

$$= \int_0^{\pi/3} \left(\sec^2\theta - 1\right) \mathrm{d}\theta = [\tan\theta - t]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3}.$$

Part (b).

$$\text{Area } R = \underbrace{\frac{1}{2}\left[\frac{2}{\sqrt{3}} + \left(\frac{2}{\sqrt{3}} - \frac{3}{4}\right)\right]\frac{3}{2}}_{\text{area of trapezium}} - \int_{1/\sqrt{3}}^{2/\sqrt{3}} \sqrt{\frac{3x^2-1}{x^2}}\,\mathrm{d}x = \frac{\pi}{3} - \frac{9}{16} \text{ units}^2.$$

Part (c).

$$\text{Volume} = \underbrace{\frac{1}{3}\pi\left(\frac{3}{2}\right)^2\left(\frac{3}{4}\right)}_{\text{volume of cone}} + \underbrace{\pi\left(\frac{2}{\sqrt{3}} - \frac{3}{4}\right)\left(\frac{3}{2}\right)^2}_{\text{volume of cylinder}} - \pi\int_{1/\sqrt{3}}^{2/\sqrt{3}} \left(\sqrt{\frac{3x^2-1}{x^2}}\right)^2 \mathrm{d}x$$
$$= 1.907 \text{ units}^3.$$

* * * * *

Problem 9. It is given that $y = \arccos(2x)\arcsin(2x)$, where $-0.5 \le x \le 0.5$, and $\arccos(2x)$ and $\arcsin(2x)$ denote their principal values.

(a) Show that $\left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 8 = 0$. Hence, find the MacLaurin series for y up to and including the term in x^3, giving the coefficients in exact form.

(b) Use your expansion from part (a) and integration to find an approximate value for $\int_{0.1}^{0.2}\left(\frac{2}{x}\right)^3 \arccos(2x)\arcsin(2x)\,\mathrm{d}x$, correct to 4 decimal places.

(c) A student, Adam, claims that the approximation in part (b) is accurate. Without performing any further calculations, justify whether Adam's claim is valid.

(d) Suggest one way to improve the accuracy of the approximated value obtained in part (b).

Solution.

Part (a). Differentiating y with respect to x,

$$y' = 2\left(\frac{\arccos(2x) - \arcsin(2x)}{\sqrt{1-4x^2}}\right).$$

Differentiating once more,

$$y'' = \frac{2}{\sqrt{1-4x^2}}\left(-\frac{4}{\sqrt{1-4x^2}}\right) + \frac{2\left[\arccos(2x) - \arcsin(2x)\right](-8x)}{-2\left(1-4x^2\right)^{3/2}}$$
$$= \frac{1}{1-4x^2}\left[-8 + 4x \cdot \frac{2\left[\arccos(2x) - \arcsin(2x)\right]}{\sqrt{1-4x^2}}\right] = \frac{1}{1-4x^2}\left(-8 + 4xy'\right).$$

Hence,

$$\left(1-4x^2\right)y'' - 4xy' + 8 = 0.$$

Differentiating with respect to x, we get

$$\left(1-4x^2\right)y''' - 12xy'' - 4y' = 0.$$

Evaluating y, y', y'' and y''' at $x = 0$, we get

$$y(0) = 0, \quad y'(0) = \pi, \quad y''(0) = -8, \quad y'''(0) = 4\pi.$$

Hence,

$$y = \pi x - 4x^2 + \frac{2}{3}\pi x^3 + \cdots.$$

.

Part (b).

$$\begin{aligned}\int_{0.1}^{0.2}\left(\frac{2}{x}\right)^3 \arccos(2x)\arcsin(2x)\,\mathrm{d}x &= 8\int_{0.1}^{0.2} x^{-3}\arccos(2x)\arcsin(2x)\,\mathrm{d}x\\ \approx 8\int_{0.1}^{0.2} x^{-3}\left[\pi x - 4x^2 + \frac{2}{3}\pi x^3\right]\mathrm{d}x &= 8\int_{0.1}^{0.2}\left(\pi x^{-2} - 4x^{-1} + \frac{2}{3}\pi x\right)\mathrm{d}x\\ = 8\left[-\frac{\pi}{x} - 4\ln|x| + \frac{2}{3}\pi x\right]_{0.1}^{0.2} &= 105.1585 \text{ (4 d.p.)}.\end{aligned}$$

Part (c). Adam's claim is valid. Since the approximation for $\arccos(2x)\arcsin(2x)$ is accurate for x near 0, and we are integrating over $(0.1, 0.2)$ (which is close to 0), the integral approximation should also be accurate.

Part (d). Consider more terms in the MacLaurin series of $y = \arccos(2x)\arcsin(2x)$ and use the improved series in the approximation for the integral.

* * * * *

Problem 10. The function f is defined by

$$f : x \mapsto 3\sin\left(2x - \frac{1}{6}\pi\right), \quad 0 \leq x \leq k.$$

(a) Show that the largest exact value of k such that f^{-1} exists is $\frac{1}{3}\pi$. Find $f^{-1}(x)$.

(b) It is given that $k = \frac{1}{3}\pi$. In a single diagram, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, labelling your graphs clearly.

The function h is defined by $h : x \mapsto 3\sin\left(2x - \frac{1}{6}\pi\right)$, $0 \leq x \leq \frac{1}{3}\pi$, $x = \frac{1}{12}\pi$.
Another function g is defined by $g : x \mapsto \left|3 - \frac{1}{x}\right|$, $-3 \leq x \leq 3$, $x \neq 0$.

(c) Explain clearly why gh exists. Find $gh(x)$ and its range.

(d) Supposing $(gh)^{-1}$ exists for some restriction, find the exact value of x for which $(gh)^{-1}(x) = 0$. Show your working clearly.

Solution.

Part (a). For f^{-1} to exist, f must be one-one. Since $\frac{1}{3}\pi$ is the first maximum point of f, it is the largest value of k.

