# The Singapore Mathematical Olympiad Compendium

https://asdia.dev/projects/smo.pdf

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# Contents

# Introduction

## About the SMO

The Singapore Mathematical Olympiad (SMO) is an annual mathematical competition organized by the Singapore Mathematical Society. The Olympiad is split into three categories, namely Junior, Senior and Open. Within each category, there are two rounds of papers (the second being an invitational round).

## About the Compendium

This compendium is written with the goal to provide clear and concise solutions to each and every problem that has appeared in the SMO. The compendium has been split into two parts: Problems and Solutions. Within each part, SMO papers are further categorized by year, category and round.

## Other Resources

Links to relevant resources, such as *Way Tan*'s reviews of Round 1 papers, and *Art of Problem Solving* threads for Round 2 questions, are provided at the beginning of solutions. A less well-known but equally enriching resource for olympiad preparation is the *SIMO Retiree Blog*.

## Contributing

The source code for this compendium can be found on GitHub at `asdia0/Compendium`. Contributions are more than welcome.

## Acknowledgements

Not all solutions presented in this compendium are original; credits will be displayed at the start of relevant solutions. The template for this compendium is adapted from Evan Chen's LaTeX style file `evan.sty`.

# Part I.

# Problems

# 1. SMO 2020

## 1.1. Open Section

### 1.1.1. Round 1 Problems

Solutions can be found in Section 6.1.1.

1. If $S$ is the sum of all the *real* roots of the equation $x^2 + \dfrac{1}{x^2} = 2020^2 + \dfrac{1}{2020^2}$ find $\lfloor S \rfloor$.

2. Find the largest positive integer $x$ that satisfies the equation

$$(\lfloor x \rfloor - 2020)^2 + (\lceil x \rceil - 2030)^2 = (\lfloor x \rfloor - \lceil x \rceil + 10)^2.$$

*(Note: If you think that the above equation has no solution in the positive integers, enter your answer as "0".)*

3. Let $S_n = \dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \cdots + \dfrac{1}{(2n-1) \times (2n+1)}$. Find the value of $n$ such that $S_n$ takes the value of 0.48.

4. Given that the three planes in the Cartesian space with equations $2x + 4y + 6z = 5$, $3x + 5y + 2z = 6$ and $8x + 14y + az = b$ have a common line of intersection, find the value of $a + b$.

5. Let $i$ be the complex number $\sqrt{-1}$, and $n$ be the smallest positive integer such that $(\sqrt{3} + i)^n = a$, where $a$ is a real number. Find the value of $\lfloor n - a \rfloor$.

6. In the three-dimensional Cartesian space, let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors along three mutually perpendicular $x$, $y$ and $z$-axes respectively. Three straight lines $l_1$, $l_2$ and $l_3$ have equations defined by

$$\begin{aligned} l_1\text{: } \mathbf{r} &= (4 + \lambda)\mathbf{i} + (5 + \lambda)\mathbf{j} + (6 + \lambda)\mathbf{k}, \\ l_2\text{: } \mathbf{r} &= (4 + 3\mu)\mathbf{i} + (5 - \mu)\mathbf{j} + (6 - 2\mu)\mathbf{k}, \\ l_3\text{: } \mathbf{r} &= (1 + 6\nu)\mathbf{i} + (2 + 2\nu)\mathbf{j} + (3 + \nu)\mathbf{k}, \end{aligned}$$

where $\mu$, $\lambda$ and $\nu$ are real numbers. If the area of the triangle enclosed by the three lines $l_1$, $l_2$ and $l_3$ is denoted by $S$, find the value of $10S^2$.

7. Given that $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^2 - b^2) = (a - b)(f(a) + f(b))$$

for all real numbers $a$ and $b$, and that $f(1) = \dfrac{1}{101}$, find the value of $\displaystyle\sum_{k=1}^{100} f(k)$.

8. Find the sum of all the positive integers $n$ such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.

*(Note: If you think that there are infinitely many such positive integers $n$ that satisfy that above conditions, enter your answer as "9999".)*

9. Assume that

$$(x+2+m)^{2019} = a_0 + a_1(x+1) + a_2(x+1)^2 + \cdots + a_{2019}(x+1)^{2019}.$$

Find the largest possible integer $m$ such that

$$(a_0 + a_2 + a_4 + \cdots + a_{2018})^2 - (a_1 + a_3 + a_5 + \cdots + a_{2019})^2 \leq 2020^{2019}.$$

10. Given that $S = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{1}{\sqrt{n(n+k)}}$, find the value of $\lfloor (S+2)^2 \rfloor$.

11. Let $A = \{1, 2, \cdots, 10\}$. Count the number of ordered pairs $(S_1, S_2)$, where $S_1$ and $S_2$ are non-intersecting and non-empty subsets of $A$ such that the largest number in $S_1$ is smaller than the smallest number in $S_2$. For example, if $S_1 = \{1, 4\}$ and $S_2 = \{5, 6, 7\}$, then $(S_1, S_2)$ is such an ordered pair.

12. Each cell of a $2020 \times 2020$ table is filled with a number which is either 1 or $-1$. For $u = 1, \ldots, 2020$, let $R_i$ be the product of all the numbers in the $i$th row and let $C_i$ be the product of all the numbers in the $i$th column. Suppose $R_i + C_i = 0$ for all $i = 1, \ldots, 2020$. What is the least number of $-1$'s in the table?

13. Assume that the sequence $\{a_k\}_{k=1}^{\infty}$ follows an arithmetic progression with $a_2 + a_4 + a_9 = 24$. Find the maximum value of $S_8 \times S_{10}$, where $S_k$ denotes the sum $a_1 + a_2 + \cdots + a_k$.

14. Consider all functions $g : \mathbb{R} \to \mathbb{R}$ satisfying the conditions that

    a) $|g(a) - g(b)| \leq |a - b|$ for any $a, b \in \mathbb{R}$;

    b) $g(g(g(0))) = 0$.

    Find the *largest* possible value of $g(0)$.

15. A sequence $\{a_i\}_{i=1}^{\infty}$ is called a *good* sequence if $\dfrac{S_{2n}}{S_n}$ is a constant for all $n \geq 1$, where $S_k$ denotes the sum $a_1 + a_2 + \cdots + a_k$. Suppose it is known that the sequence $\{a_i\}_{i=1}^{\infty}$ is a *good* sequence that follows an arithmetic progression. Determine $a_{2020}$ if $a_1 = 1 \neq a_2$.

16. Determine the smallest positive integer $p$ such that the system

$$\begin{cases} 6x + 4y + 3z = 0 \\ 4xy + 2yz + pxz = 0 \end{cases}$$

has more the one set of real solutions in $x$, $y$, $z$.

17. Let $ABC$ be a triangle with $a = BC$, $b = AC$ and $c = AB$. It is given that $c = 100$ and

$$\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{4}{3}.$$

Let $P$ be a point on the inscribed circle of $\triangle ABC$. Find the maximum value of

$$PA^2 + PB^2 + PC^2.$$

18. Find the largest positive integer $n$ less than 2020 such that $\binom{n-1}{k} - (-1)^k$ is divisible by $n$ for $k = 0, 1, \ldots, n-1$.

19. Assume that $\{a_k\}_{k=1}^{\infty}$ is a sequence with the property that for any distinct positive integers $m$, $n$, $p$, $q$ with $m+n=p+q$, the following equality always holds:

$$\frac{a_m+a_n}{(a_m+1)(a_n+1)}=\frac{a_p+a_q}{(a_p+1)(a_q+1)}.$$

Given $a_1=0$ and $a_2=\dfrac{1}{2}$, determine $\dfrac{1}{1-a_5}$.

*(Hint: Consider* $c_k=\dfrac{1}{a_k+1}-\dfrac{1}{2}$ *for all positive integer* $k$*.)*

20. In the triangle $ABC$, the incircle touches the sides $BC$, $CA$, $AB$ at $D$, $E$, $F$ respectively. The line segments $ED$ and $AB$ are extended to intersect at $P$ such that $AB=BP=PD$. Suppose $CA=9$. Find the value of $[ABC]^2$, where $[ABC]$ is the area of the triangle $ABC$.

21. In an acute-angled triangle $ABC$, $AB=75$, $AC=53$, the external bisector of $\angle A$ on $CA$ produced meets the circumcircle of triangle $ABC$ at $E$, and $F$ is the foot of the perpendicular from $E$ onto $AB$. Find the value of $AF\times FB$.

22. Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence with $a_k<a_{k+1}$ for all $k=1,2,3,\cdots$ formed by arranging all the terms in the set $\left\{2^r+2^s+2^t : 0\le r<s<t\right\}$ in increasing order. Find the largest value of the integer $n$ such that $a_n\le 2020$.

23. Let $n$ be a positive integer and $S$ be the set of all numbers that can be written in the form $\displaystyle\sum_{i=2}^{k} a_{i-1}a_i$ with $a_1,\ldots,a_k$ being positive integers that sum to $n$. Suppose the average value of all the numbers in $S$ is 88199. Determine $n$.

24. Let $x$, $y$, $z$ and $w$ be real numbers such that $x+y+z+w=5$. Find the minimum value of $(x+5)^2+(y+10)^2+(z+20)^2+(w+40)^2$.

25. Let $p$ and $q$ be positive integers satisfying the equation $p^2+q^2=3994(p-q)$. Determine the largest possible value of $q$.

# 2. SMO 2021

## 2.1. Open Section

### 2.1.1. Round 1 Problems

Solutions can be found in Section 7.1.1.

1. It is given that $\frac{\pi}{2} < \beta < \alpha < \frac{3\pi}{4}$, $\cos(\alpha - \beta) = \frac{12}{13}$ and $\sin(\alpha + \beta) = -\frac{3}{5}$. Find $\lfloor |2021 \sin(2\alpha)| \rfloor$.

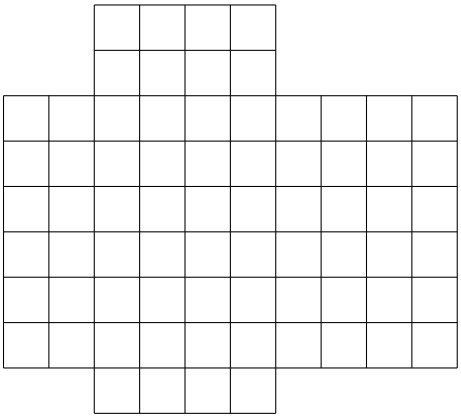
2. Find the number of solutions of the equation $|x - 3| + |x - 5| = 2$.

   *(Note: If you think that there are infinitely many solutions, enter your answer as "99999".)*

3. Evaluate $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots + 10 \times 11 \times 12$.

4. It is given that the solution of the inequality $\sqrt{81 - x^4} \leq kx + 1$ is $a \leq x \leq b$ with $b - a = 2$, where $k > 0$. Determine $\lfloor k \rfloor$.

5. The figure below shows a cross that is cut out from a $10 \times 9$ rectangular board.



   Find the total number of rectangles in the above figure.

   *(Note: A square is a rectangle.)*

6. Consider all polynomials $P(x, y)$ in two variables such that $P(0, 0) = 2020$ and for all $x$ and $y$, $P(x, y) = P(x + y, y - x)$. Find the largest possible value of $P(1, 1)$.

7. In the three-dimensional Cartesian space with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denoting the unit vectors along three perpendicular directions in a clockwise manner, the line $l$ with equation given by $\mathbf{r} \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 5\mathbf{i} - 13\mathbf{j} + 7\mathbf{k}$ intersects the plane $\Pi$ with equation $x + y + z = 16$ at the point $(a, b, c)$. Find the value of $a + b + c$.

8. Find the minimum value of $(x + 7)^2 + (y + 2)^2$ subject to the constraint $(x - 5)^2 + (y - 7)^2 = 4$.

9. Find the largest possible value $\alpha^4 + \beta^4 + \gamma^4$ among all possible sets of numbers $(\alpha, \beta, \gamma)$ that satisfy the equations

$$\begin{aligned} \alpha + \beta + \gamma &= 2 \\ \alpha^2 + \beta^2 + \gamma^2 &= 14 \\ \alpha^3 + \beta^3 + \gamma^3 &= 20. \end{aligned}$$

10. If $p$ is the product of all the non-zero real roots of the equation

$$\sqrt[9]{x^7 + 30x^5} = \sqrt[7]{x^9 - 30x^5},$$

find $\lfloor |p| \rfloor$.