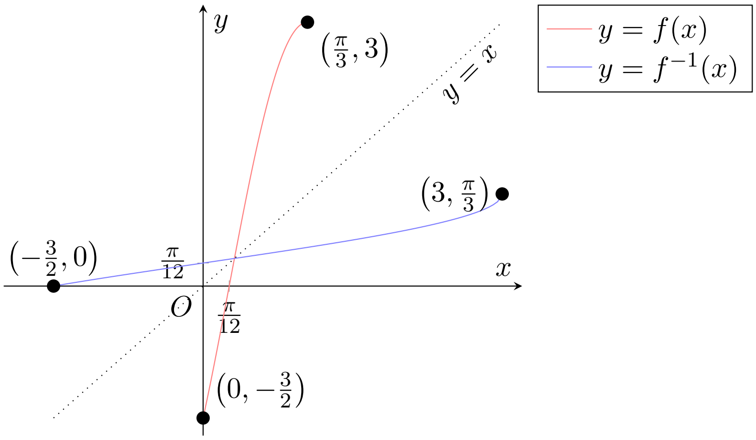
Let $y = f(x)$.

$$3\sin\left(2x - \frac{1}{6}\pi\right) = y \implies \sin\left(2x - \frac{1}{6}\pi\right) = \frac{y}{3}$$
$$\implies 2x - \frac{1}{6}\pi = \arcsin\left(\frac{y}{3}\right) \implies x = \frac{1}{2}\arcsin\left(\frac{y}{3}\right) + \frac{1}{12}\pi.$$

Hence,

$$f^{-1}(x) = \frac{1}{2}\arcsin\left(\frac{x}{3}\right) + \frac{1}{12}\pi.$$

Part (b).



Part (c). Since $R_h = [-\frac{3}{2}, 0) \cup (0, 3]$ and $D_g = [-3, 0) \cup (0, 3]$, we have $R_h \subseteq D_g$, whence gh exists.

$$gh(x) = g\left[3\sin\left(2x - \frac{1}{6}\pi\right)\right] = \left|3 - \frac{1}{3\sin\left(2x - \frac{1}{6}\pi\right)}\right|.$$

When $h(x) = \frac{1}{3}$, $gh(x) = 0$. When $x \to \frac{1}{12}\pi$, $gh(x) \to \infty$. Hence, $R_{gh} = [0, \infty)$.

Part (d).

$$gh^{-1}(x) = 0 \implies x = gh(0) = \left|3 - \frac{1}{3\sin\left(-\frac{1}{6}\pi\right)}\right| = \frac{11}{3}.$$

* * * * *

Problem 11.

(a) (i) Express $\frac{2x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+c}{x^2+1}$, where A, B and C are constants to be found.

(ii) Evaluate $\int_0^1 \frac{\ln(1+x^2)}{(x+1)^2}\,\mathrm{d}x$, giving your answer in the form $a\pi - \ln b$, where a and b are positive constants to be found.

(b) Find $\int \sin^3(kx)\,\mathrm{d}x$, where k is a constant.

Solution.

Part (a).

Part (a)(i). Clearing denominators, we have

$$2x = A(x^2+1) + (Bx+C)(x+1) = (A+B)x^2 + (B+C)x + (A+C).$$

Comparing coefficients of x^2, x and constant terms, we have

$$\begin{cases} A + B = 0 \\ B + C = 2 \\ A + C = 0 \end{cases}$$

Hence, $A = -1$, $B = 1$ and $C = 1$, giving

$$\frac{2x}{(x+1)(x^2+1)} = -\frac{1}{x+1} + \frac{x+1}{x^2+1}.$$

Part (a)(ii). Note that
$$\frac{x+1}{x^2+1} = \frac{1}{2}\left(\frac{2x}{x^2+1} + \frac{2}{x^2+1}\right).$$

Hence,

$$\int_0^1 \frac{2x}{(x+1)(x^2+1)} \, \mathrm{d}x = \int_0^1 \left[-\frac{1}{x+1} + \frac{1}{2}\left(\frac{2x}{x^2+1} + \frac{2}{x^2+1}\right)\right] \mathrm{d}x$$
$$= \left[-\ln|x+1| + \frac{1}{2}\left(\ln(x^2+1) + 2\arctan x\right)\right]_0^1 = -\frac{\ln 2}{2} + \frac{\pi}{4}.$$

Thus,

$$\int_0^1 \frac{\ln(1+x^2)}{(x+1)^2} \, \mathrm{d}x = \left[-\frac{\ln(1+x^2)}{x+1}\right]_0^1 + \int_0^1 \frac{2x}{(x+1)(x^2+1)} \, \mathrm{d}x$$
$$= -\frac{\ln 2}{2} + \left(-\frac{\ln 2}{2} + \frac{\pi}{2}.\right) = -\ln 2 + \frac{\pi}{4}.$$

Hence, $a = \frac{1}{4}$ and $b = 2$.

Part (b). Note that $\sin 3u = 3\sin u - 4\sin^3 u$, whence $\sin^3 u = \frac{3\sin u - \sin 3u}{4}$.

$$\int \sin^3(kx) \, \mathrm{d}x = \int \frac{3\sin(kx) - \sin(3kx)}{4} \, \mathrm{d}x = \frac{-3\cos(kx) + \frac{1}{3}\cos(3kx)}{4k} + C.$$

* * * * *

Problem 12. Alan wants to sign up for a triathlon competition which requires him to swim for 1.5 km, cycle for 30 km and run for 10 km. He plans a training programme as follows: In the first week, Alan is to swim 400 m, cycle 1 km and run 400 m. Each subsequent week, he will increase his swimming distance by 50 m, his cycling distance by 15% and his running distance by r%.

(a) Given that Alan will run 829.44 m in Week 5, show that $r = 20$. Hence, determine the distance that Alan will run in Week 20, giving your answer to the nearest km.

(b) Determine the week when Alan first achieves the distances required for all three categories of the competition.

During a particular running practice, Alan plans to run q metres in the first minute. The distance he covers per minute will increase by 80 m for the next four minutes. Subsequently, he will cover 6% less distance in a minute than that in the previous minute.

(c) Show that the distance, in metres, Alan will cover in the sixth minute is $0.94q + 300.8$, and hence find the minimum value of q, to the nearest metre, such that he can eventually complete 10 km.

While training, Alan suffers from inflammation and needs medication. The concentration of the medicine in the bloodstream after administration can be modelled by the recurrence relation
$$C_{n+1} = C_n \mathrm{e}^{-\left(p + \frac{1}{n+100}\right)},$$

where n represents the number of complete hours from which the medicine is first taken and p is the decay constant.

(d) The dosage of the medicine prescribed for Alan is 200 mg and the concentration of the medicine dropped to approximately 168 mg one hour later. It is given that his pain will be significant once the concentration falls below 60 mg. Determine after which complete hour he would feel significant pain and should take his medicine again.

Solution.