11. Let $S$ be the sum of a convergent geometric series with first term 1. If the third term of the series is the arithmetic mean of the first two terms, find $\lfloor 3S + 4 \rfloor$.

12. Given that $\sin\alpha + \sin\beta = \dfrac{1}{10}$, and $\cos\alpha + \cos\beta = \dfrac{1}{9}$, find $\left\lfloor \tan^2(\alpha + \beta) \right\rfloor$.

13. Determine the number of positive integers that are divisible by 2021 and has exactly 2021 divisors (including 1 and itself).

14. Let $S = \displaystyle\sum_{k=0}^{25} \binom{100}{4k} - 2^{98}$. Find $\left\lfloor \left| \dfrac{S}{2^{48}} \right| \right\rfloor$.

15. Assume that $ABC$ is an acute triangle with $\sin(A + B) = \dfrac{3}{5}$ and $\sin(A - B) = \dfrac{1}{5}$. If $AB = 2022(\sqrt{6} - 2)$, determine $\lfloor h \rfloor$, where $h$ is the height of the triangle from $C$ on $AB$.

16. Let $a_1, a_2, \cdots$ be a sequence with $a_1 = 1$ and $a_{n+1} = \dfrac{n+2}{n} S_n$ for all $n = 1, 2, \cdots$, where $S_n = a_1 + a_2 + \cdots + a_n$. Determine the minimum integer $n$ such that $a_n \geq 2021$.

17. Each card of a stack of 101 cards has one side coloured red and the other coloured blue. Initially all cards have the red side facing up and stacked together in a deck. On each turn, Ah Meng takes 8 cards on the top, flip them over, and place them to the bottom deck. Determine the minimum number of turns required so that all the cards have the red sides facing up again.

18. Let $ABC$ be a triangle with $AB = 10$ and $\dfrac{\cos A}{\cos B} = \dfrac{AC}{BC} = \dfrac{4}{3}$. Let $P$ be a point on the inscribed circle of triangle $ABC$. Find the largest possible value of $PA^2 + PB^2 + PC^2$.

19. A basket contains 19 apples labelled by the numbers $2, 3, \ldots, 20$, and 19 bananas labelled by the numbers $2, 3, \ldots, 20$. Ah Beng picks $m$ apples and $n$ bananas from the basket. However, he needs to ensure that for any apple labelled $a$ and any banana labelled $b$ that he picks, $a$ and $b$ are relatively prime. Determine the largest possible value of $mn$.

20. Let $p(x) = ax^2 - bx + c$ be a polynomial where $a$, $b$, $c$ are positive integers and $p(x)$ has two distinct roots in $(0, 1)$. Determine the least possible value of $abc$.

21. In the triangle $ABC$, $\angle A > 90°$, the incircle touches the side $BC$ and $AC$ at $A_1$ and $B_1$ respectively. The line $A_1B_1$ meets the extension of $BA$ at $X$ such that $CXB = 90°$. Suppose $BC^2 = AB^2 + BC \cdot AC$. Find the size of $\angle A$ in degrees.

22. Find the number of positive integers $n$ such that $7n - 16$ divides $n \cdot 13^{2019}$.

23. In the acute triangle $ABC$, $P$ is a point on $AB$, $Q$ is a point on $AC$ such that $BP + CQ = PQ$. The bisector of $\angle A$ meets the circumcircle of the triangle $ABC$ at the point $R$ distinct from $A$. Suppose $\angle PRQ = 52.5°$. Find the size of $\angle BAC$ in degrees.

24. Let $S = \displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x$. Determine the value of $\lfloor S^2 \rfloor$.

25. Let $p$, $q$, $r$ be positive numbers with $p - r = 4q$ and $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be two sequences defined by $a_1 = p$, $b_1 = q$ and for $n \geq 2$,

$$a_n = pa_{n-1}, \quad b_n = qa_{n-1} + rb_{n-1}.$$

Find the value of $\displaystyle\lim_{n\to\infty} \frac{\sqrt{a_n^2 + (3b_n)^2}}{b_n}$.

# 3. SMO 2022

## 3.1. Open Section

### 3.1.1. Round 1 Problems

Solutions can be found in Section 8.1.1.

1. If $S = \sum_{k=-2021}^{2021} \frac{1}{10^k + 1}$, find $\lfloor 2S \rfloor$.

2. All the positive integers $1, 2, 3, 4, \cdots$, are grouped in the following way: $G_1 = \{1, 2\}$, $G_2 = \{3, 4, 5, 6\}$, $G_3 = \{7, 8, 9, 10, 11, 12, 13, 14\}$, that is, the set $G_n$ contains the next $2^n$ positive integers listed in ascending order after the set $G_{n-1}$, $n > 1$. If $S$ is the sum of all the positive integers from $G_1$ to $G_8$, find $\lfloor \frac{S}{100} \rfloor$.

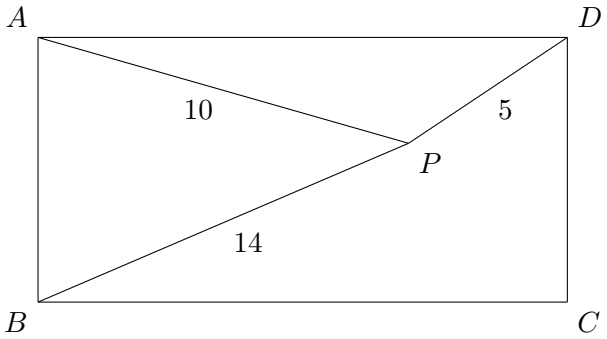
3. A sequence of one hundred positive integers $x_1, x_2, x_3, \cdots, x_{100}$ are such that

$$(x_1)^2 + (2x_2)^2 + (3x_3)^2 + (4x_4)^2 + \cdots + (100x_{100})^2 = 338350.$$

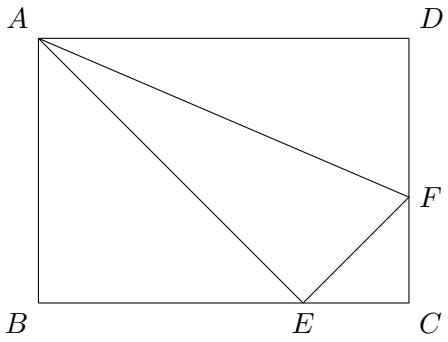
   Find the largest possible value of $x_1 + x_2 + x_3 + \cdots + x_{100}$.