Part (a). Let R_n m be the distance ran in the nth week. We have $R_1 = 400$ and $R_{n+1} = \left(1 + \frac{r}{100}\right) R_n$, whence

$$R_n = 400\left(1 + \frac{r}{100}\right)^{n-1}.$$

Since $R_5 = 829.44$, we have

$$400\left(1 + \frac{r}{100}\right)^4 = 829.44 \implies \left(1 + \frac{r}{100}\right)^4 = 2.0736 \implies 1 + \frac{r}{100} = 1.2 \implies r = 20.$$

Hence,

$$R_20 = 400 \cdot 1.2^19 = 12779.2 = 13000,$$

rounded to the nearest thousand. Hence, Alan will run 13 km in week 20.

Part (b). Let S_n, C_n be the distance swam and cycled in week n, respectively, in metres.

We have $S_1 = 400$ and $S_{n+1} = S_n + 50$, whence

$$S_n = 400 + (n-1)50.$$

Consider $S_n \geq 1500$. Then $n \geq 23$.

We have $C_1 = 1000$ and $C_{n+1} = 1.15C_n$, whence

$$C_n = 1000\left(1.15^{n-1}\right).$$

Consider $C_n \geq 30000$. Then $n \geq \log_{1.15} 30 = 25.3$.

Consider $R_n \geq 10000$. Then $n \geq 1 + \log_{1.2} 25 = 18.7$.

Hence, the minimum n required is 26. Thus, in week 26, Alan will achieve all distances required.

Part (c). The distance Alan will run in the 6th minute is $(q+4\cdot 80)(1-0.06) = 0.94q+300.8$.

Let d_n be the distance travelled in the nth minute, where $n \geq 6$. We have $d_6 = 0.94q + 300.8$ and $d_{n+1} = (1 - 0.06)d_n$, whence

$$d_n = 0.94^{n-6} d_6.$$

The total distance Alan will eventually travel is thus given by

$$5q + 80(4 + 3 + 2 + 1) + \sum_{n=6}^{\infty} (0.94)^{n-6} d_6 = 5q + 800 + \frac{0.94q + 300.8}{1 - 0.94}.$$

Let the above expression be greater than 10000. Then $q \geq 202.581$. Hence, $\min q = 203$.

Part (d). We have $C_0 = 200$ and $C_1 = 168$. We hence have

$$168 = 200\mathrm{e}^{-(p+\frac{1}{100})},$$

whence $p = 0.16435$. Using G.C., the first time $C_n \leq 60$ occurs when $n = 7$. Thus, after 7 complete hours, Alan will feel significant pain and should take his medicine again.

JC1 Promotional Examination - H2 Further Mathematics 9649

Problem 1. Given that $z = f(x, y)$ is a differentiable function and $f(x, y) = k$ is the level curve of f that passes through the point P, show that the gradient vector ∇f is perpendicular to the tangent of the level curve at P.

Solution. Let x and y be parametrized t. Implicitly differentiating $f(x, y) = k$ with respect to t, we obtain

$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \implies \begin{pmatrix}\partial f/\partial x\\ \partial f/\partial y\end{pmatrix} \cdot \begin{pmatrix}\mathrm{d}x/\mathrm{d}t\\ \mathrm{d}y/\mathrm{d}t\end{pmatrix} = 0.$$

Observe that $\langle \partial f/\partial x,\, \partial f/\partial y\rangle$ is exactly ∇f. Taking $t = x$, we also have $\langle \mathrm{d}x/\mathrm{d}t,\, \mathrm{d}y/\mathrm{d}t\rangle = \langle 1,\, \mathrm{d}y/\mathrm{d}x\rangle$, which is the tangent vector of the level curve at P. Since the dot product of the two vectors is 0, they must be perpendicular.

* * * * *

Problem 2. A harvesting model is given by $\frac{\mathrm{d}P}{\mathrm{d}t} = (20 - P)\left(P^2 - 30P + h\right)$ where P is the population of the resource at time t and h is the constant harvest rate. It is given that at $t = 0$, $P = 50$. Find the range of values of h such that $P = 20$ in the long run.

Solution. Let $f(P) = (20 - P)\left(P^2 - 30P + h\right)$. For $P = 20$ in the long run, we need $f(P) < 0$ for all $P \in (20, 50]$. Observe that $20 - P < 0$ for all $P \in (20, 50]$. We hence need $P^2 - 30P + h > 0$ for all $P \in (20, 50]$. However, observe that $P^2 - 30P + h$ is strictly increasing after $P > 15$. Thus, we only require the quadratic to be non-negative at $P = 20$, whence $20^2 - 30(20) + h \geq 0$, giving $h \geq 200$.

* * * * *

Problem 3.

(a) Describe the locus given by $|2\mathrm{i} + 1 + \mathrm{i}z| = |4\mathrm{i} - 5 - z|$.

S is the set of complex numbers z for which

$$|2\mathrm{i} + 1 + \mathrm{i}z| \geq |4\mathrm{i} - 5 - z| \text{ and } |z + 2 - 3\mathrm{i}| \leq \sqrt{13}.$$

(b) On a single Argand diagram, shade the region corresponding to S.

(c) Find the set of values of $\arg(z - 8\mathrm{i})$.

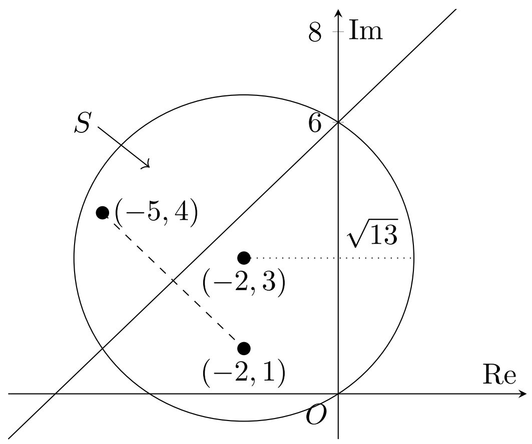
Solution.

Part (a). Note that $|2\mathrm{i} + 1 + \mathrm{i}z| = |2 - \mathrm{i} + z| = |z - (-2 + \mathrm{i})|$. We hence have

$$|z - (-2 + \mathrm{i})| = |z - (-5 + 4\mathrm{i})|,$$

which describes the perpendicular bisector of the line segment joining $(-2, 1)$ and $(-5, 4)$.

Part (b).



Part (c). Clearly, $\min \arg(z - 8\mathrm{i}) = -\frac{\pi}{2}$ (where $z = 6\mathrm{i}$). Let $A(-2, 3)$ and $B(0, 8)$. Let C be the point on the circle such that BC is tangent to the circle. We have $\angle ABO = \arctan \frac{2}{5}$. Since $AB = \sqrt{5^2 + 2^2} = \sqrt{29}$, we also have $\angle CBA = \arcsin \frac{\sqrt{13}}{\sqrt{29}}$. Thus,

$$\max \arg(z - 8\mathrm{i}) = -\frac{\pi}{2} - \arctan \frac{2}{5} - \arcsin \frac{\sqrt{13}}{\sqrt{29}} = -2.68,$$

whence

$$-2.68 \leq \arg(z - 8\mathrm{i}) \leq -\frac{\pi}{2}.$$

* * * * *

Problem 4. A tuition agency is designing a revision programme to help students to prepare for their A-level examinations. There are three types of revisions that the agency can run each day: Basic, Intermediate and Challenging. Due to resource constraints, the Challenging revision cannot be run consecutively. The revision programme consists of n days. Let a_n be the number of possible programmes for the duration.

(a) Explain why $a_n = 2(a_{n-1} + a_{n-2})$.

(b) Solve the recurrence relation and find a_n in terms of n.

(c) For a 10 days revision programme, given that both the 1st and 10th days consist of the Challenging revision, find the number of possible programmes.

Solution.

Part (a). Suppose the nth day ran Basic or Intermediate. The agency was hence free to run any programme on the $n - 1$th day, thus contributing $2 \cdot a_{n-1}$ total programmes towards a_n.

Now suppose the nth day ran Challenging. The agency could have only ran Basic or Intermediate on the $n - 1$th day. This hence contributes $1 \cdot 2 \cdot a_{n-2}$ total programmes towards a_n.

Hence, $a_n = 2a_{n-1} + 2a_{n-2} = 2(a_{n-1} + a_{n-2})$.

Part (b). Consider the characteristic polynomial of the second-order recurrence relation:

$$x^2 - 2x - 2 = 0 \implies x = 1 \pm \sqrt{3}.$$

Hence,

$$a_n = A\left(1+\sqrt{3}\right)^n + B\left(1-\sqrt{3}\right)^n$$

for some constants A and B.

Observe that $a_0 = 1$ (since there is only one way to do nothing). This gives $A + B = 1$.
Also observe that $a_1 = 3$. Hence, $(A+B) + \sqrt{3}(A-B) = 3$.
Solving, we get $A = \frac{1}{2} + \frac{1}{\sqrt{3}}$ and $B = \frac{1}{2} - \frac{1}{\sqrt{3}}$. Thus,

$$\alpha_n = \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\left(1+\sqrt{3}\right)^n + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(1-\sqrt{3}\right)^n.$$

Part (c). Since the first day ran Challenging, there are only two options for the second day (Basic and Intermediate). Likewise, since the tenth day ran Challenging, there are only two options for the ninth day. The remaining six days are free. This gives a total of $2 \cdot 2 \cdot a_6 = 1792$ possibilities.

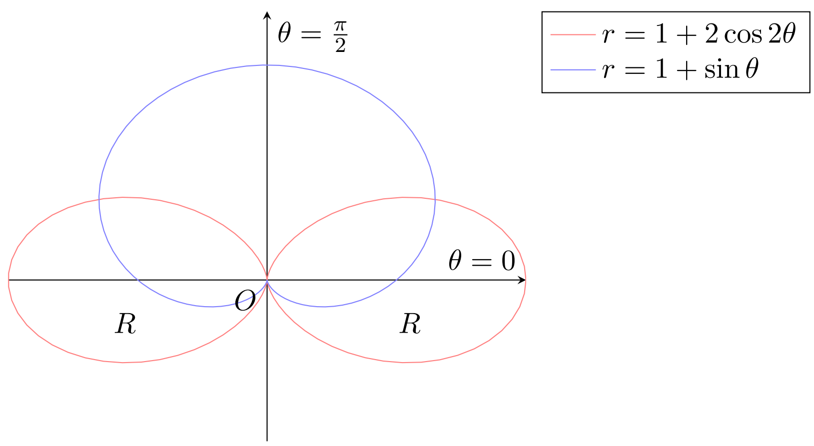
* * * * *

Problem 5. The curve C_1 has equation

$$\left(x^2+y^2\right)^{3/2} = 2x^2.$$

(a) Show that the polar equation of C_1 is $r = 1 + 2\cos 2\theta$.

The curve C_2 has polar equation $r = 1 + \sin\theta$.
The diagram below shows the region R enclosed by C_1 and C_2.



(b) Find the exact area of R.

(c) Find the perimeter of R.

Solution.

Part (a).

$$\left(x^2+y^2\right)^{3/2} = 2x^2 \implies \left(r^2\right)^{3/2} = 2(r\cos\theta)^2 \implies r = 2\cos^2\theta = 1 + 2\cos 2\theta.$$

Part (b). Observe that $t = -\frac{\pi}{2}$ is tangent to the pole to both C_1 and C_2. Now consider the intersections of C_1 and C_2.

$$2\cos^2\theta = 2 - 2\sin^2\theta = 1 + \sin\theta \implies 2\sin^2\theta + \sin\theta - 1 = (2\sin\theta - 1)(\sin\theta + 1) = 0.$$

We hence have $\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$, whence $\theta = \frac{1}{6}\pi, \frac{5}{6}\pi, -\frac{1}{2}\pi$.

$$\begin{aligned}
\text{Area } R &= 2\left(\frac{1}{2}\int_{-\pi/2}^{\pi/6}\left[(1+\cos 2\theta)^2-(1+\sin\theta)^2\right]\,\mathrm{d}\theta\right)\\
&= \int_{-\pi/2}^{\pi/6}\left(1+2\cos 2\theta+\cos^2 2\theta-1-2\sin\theta-\sin^2\theta\right)\,\mathrm{d}\theta\\
&= \int_{-\pi/2}^{\pi/6}\left(2\cos 2\theta+\frac{1+\cos 4\theta}{2}-2\sin\theta-\frac{1-\cos 2\theta}{2}\right)\,\mathrm{d}\theta\\
&= \int_{-\pi/2}^{\pi/6}\left(\frac{5}{2}\cos 2\theta+\frac{1}{2}\cos 4\theta-2\sin\theta\right)\,\mathrm{d}\theta\\
&= \left[\frac{5}{4}\sin 2\theta+\frac{1}{8}\sin 4\theta+2\cos\theta\right]_{-\pi/2}^{\pi/6} = \frac{27}{16}\sqrt{3}\text{ units}^2.
\end{aligned}$$