4. Let $a$ and $b$ be two real numbers satisfying $a < b$, and such that for each real number $m$ satisfying $a < m < b$, the circle $x^2 + (y - m)^2 = 25$ meets the parabola $4y = x^2$ at four distinct points in the Cartesian plane. Let $S$ be the maximum possible value of $b - a$. Find $\lfloor 4S \rfloor$.

5. Let $P$ be a point within a rectangle $ABCD$ such that $PA = 10$, $PB = 14$ and $PD = 5$, as shown below. Find $\lfloor PC \rfloor$.



6. In the diagram below, the rectangle $ABCD$ has area 180 and both triangles $ABE$ and $ADF$ have areas 60. Find the area of triangle $AEF$.

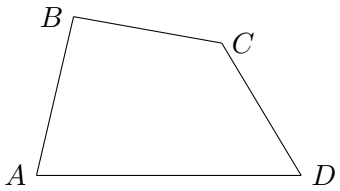



7. A tetrahedron in $\mathbb{R}^3$ has one vertex at the origin $O$ and other vertices at the points $A(6,0,0)$, $B(4,2,4)$ and $C(3,2,6)$. If $x$ is the height of the tetrahedron from $O$ to the plane $ABC$, find $\lfloor 5x^2 \rfloor$.

8. Let $x$ and $y$ be real numbers such that $(x-2)^2+(y-3)^2=4$. If $S$ is the largest possible value of $x^2+y^2$, find $\lfloor (S-17)^2 \rfloor$.

9. Let $S$ be the maximum value of $w^3-3w$ subject to the condition that $w^4+9 \leq 10w^2$. Find $\lfloor S \rfloor$.

10. In the quadrilateral $ABCD$ below, it is given that $AB=BC=CD$ and $\angle ABC = 80^\circ$ and $\angle BCD = 160^\circ$. Suppose $\angle ADC = x^\circ$. Find the value of $x$.




11. Let $a$, $b$, $c$ be integers with $ab+c=49$ and $a+bc=50$. Find the largest possible value of $abc$.

12. Find the largest possible value of $|a|+|b|$, where $a$ and $b$ are coprime integers (i.e., $a$ and $b$ are integers which have no common factors larger than 1) such that $\frac{a}{b}$ is a solution of the equation below:

$$\sqrt{4x+5-4\sqrt{x+1}}+\sqrt{x+2-2\sqrt{x+1}}=1.$$

13. Let $S$ be the set of real solutions $(x,y,z)$ of the following system of equations:

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y, \\ \dfrac{4y^2}{1+4y^2} = z, \\ \dfrac{4z^2}{1+4z^2} = x. \end{cases}$$

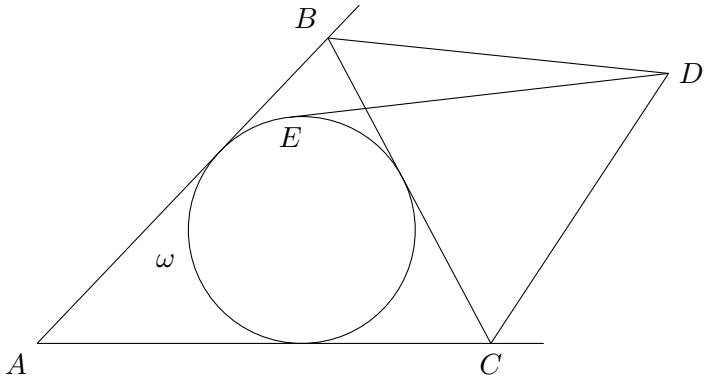
For each $(x,y,z) \in S$, define $m(x,y,z) = 2000(|x|+|y|+|z|)$. Determine the maximum value of $m(x,y,z)$ over all $(x,y,z) \in S$.

14. Assume that $t$ is a positive solution to the equation

$$t = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + t}}}}.$$

Determine the value of $t^4 - t^3 - t + 10$.

15. In the triangle $ABC$ shown in the diagram below, the external angle bisectors of $\angle B$ and $\angle C$ meet at the point $D$. The tangent from $D$ to the incircle $\omega$ of the triangle $ABC$ touches $\omega$ at $E$, where $E$ and $B$ are on the same side of the line $AD$. Suppose $\angle BEC = 112°$. Find the size of $\angle A$ in degrees.



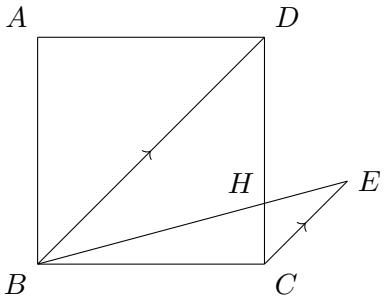

16. Find the largest integer $n$ such that $n^2 + 5n - 9486 = 10s(n)$, where $s(n)$ is the product of all digits of $n$ in the decimal representation of $n$.

(For example, $s(481) = 4 \times 8 \times 1 = 32$.)

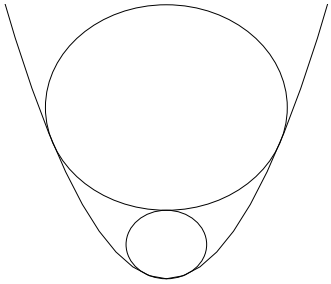
17. Find the number of integer solutions to the equation $19x + 93y = 4xy$.

18. Find the number of integer solutions to the equation $x_1 + x_2 - x_3 = 20$ with $x_1 \geq x_2 \geq x_3 \geq 0$.

19. In the diagram below, $E$ is a point outside a square $ABCD$ such that $CE$ is parallel to $BD$, $BE = BD$, and $BE$ intersects $CD$ at $H$. Given $BE = \sqrt{6} + \sqrt{2}$, find the length of $DH$.