Part (c). Observe that for C_1, $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2\sin 2\theta$, while for C_2, $\frac{\mathrm{d}r}{\mathrm{d}\theta} = \cos\theta$. Hence, the perimeter of R is

$$2\int_{-\pi/2}^{\pi/6}\left(\sqrt{(1+\cos 2\theta)^2+(-2\sin 2\theta)^2}+\sqrt{(1+\sin\theta)^2+\cos^2\theta}\right)\,\mathrm{d}\theta = 11.7\text{ units}.$$

* * * * *

Problem 6. A civil engineer is designing a decorative water feature for a garden. The profile of the water feature is modelled by the curve $y = \sin(x^2)$, for $-\sqrt{\pi} \leq x \leq \sqrt{\pi}$. The region R is bounded by this curve and the x-axis.

(a) To create the actual water feature, the region R is rotated by π radians about the y-axis forming a symmetrical, bowl-shaped structure. Find the exact volume of the water feature.

(b) Water is poured into the bowl of the water feature at a rate of 2 units^3 per second. Given that the bowl is initially empty, find the rate of change of the depth of the water when the depth is at $\frac{\sqrt{3}}{2}$ units.

(c) The engineer also needs to know the total length of the curved surface of the profile of the water feature. Estimate, to 4 decimal places, the arc length of the profile curve from $x = 0$ to $x = 1.25$ using Simpson's rule with 4 strips.

Solution.

Part (a).

$$\text{Volume} = 2\pi\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x = \pi\left[-\cos(x^2)\right]_0^{\sqrt{\pi}} = 2\pi\text{ units}^3.$$

Part (b). Let the depth of the water be h. We have $V = \pi\int_0^h \arcsin(y)\,\mathrm{d}y$, whence $\frac{\mathrm{d}V}{\mathrm{d}h} = \pi\arcsin(h)$. Hence,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V/\mathrm{d}t}{\mathrm{d}V/\mathrm{d}h} = \frac{2}{\pi\arcsin(h)}.$$

When $h = \frac{\sqrt{3}}{2}$, we have $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{2}{\pi\cdot\pi/3}$. Thus, the rate of change of the depth of water is $\frac{6}{\pi^2}$ units/s.

Part (c). Note that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\cos\left(x^2\right)$. Hence, the arc length is given by

$$\int_0^{1.25} \sqrt{1 + (2x\cos(x^2))^2}\,\mathrm{d}x.$$

Let $f(x) = \sqrt{1 + (2x\cos(x^2))^2}$. By Simpson's rule, the arc length is approximately

$$\frac{1.25 - 0}{3 \cdot 4}\left[f(0) + f\left(\frac{5}{16}\right) + 2f\left(\frac{10}{16}\right) + 4f\left(\frac{15}{16}\right) + f(1.25)\right] = 1.6671 \text{ units (4 d.p.)}.$$

* * * * *

Problem 7. It is given that $y = f(x)$ satisfies the following differential equation:

$$\left(x^3 + 1\right) y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y^2 = 2, \quad \text{where } y = 2 \text{ when } x = 0.$$

(a) By using the substitution $z = y^2$, solve the differential equation and find the value of y when $x = 1$.

(b) A point on the graph, initially at $x = 1$, varies such that x is increasing at a rate of $\mathrm{e}^{\sqrt{t}}$ units/s, where t represents time in seconds. Show that $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{19}{12}\mathrm{e}^{\sqrt{t}}$ at that instance and use the Euler's method with step length 0.2 to find an approximation of the value of y when $t = 1$.

(c) Explain whether the approximation in part (b) is an underestimation or an overestimation of the true value.

Solution.

Part (a). Note that $z' = 2y \cdot y'$. Hence,

$$\left(x^3 + 1\right) y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y^2 = \left(x^3 + 1\right)\frac{1}{2}z' + 3x^2z = 2 \implies z' + \frac{6x^2}{x^3 + 1}z = \frac{4}{x^3 + 1}.$$

The integrating factor is hence

$$\text{I.F.} = \exp\int \frac{6x^2}{x^3 + 1}\,\mathrm{d}x = \exp\left(2\ln\left(x^3 + 1\right)\right) = \left(x^3 + 1\right)^2.$$

Multiplying through, we get

$$\left(x^3 + 1\right)^2 z' + 6x^2\left(x^3 + 1\right) z = \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(x^3 + 1\right)^2 z\right] = 4\left(x^3 + 1\right)$$

$$\implies \left(x^3 + 1\right)^2 z = \int 4\left(x^3 + 1\right)\,\mathrm{d}x = x^4 + 4x + C \implies y^2 = \frac{x^4 + 4x + C}{\left(x^3 + 1\right)^2}.$$

When $x = 0$, $y = 2$, giving $C = 4$. Hence,

$$y = \sqrt{\frac{x^4 + 4x + 4}{\left(x^3 + 1\right)^2}}.$$

When $x = 1$, $y = \frac{3}{2}$.

Part (b).

$$\left(x^3+1\right)y\frac{\mathrm{d}y}{\mathrm{d}x}+3x^2y^2=\left(1^3+1\right)\left(\frac{3}{2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+3\left(1^2\right)\left(\frac{3}{2}\right)^2=2$$
$$\implies \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{19}{12} \implies \frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{19}{12}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{19}{12}\mathrm{e}^{\sqrt{t}}.$$

Let $f(y,t)=-\frac{19}{12}\mathrm{e}^{\sqrt{t}}$, $t_0=0$, $y_0=1.5$. Using Euler's method with $h=0.2$,

$$y_{n+1}=y_n+0.2f(t_n,y_n)$$

Using G.C.,

$$y_1=1.1833,\quad y_2=0.68808,\quad y_3=0.09204,\quad y_4=-0.59503,\quad y_5=-1.36958.$$

Hence, when $t=1$, $y\approx-1.37$.

Part (c). Observe that $\sqrt{t}>0$ and $\mathrm{e}^{\sqrt{t}}>0$ for all $t\in\mathbb{R}$. Thus,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=-\frac{19}{12}\frac{1}{2\sqrt{t}}\mathrm{e}^{\sqrt{t}}<0,$$

whence y is concave downwards, making the approximation an overestimate.