20. The diagram below shows the region $R = \{(x, y) \in \mathbb{R}^2 | y \geq \frac{1}{2}x^2\}$ on the $xy$-plane bounded by the parabola $y = \frac{1}{2}x^2$. Let $C_1$ be the largest circle lying inside $R$ with its lowest point at the origin. Let $C_2$ be the largest circle lying inside $R$ and resting on top of $C_1$. Find the sum of radii of $C_1$ and $C_2$.



21. Find the smallest positive integer $x$ such that $3x^2 + x = 4y^2 + y$ for some positive integer $y$.

22. A group of students participates in some sports activities among 6 different types of sports. It is known that for each sport activity there are exactly 100 students in the group participating in it; and the union of all the sports activities participated by any two students is NOT the entire set of 6 sports activities. Determine the minimum number of students in the group.

23. Let $p$ and $q$ be positive prime integers such that $p^3 - 5p^2 - 18p = q^9 - 7q$. Determine the smallest value of $p$.

24. Given that $a$, $b$, $c$ are positive real numbers such that $a+b+c = 9$, find the maximum value of $a^2b^3c^4$.

25. Let $\mathbb{R}^+$ be the set of all positive real numbers. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying
$$xyf(x)\big(f(y) - f(yf(x))\big) = 1$$
for all $x, y \in \mathbb{R}^+$. Find $f(\frac{1}{2022})$.

### 3.1.2. Round 2 Problems

Solutions can be found in Section 8.1.2.

1. For $\triangle ABC$ and its circumcircle $\omega$, draw the tangents at $B$, $C$ to $\omega$ meeting at $D$. let the line $AD$ meet the circle with centre $D$ and radius $DB$ at $E$ inside $\triangle ABC$. Let $F$ be the point on the extension of $EB$ and $G$ be the point on the segment $EC$ such that $\angle AFB = \angle AGE = \angle A$. Prove that the tangent at $A$ to the circumcircle of $\triangle AFG$ is parallel to $BC$.

2. Prove that if the length and breadth of a rectangle are both odd integers, then there does not exist a point $P$ inside the rectangle such that each of the distances from $P$ to the 4 corners of the rectangle is an integer.

3. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying

$$m!! + n!! \mid f(m)!! + f(n)!!$$

for each $m, n \in \mathbb{Z}^+$, where $n!! = (n!)!$ for all $n \in \mathbb{Z}^+$.

4. Let $n$, $k$, $1 \leq k \leq n$ be fixed integers. Alice has $n$ cards in a row, where the card in position $i$ has the label $i+k$ (or $i+k-n$ if $i+k > n$). Alice starts by colouring each card either red or blue. Afterwards, she is allowed to make several moves, where each move consists of choosing two cards of different colours and swapping them. Find the minimum number of moves she has to make (given that she chooses the colouring optimally) to put the cards in order (i.e. card $i$ is at position $i$).

5. Let $n \geq 2$ be a positive integer. For any integer $a$, let $P_a(x)$ denote the polynomial $x^n + ax$. Let $p$ be a prime number and define the set $S_a$ as the set of residues mod $p$ that $P_a(x)$ attains. That is,

$$S_a = \{b \mid 0 \leq b \leq p-1, \text{ and there is } c \text{ such that } P_a(x) \equiv p \pmod{p}\}.$$

Show that the expression $\frac{1}{p-1} \sum_{a=1}^{p-1} |S_a|$ is an integer.

# 4. SMO 2023

## 4.1. Senior Section

### 4.1.1. Round 1 Problems

Solutions can be found in Section 9.1.1.

**Multiple Choice Questions**

1. Find the value of $m$ such that $2x^2 + 3x + m$ has a minimum value of 9.

   (A) $\frac{9}{8}$ (B) $-\frac{9}{8}$ (C) $\frac{81}{8}$ (D) $-\frac{81}{8}$ (E) $\frac{63}{8}$

2. Which of the following is true?

   (A) $\sin(105°) - \cos(105)^\circ = \frac{\sqrt{3}}{2}$

   (B) $\sin(105°) - \cos(105)^\circ = \frac{\sqrt{3}}{\sqrt{2}}$

   (C) $\sin(105°) + \cos(105)^\circ = \frac{1}{2}$

   (D) $\sin(105°) + \cos(105)^\circ = \frac{1}{\sqrt{3}}$

   (E) None of the above

3. If $\log_{\sqrt{2}} x = 10 - 3\log_{\sqrt{2}} 10$, find $x$.

   (A) 0.32 (B) 0.032 (C) 0.0032 (D) 0.64 (E) 0.064

4. If $(x-5)^2 + (y-5)^2 = 5^2$, find the smallest value of $(x+5)^2 + (y+5)^2$.

   (A) $225 - 100\sqrt{2}$

   (B) $225 + 100\sqrt{2}$

   (C) $225\sqrt{2}$

   (D) $100\sqrt{2}$

   (E) None of the above

5. Suppose $\cos(180° + x) = \frac{4}{5}$, where $90° < x < 180°$. Find $\tan(2x)$.

   (A) $\frac{24}{7}$ (B) $\frac{7}{24}$ (C) $-\frac{24}{7}$ (D) $-\frac{7}{24}$ (E) $-\frac{24}{25}$

**Short Questions**

6. Suppose the roots of $x^2 + 11x + 3 = 0$ are $p$ and $q$, and the roots of $x^2 + Bx - C = 0$ are $p + 1$ and $q + 1$. Find $C$.

7. If the smallest possible value of $(A-x)(23-x)(A+x)(23+x)$ is $-(48)^2$, find the value of $A$ if $A>0$.

8. Find the smallest positive odd integer greater than 1 that is a factor of
$$(2023)^{2023}+(2026)^{2026}+(2029)^{2029}.$$

9. Find the remainder of $7^{2023}+9^{2023}$ when divided by 64.

10. Let $x,y,z>1$ and let $A$ be a positive number such that $\log_x A=30$, $\log_y A=50$ and $\log_{xy}(Az)=150$. Find $(\log_A z)^2$.

11. Find the largest integer that is less than
$$\frac{3^{10}-2^{10}}{10!}\left(\frac{1}{1!9!2}+\frac{1}{2!8!2^2}+\frac{1}{3!7!2^3}+\cdots+\frac{1}{9!1!2^9}\right)^{-1}.$$
Here, $n!=n\cdot(n-1)\cdots3\cdot2\cdot1$. For example, $5!=5\cdot4\cdot3\cdot2\cdot1=120$.

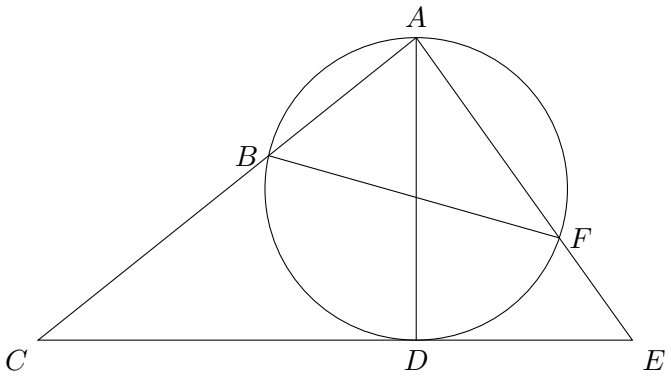
12. Consider the following simultaneous equations:
$$xy^2+xyz=91$$
$$xyz-y^2z=72$$
where $x$, $y$, and $z$ are positive integers. Find the maximum value of $xz$.

13. Let $x$ be a real number such that
$$\frac{\sin^4 x+\cos^4 x}{\sin^2 x+\cos^4 x}=\frac{8}{11}.$$
Assuming $\sin^2 x>\dfrac{1}{2}$, find the value of $\sqrt{28}\left(\sin^4 x-\cos^4 x\right)$.

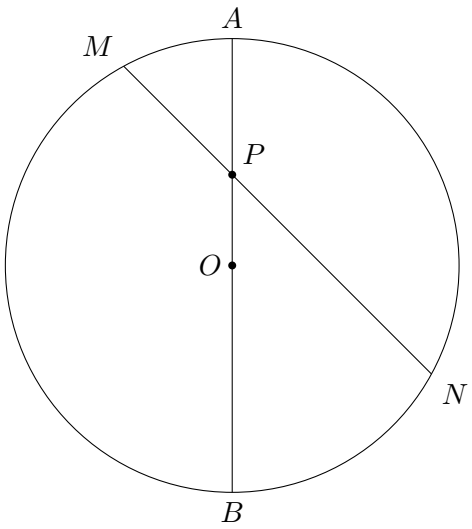
14. A sequence $a_1,a_2,\ldots$, is defined by
$$a_1=5,\quad a_2=7,\quad a_{n+1}=\frac{a_n+1}{a_{n-1}}\text{ for }n\geq2.$$
Find the value of $100\times a_{2023}$.

15. Let $C$ be a constant such that the equation $5\cos x+6\sin x-C=0$ have two distinct roots $a$ and $b$, where $0<b<a<\pi$. Find the value of $61\times\sin(a+b)$.

16. In the diagram below, $CE$ is tangent to the circle at point $D$, $AD$ is the diameter of the circle, and $ABC$, $AFE$ are straight lines. It is given that $\frac{AB}{AC}=\frac{16}{41}$ and $\frac{AF}{AE}=\frac{49}{74}$. Let $\tan(\angle CAE)=\frac{m}{n}$, where $m$, $n$ are positive integers and $\frac{m}{n}$ is a fraction in its lowest form. Find the sum $m+n$.



17. In the diagram below, $AB$ is a diameter of the circle with centre $O$, $MN$ is a chord of the circle that intersects $AB$ at $P$, $\angle BON$ and $\angle MOA$ are acute angles, $\angle MPA = 45°$, $MP = \sqrt{56}$, and $NP = 12$. Find the radius of the circle.




18. Let $f(x) = \cos^2\left(\frac{\pi x}{2}\right)$. Find the value of

$$f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \cdots + f\left(\frac{2021}{2023}\right) + f\left(\frac{2022}{2023}\right).$$

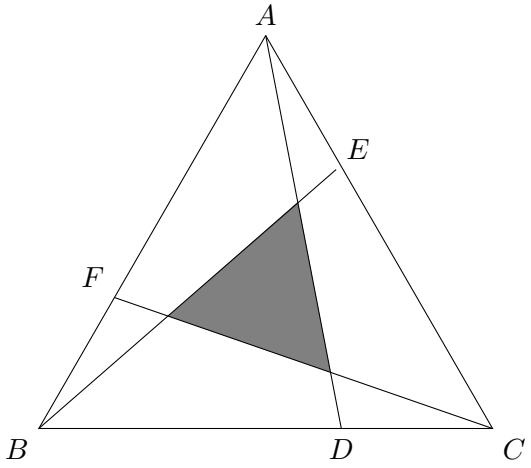
19. Find the remainder when $3^{2023}$ is divided by 215.

20. Find the sum of the prime divisors of 64000027.

21. Let $\triangle ABC$ be an equilateral triangle. $D$, $E$, $F$ are points on the sides such that

$$BD : DC = CE : EA = AF : FB = 2 : 1.$$

Suppose the area of the triangle bounded by $AD$, $BE$ and $CF$ is 2023. Find the area of $\triangle ABC$.




22. Find the number of possible ways of arranging 10 ones and 11 zeros in a row such that there are in total 13 strings of ones and zeros. For example,

1110001001110001

consists of 4 strings of ones and 3 strings of zeros.

23. Suppose there exist numbers $a$, $b$, $c$ and a function $f$ such that for any real numbers $x$ and $y$,

$$f(x+y)+f(x-y)=2f(x)+2f(y)+ax+by+c.$$

It is given that

$$f(2)=3, \quad f(3)=-5, \quad \text{and} \quad f(5)=7.$$

Find the value of $f(123)$.

24. Let $f$ be a function such that for any non-zero number $x$,

$$6xf(x)+5x^2f(1/x)+10=0.$$

Find the value of $f(10)$.