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Problem 8.

(a) Use De Moivre's theorem to find a polynomial expression for $\cos 5\theta$ in terms of u, where $u=\cos\theta$.

(b) Write down the five values of θ, $0\le\theta\le\pi$, for which $\cos 5\theta=0$.

(c) Find in trigonometric form, the roots of the equation $16z^4-20z^2+5=0$.

(d) Express the roots found in part (c) in exact surd form. Hence, find the value of

$$\sin^2\frac{\pi}{10}+\sin^2\frac{3\pi}{10}+\sin^2\frac{7\pi}{10}+\sin^2\frac{9\pi}{10}.$$

Solution.

Part (a).

$$\begin{aligned}\cos 5\theta&=\operatorname{Re}\left(\cos\theta+\mathrm{i}\sin\theta\right)^5\\&=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\\&=\cos^5\theta-10\cos^3\theta\left(1-\cos^2\theta\right)+5\cos\theta\left(1-\cos^2\theta\right)^2\\&=\cos^5\theta-10\cos^3\theta\left(1-\cos^2\theta\right)+5\cos\theta\left(1-2\cos^2\theta+\cos^4\theta\right)\\&=16\cos^5\theta-20\cos^3\theta+5\cos\theta=16u^5-20u^3+5u.\end{aligned}$$

Part (b).

$$\theta=\frac{\pi}{10},\frac{3\pi}{10},\frac{5\pi}{10},\frac{7\pi}{10},\frac{9\pi}{10}.$$

Part (c).

$$16z^4-20z^2+5=0\implies 16z^5-20z^3+5z=0,\quad z\neq0.$$

Let $z=\cos\theta$. Then $\cos 5\theta=0$, whence $\theta=\frac{\pi}{10},\frac{3\pi}{10},\frac{5\pi}{10},\frac{7\pi}{10},\frac{9\pi}{10}$. Hence,

$$z=\cos\frac{\pi}{10},\cos\frac{3\pi}{10},\cos\frac{7\pi}{10},\cos\frac{9\pi}{10}.$$

Note that we reject $z=\cos\frac{5\pi}{10}$ since $z\neq0$.

Part (d). By the quadratic formula,

$$z^2 = \frac{5 \pm \sqrt{5}}{8} \implies z = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}}.$$

The corresponding trigonometric forms are

$$\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}, \quad \cos\frac{3\pi}{10} = \sqrt{\frac{5-\sqrt{5}}{8}}$$
$$\cos\frac{7\pi}{10} = -\sqrt{\frac{5-\sqrt{5}}{8}}, \quad \cos\frac{9\pi}{10} = -\sqrt{\frac{5+\sqrt{5}}{8}}.$$

Hence,

$$\sin^2\frac{\pi}{10} + \sin^2\frac{3\pi}{10} + \sin^2\frac{7\pi}{10} + \sin^2\frac{9\pi}{10} = 4 - \left(\cos^2\frac{\pi}{10} + \cos^2\frac{3\pi}{10} + \cos^2\frac{7\pi}{10} + \cos^2\frac{9\pi}{10}\right)$$
$$= 4 - \left(\frac{5+\sqrt{5}}{8} + \frac{5-\sqrt{5}}{8} + \frac{5-\sqrt{5}}{8} + \frac{5+\sqrt{5}}{8}\right) = 4 - \frac{20}{8} = \frac{3}{2}.$$

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Problem 9. Suppose the complex number w is a root of the equation $z^9 - 1 = 0$.

(a) (i) Express all the roots of this equation in the form w^n, $n \in \mathbb{Z}$, $0 \leq n \leq 8$, where w is a complex number to be determined.

(ii) Show that $\sum_{r=0}^{8} w^r = 0$.

(iii) Show that $w^2 + w^7 = 2\cos\frac{4\pi}{9}$.

(iv) Using the results above, deduce that $16\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{6\pi}{9}\cos\frac{8\pi}{9} = 1$.

Let point O be the origin, and points A, B and C represent the complex numbers w^2, $2\mathrm{i}w^2$ and $\frac{w}{u^*}$ respectively, where $u = \frac{1}{3}\left(\cos\frac{5\pi}{18} - \mathrm{i}\sin(5\pi)18\right)$.

(b) (i) Find the modulus and arguments of the complex numbers $\frac{w}{u^*}$ and $2iw^2$, and illustrate the points A, B and C on a clearly labelled Argand diagram.

(ii) Find the area of triangle ABC.

Solution.

Part (a).

Part (a)(i).

$$z^9 - 1 = 0 \implies z^9 = \mathrm{e}^{2\pi n\mathrm{i}} \implies z = \mathrm{e}^{2\pi n\mathrm{i}/9}.$$

Hence, the roots are w^n, where $w = \mathrm{e}^{2\pi\mathrm{i}/9}$.

Part (a)(ii).

$$\sum_{r=0}^{8} w^r = \frac{w^9 - 1}{w - 1} = \frac{1 - 1}{w - 1} = 0$$

Part (a)(iii).

$$w^2 + w^7 = w^2 + w^{-2} = 2\cos\left(2 \cdot \frac{2\pi}{9}\right) = 2\cos\frac{4\pi}{9}$$

Part (a)(iv).

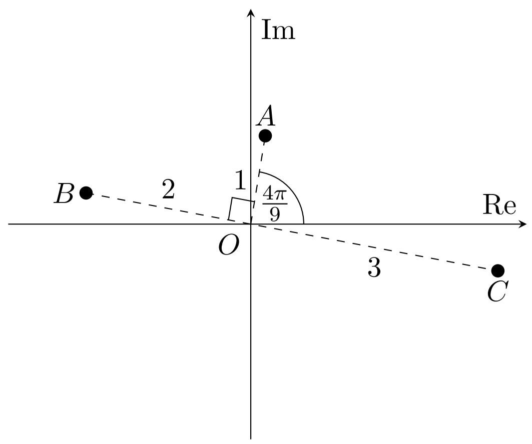
$$\begin{aligned}
&16\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{6\pi}{9}\cos\frac{8\pi}{9}\\
&=\left(w+w^8\right)\left(w^2+w^7\right)\left(w^3+w^6\right)\left(w^4+w^5\right)\\
&=\left(w^5+w^6+w^{12}+w^{13}\right)\left(w^5+w^8+w^{10}+w^{13}\right)\\
&=\left(w^{10}+w^{13}+w^{15}+w^{18}\right)+\left(w^{11}+w^{14}+w^{16}+w^{19}\right)\\
&\quad+\left(w^{17}+w^{20}+w^{22}+w^{25}\right)+\left(w^{18}+w^{21}+w^{23}+w^{26}\right)\\
&=\left(w+w^4+w^6+1\right)+\left(w^2+w^5+w^7+w\right)\\
&\quad+\left(w^8+w^2+w^4+w^7\right)+\left(1+w^3+w^5+w^8\right)\\
&=2\left(1+w+w^2+w^3+w^4+w^5+w^6+w^7+w^8\right)-\left(w^3+w^6\right)\\
&=-2\cos\frac{6\pi}{9}=1.
\end{aligned}$$

Part (b).