25. Find the number of triangles such that all the sides are integers and the area equals the perimeter (in number).

### 4.1.2. Round 2 Problems

Solutions can be found in Section 9.1.2.

1. Let $ABCD$ be a square, $E$ be a point on the side $DC$, $F$ and $G$ be the feet of the altitudes from $B$ to $AE$ and from $A$ to $BE$ respectively. Suppose $DF$ and $CG$ intersect at $H$. Prove that $\angle AHB = 90°$.

2. Find all positive integers $k$ such that there exists positive integers $a$, $b$ such that

$$a^2 + 4 = \left(k^2 - 4\right) b^2.$$

3. Let $n$ be a positive integer. There are $n$ islands with $n-1$ bridges connecting them such that one can travel from any island to another. One afternoon, a fire breaks out in one of the islands. Every morning, it spreads to all neighbouring islands. (Two islands are neighbours if they are connected by a bridge.) To control the spread, one bridge is destroyed every night until the fire has nowhere to spread to the next day. Let $X$ be the minimum possible number of bridges one has to destroy before the fire stops spreading. Find the maximum possible value of $X$ over all possible configurations of bridges and islands where the fire starts at.

4. Find all positive integers $m$, $n$ satisfying $n! + 2^{n-1} = 2^m$.

5. Colour a $20000 \times 20000$ square grid using 2000 different colours with 1 colour in each square. Two squares are neighbours if they share a vertex. A path is a sequence of squares so that 2 successive squares are neighbours. Mark $k$ of the squares. For each unmarked square $x$, there is exactly 1 marked square $y$ of the same colour so that $x$ and $y$ are connected by a path of squares of the same colour. For any 2 marked squares of the same colour, any path connecting them must pass through squares of all the colours. Find the maximum value of $k$.

## 4.2. Open Section

### 4.2.1. Round 1 Problems

Solutions can be found in Section 9.2.1.

1. The graph $C$ with equation $y = \dfrac{ax^2 + bx + c}{x + 2}$ has an oblique asymptote with equation $y = 4x - 6$ and one of the stationary points at $x = -4$. Find the value of $a + b + c$.

2. If $x = \dfrac{1}{1} + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \dfrac{1}{1+2+3+4} + \cdots + \dfrac{1}{1+2+3+\cdots+100}$, find the value of $\lfloor 1010x \rfloor$.

3. The set of all possible values of $x$ for which the sum of the infinite series

$$1 + \frac{1}{6}\left(x^2 - 5x\right) + \frac{1}{6^2}\left(x^2 - 5x\right)^2 + \frac{1}{6^3}\left(x^2 - 5x\right)^3 + \cdots$$

exists can be expressed as $(a, b) \cup (c, d)$, where $a < b < c < d$. Find $d - a$.

4. Find the value of $\lfloor y \rfloor$, where $y = \displaystyle\sum_{k=0}^{\infty} (2k + 1)(0.5)^{2k}$.

*(Hint: Consider the series expansion of $(1 - x)^{-2}$)*

5. The solution of the inequality $|x - 1| + |x + 1| < ax + b$ is $-1 < x < 2$. Find the value of $\lfloor a + b \rfloor$.

6. The equation $x^4 - 4x^2 + qx - r = 0$ has three equal roots. Find the value of $\left\lfloor \dfrac{3q^2}{r^2} \right\rfloor$.

7. The parabolas $y = x^2 - 16x + 50$ and $x = y^2$ intersect at 4 distinct points which lie on a circle centred at $(a, b)$. Find $|a - b|$.

8. In the 3-dimensional Euclidean space with origin $O$ and three mutually perpendicular $x$-, $y$- and $z$-axes, two planes $x + y + 3z = 4$ and $2x - z = 6$ intersect at the line $\mathbf{r} \times \begin{pmatrix} -1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ c \\ d \end{pmatrix}$. Find the value of $|a + b + c + d|$.

9. Let $x$, $y$, $z$ be real numbers with $3x + 4y + 5z = 100$. Find the minimum value of $x^2 + y^2 + z^2$.

10. Find the area of the region represented by the equation $\lfloor x \rfloor + \lfloor y \rfloor = 1$ in the region $0 \le x < 2$.

*(Note: If you think that there is no area defined by the graph, enter "0"; if you think that the area is infinite, enter "9999".)*

11. Let $ABC$ be a triangle satisfying the following conditions that $\angle A + \angle C = 2\angle B$, and $\dfrac{1}{\cos A} + \dfrac{1}{\cos C} = \dfrac{-\sqrt{2}}{\cos B}$. Determine the value of $\dfrac{2022 \cos\left(\frac{A-C}{2}\right)}{\sqrt{2}}$.

12. Find $x$ which satisfies the following equation

$$\frac{x - 2019}{1} + \frac{x - 2018}{2} + \frac{x - 2017}{3} + \cdots + \frac{x + 2}{2022} + \frac{x + 3}{2023} = 2023.$$

13. Assume that $x$ is a positive number such that $x - \frac{1}{x} = 3$ and
$$\frac{x^{10} + x^8 + x^2 + 1}{x^{10} + x^6 + x^4 + 1} = \frac{m}{n},$$
where $m$ and $n$ are positive integers without common factors larger than 1. Determine the value of $m + n$.

14. Consider the set of all possible pairs $(x, y)$ of real numbers that satisfy $(x-4)^2 + (y-3)^2 = 9$. If $S$ is the largest possible value of $\frac{y}{x}$, find the value of $\lfloor 7S \rfloor$.

15. Let $x$, $y$ be positive integers with $16x^2 + y^2 + 7xy \leq 2023$. Find the maximum value of $4x + y$.

16. Let $x$ be the largest real number such that
$$\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x.$$
Determine the value of $(2x-1)^4$.

17. Two positive integers $m$ and $n$ differ by 10 and the digits in the decimal representation of $mn$ are all equal to 9. Determine $m + n$.

18. Let $\{a_n\}$ be a sequence of positive numbers, and let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. For any positive integer $n$, let $b_n = \frac{1}{2}\left(\frac{a_{n+1}}{a_n} + \frac{a_n}{a_{n+1}}\right)$. Given that $\frac{a_n + 2}{2} = \sqrt{2S_n}$ holds for all positive integers $n$, determine the limit $\lim_{n\to\infty}(b_1 + b_2 + \cdots + b_n - n)$.