Part (b)(i). Note that $w^2=e^{4\pi i/9}$. Hence, $\left|w^2\right|=1$ and $\arg w^2=\frac{4\pi}{9}$. Likewise, $\left|2iw^2\right|=2$ and $\arg\left(2iw^2\right)=\frac{4\pi}{9}+\frac{\pi}{2}=\frac{17}{18}\pi$. Lastly, note that

$$\frac{w}{u^*}=\frac{uw}{uu^*}=\frac{1/3\cdot e^{-5\pi i/18}\cdot e^{2\pi i/9}}{1/9}=3e^{-i\pi/18},$$

whence $\left|\frac{w}{u^*}\right|=3$ and $\arg\left(\frac{w}{u^*}\right)=-\frac{\pi}{18}$.



Part (b)(ii). Observe that B, O and C are collinear. Hence,

$$[\triangle ABC]=\frac{1}{2}(1)(2+3)=\frac{5}{2}\text{ units}^2.$$

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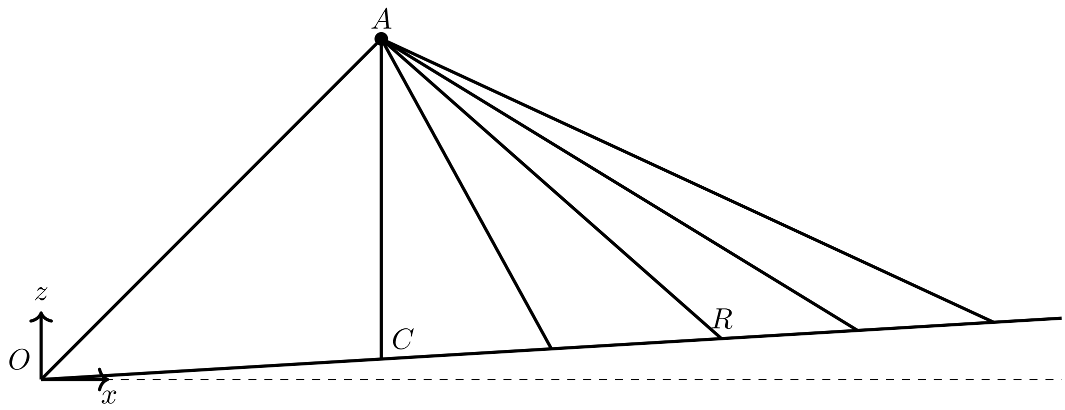
Problem 10. The diagram below shows the elevation view of a single vertical tower cable-stayed-inclined bridge which stretches across a river. The bridge deck is supported by the tower, a main cable, and four smaller cables. Points are defined relative to an origin O, the point of intersection between the main cable and the deck. The x-, y- and z-axes are in the directions east, north and vertically upwards respectively, with units in metres. The deck of the bridge is modelled as a plane. Points P and Q are on this plane and have coordinates $(20,0,1)$ and $(40,4,2)$ respectively.



(a) Find the Cartesian equation of the plane.

Point A is at the top of the vertical tower and has coordinates $(20, 1, 20)$. Point C is the intersection of the tower and the deck. The tower and the five cables are attached on the deck along the line passing through Points O and C.

(b) The bridge is considered stable if the distance between C and the foot of perpendicular from A to the deck does not exceed 1 m. Comment whether the bridge is stable. Show your working clearly.

(c) One of the cables, which is installed at a point R, has the same length as the main cable. Find the coordinates of R.

(d) Find the acute angle that the deck makes with the horizontal plane.

Solution.

Part (a). Observe that $\langle 20,\, 0,\, 1\rangle$ and $\langle 40,\, 4,\, 2\rangle$ are parallel to the plane. Note that

$$\begin{pmatrix}20\\0\\1\end{pmatrix} \times \begin{pmatrix}40\\4\\2\end{pmatrix} = \begin{pmatrix}-4\\0\\80\end{pmatrix} \parallel \begin{pmatrix}-1\\0\\20\end{pmatrix}.$$

Hence, the vector equation of the plane is

$$\mathbf{r} \cdot \begin{pmatrix}-1\\0\\20\end{pmatrix} = 0,$$

whence the Cartesian equation is $-x + 20z = 0, y \in \mathbb{R}$.

Part (b). Observe that $\overrightarrow{AC} = k\,\langle 0,\, 0,\, 1\rangle$ for some $k \in \mathbb{R}$. Hence, $\overrightarrow{OC} = \langle 20,\, 1,\, 20 - k\rangle$. Since C lies on the deck, we have

$$\begin{pmatrix}20\\1\\20-k\end{pmatrix} \cdot \begin{pmatrix}-1\\0\\20\end{pmatrix} = 0,$$

whence $k = 19$ and $\overrightarrow{OC} = \langle 20,\, 1,\, 1\rangle$.

Let F be the foot of perpendicular of A to the deck. Note that $\overrightarrow{OF} \cdot \langle -1,\, 0,\, 20\rangle = 0$ and $\overrightarrow{AF} = \lambda\,\langle -1,\, 0,\, 20\rangle$ for some $\lambda \in \mathbb{R}$. Thus,

$$\left[\begin{pmatrix}20\\1\\20\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\20\end{pmatrix}\right] \cdot \begin{pmatrix}-1\\0\\20\end{pmatrix} = 0, \implies \lambda = -\frac{380}{401} \implies \overrightarrow{OF} = \frac{1}{401}\begin{pmatrix}8400\\401\\420\end{pmatrix}.$$

Hence,

$$\overrightarrow{FC} = \begin{pmatrix} 20 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{401}\begin{pmatrix} 8400 \\ 401 \\ 420 \end{pmatrix} \implies \left|\overrightarrow{FC}\right| = \sqrt{0.948^2 + 0^2 + (-0.0474)^2} = 0.949 < 1.$$

The bridge is thus stable.

Part (c). We have $\left|\overrightarrow{AR}\right| = \left|\overrightarrow{OA}\right| = \sqrt{20^2 + 1^2 + 20^1} = \sqrt{801}$. Since O, C and R are collinear, we also have $\overrightarrow{OR} = \mu \langle 20, 1, 1\rangle$ for some $\mu \in \mathbb{R}$. Thus,

$$\left|\overrightarrow{AR}\right| = \left|\mu\begin{pmatrix} 20 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 20 \\ 1 \\ 20 \end{pmatrix}\right| = \sqrt{(20\mu - 20)^2 + (\mu - 1)^2 + (\mu - 20)^2} = \sqrt{801}.$$

Using G.C., $\mu = 2.09$, whence $R(41.9, 2.09, 2.09)$.

Part (d). Let θ be the acute angle between the deck and the horizontal. Note that the horizontal plane has normal vector $\langle 0, 0, 1\rangle$. Thus,

$$\cos\theta = \frac{|\langle 0, 0, 1\rangle - \langle -1, 0, 20\rangle|}{|\langle 0, 0, 1\rangle|\,|\langle -1, 0, 20\rangle|} = \frac{20}{\sqrt{401}} \implies \theta = 2.9^\circ \text{ (2 d.p.)}.$$

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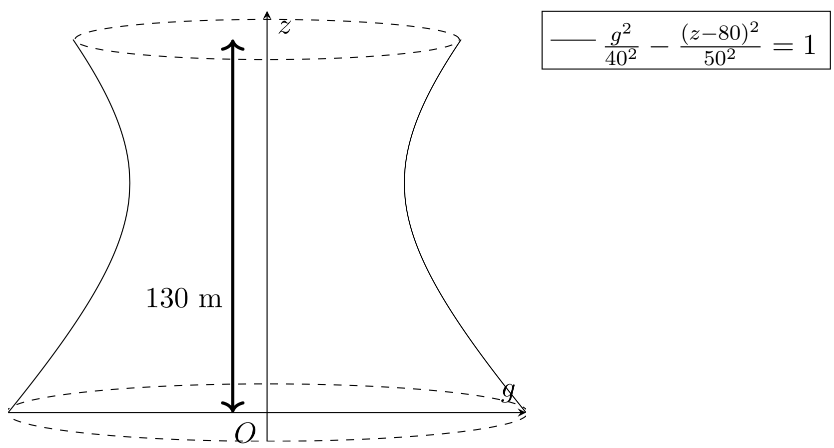
Problem 11. A nuclear reactor plant is built to meet the increased demand for electricity due to a particular country's economic developments. The cooling tower of the nuclear reactor is as shown in the figure below. The curved surface area of the cooling tower is modelled by rotating the region enclosed by a part of a hyperbolic curve about an axis. The height of the tower is 130 m.



The equation of the hyperbolic curve is given as $\frac{g^2}{40^2} - \frac{(z-80)^2}{50^2} = 1$ where g is the axis that represents the ground and z is the axis that represents the height of the reactor. The curve surface area of the tower is formed by rotating the region bounded by the hyperbolic curve, the line $z = 130$ and the g axis about the z-axis by π radians. The external curved surface area of the tower is to be painted with weather resistant paint.

(a) Find the external curved surface area of the tower. Leave your answer to the nearest m^2.

The ground is now represented by the x-y plane.

(b) Find the Cartesian equation that models the surface of the tower in terms of z, y and z.

Before the paint can be applied, a robot is programmed to go around the tower to clean and polish its surface. Assuming that the robot is negligible compared to the tower, it can be viewed as a point on the curved surface of the tower.

(c) Given that the robot is at $(40, 40, 30)$ and is about the move in the direction of $\langle 3, -4\rangle$ parallel to the x-y plane, determine whether the robot will be ascending or descending in height.

The robot is now at $(40, 40, 130)$ on the surface of the tower. A signal needs to be transmitted from the ground to the robot such that the signal travels in a straight line and its direction must be normal to the surface of the tower where the robot is at.

(d) Find the coordinates on the ground where the signal can be transmitted to the robot.

Solution.

Part (a). Note that $g = \sqrt{40^2\left[\frac{(z-80)^2}{50^2}+1\right]}$. Using G.C.,

$$\text{Area} = 2\pi\int_0^{130} g\sqrt{1+\left(\frac{\mathrm{d}g}{\mathrm{d}z}\right)^2}\,\mathrm{d}z \approx 45552 \text{ units}^2.$$

Part (b). For every constant value of z, we will have the value of g such that $x^2+y^2=g^2$. Hence,

$$\frac{x^2+y^2}{40^2}-\frac{(z-80)^2}{50^2}=1.$$

Part (c). Implicitly differentiating the above expressing with respect to x and y, we have

$$\frac{\partial z}{\partial x}=\left(\frac{5}{4}\right)^2\frac{x}{z-80},\quad \frac{\partial z}{\partial y}=\left(\frac{5}{4}\right)^2\frac{y}{z-80}.$$

Evaluating at $(40, 40, 30)$, we have that $\nabla z = -\frac{5}{4}\langle 1,\, 1\rangle$. Hence,

$$\nabla z\cdot\frac{1}{\sqrt{3^2+5^2}}\begin{pmatrix}3\\-4\end{pmatrix}=\frac{1}{4}>0.$$

Thus, the robot is ascending.

Part (d). At $(40, 40, 130)$, we have $\nabla z = \langle 5/4,\, 5/4\rangle$. The equation of the tangent plane at that point is hence

$$z = 130+\frac{5}{4}(x-40)+\frac{5}{4}(y-40),$$

which has vector equation

$$\mathbf{r}\cdot\begin{pmatrix}5\\5\\-4\end{pmatrix}=-120.$$

The line of the signal is hence given by

$$\mathbf{r}=\begin{pmatrix}40\\40\\130\end{pmatrix}+\lambda\begin{pmatrix}5\\5\\-4\end{pmatrix},\quad \lambda\in\mathbb{R}.$$

Setting $z=0$, we have $\lambda = 130/4$, whence $x=y=405/2$. The required coordinates are thus $\left(\frac{405}{2}, \frac{405}{2}, 0\right)$.