19. Let $ABC$ be a triangle with $AB = c$, $AC = b$ and $BC = a$, and satisfies the conditions $\tan C = \frac{\sin A + \sin B}{\cos A + \cos B}$, $\sin(B - A) = \cos C$ and that the area of triangle $ABC = 3 + \sqrt{3}$. Determine the value of $a^2 + c^2$.

20. **[VOID]** Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(0) = 4$ and that
$$g(xy + 1) = g(x)g(y) - g(y) - x + 2023.$$
Find the value of $g(2023)$.

21. In the triangle $ABC$, $D$ is the midpoint of $AC$, $E$ is the midpoint of $BD$, and the lines $BA$ and $CE$ are tangent to the circumcircle of the triangle $ADE$ at $A$ and $E$ respectively. Suppose the circumradius of the triangle $AED$ is $(\frac{64}{7})^{1/4}$. Find the area of the triangle $ABC$.

22. $ABCD$ is a parallelogram such that $\angle ABC < 90^\circ$ and $\sin \angle ABC = \frac{4}{5}$. The point $K$ is on the extension of $BC$ such that $DC = DK$; the point $L$ is on the extension of $DC$ such that $BC = BL$. The bisector of $\angle CDK$ intersects the bisector of $\angle LBC$ at $Q$. Suppose the circumradius of the triangle $ABD$ is 25. Find the length of $KL$.

23. A group of 200 monkeys is given the task of picking up all 3000 peanuts on the ground. Determine the maximum number $k$ such that there must be $k$ monkeys picking up the same number of peanuts. (It is possible that some lazy monkeys may not pick up any peanuts at all).

24. A chain of $n$ identical circles $C_1, C_2, \ldots, C_n$ of equal radii and centres on the $x$-axis lie inside the ellipse $E : \frac{x^2}{2023} + \frac{y^2}{333} = 1$ such that $C_1$ is tangent to $E$ internally at $(-\sqrt{2023}, 0)$, $C_n$ is tangent to $E$ internally at $(\sqrt{2023}, 0)$, and $C_i$ is tangent to $C_{i+1}$ externally for $i = 1, \ldots, n-1$. Determine the smallest possible value of $n$.

25. Let $p > 2023$ be a prime. Determine the number of positive integers $n$ such that $(n-p)^2 + 2023(2023 - 2n - 2p)$ is a perfect square.

### 4.2.2. Round 2 Problems

Solutions can be found in Section 9.2.2.

1. In a scalene triangle $ABC$ with centroid $G$ and circumcircle $\omega$ centred at $O$, the extension of $AG$ meets $\omega$ at $M$; lines $AB$ and $CM$ intersect at $P$; and lines $AC$ and $BM$ intersect at $Q$. Suppose the circumcentre $S$ of the triangle $APQ$ lies on $\omega$ and $A$, $O$, $S$ are collinear. Prove that $\angle AGO = 90°$.

2. A grid of cells is tiled with dominoes such that every cell is covered by exactly one domino. A subset $S$ of dominoes is chosen. Is it true that at least one of the following two statements is false?

   a) There are 2022 more horizontal dominoes than vertical dominoes in $S$.

   b) The cells covered by the dominoes in $S$ can be tiled completely and exactly by L-shaped tetrominoes.

3. Let $n \geq 2$ be a positive integer. For a positive integer $a$, let $Q_a(x) = x^n + ax$. Let $p$ be a prime and let $S_a = \{b \mid 0 \leq b \leq p-1, \exists c \in \mathbb{Z}, Q_a(c) \equiv b \pmod{p}\}$. Show that $\frac{1}{p-1}\sum_{a=1}^{p-1} |S_a|$ is an integer.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$, such that

$$f(x+y)((f(x)-f(y))^2 + f(xy)) = f(x^3) + f(y^3)$$

   for all integers $x$, $y$.

5. Determine all real numbers $x$ between 0 and 180 such that it is possible to partition an equilateral triangle into finitely many triangles, each of which has an angle of $x°$.

# 5. SMO 2024

## 5.1. Junior Section

### 5.1.1. Round 1 Problems

Solutions can be found in Section 10.1.1.

**Multiple Choice Questions**

1. If $x^2 + 4x + 16 = 0$, what is the value of $x^3$?

   (A) 4 (B) 8 (C) 16 (D) 64 (E) 128

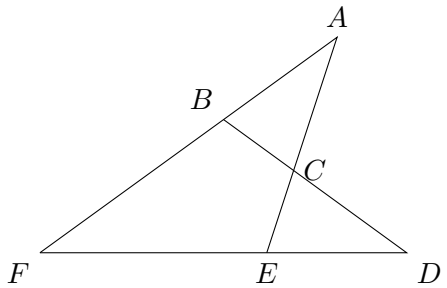
2. Let $a$ be a real number that satisfies $-1 < a < 0$. Which of the following is true?

   (A) $\pi^a < \frac{1}{\pi} < (\frac{1}{\pi})^a$
   (B) $\pi^a < (\frac{1}{\pi})^a < \frac{1}{\pi}$
   (C) $\frac{1}{\pi} < (\frac{1}{\pi})^a < \pi^a$
   (D) $\frac{1}{\pi} < \pi^a < (\frac{1}{\pi})^a$
   (E) $(\frac{1}{\pi})^a < \pi^a < \frac{1}{\pi}$

3. How many non-congruent triangles are there whose sides have integer lengths and the longest side has length 10 units?

   (A) 25 (B) 30 (C) 35 (D) 40 (E) 45

4. In the diagram below, the points $B$ and $E$ lie on $AF$ and $DF$ respectively, and $AE$ and $BD$ intersect at $C$. If $AB = AC$, $BD = BF$ and $EA = EF$, find $\angle BAC$.




   (A) 30° (B) 33° (C) 36° (D) 38° (E) 40°

5. Let $x$, $y$ and $z$ be real numbers such that $x \neq 0$, $y - z \neq 0$ and $z + x \neq 0$. If $\frac{2}{x} = \frac{4}{y-z} = \frac{5}{z+x}$, find the value of $\frac{7x-y}{y+2z}$.

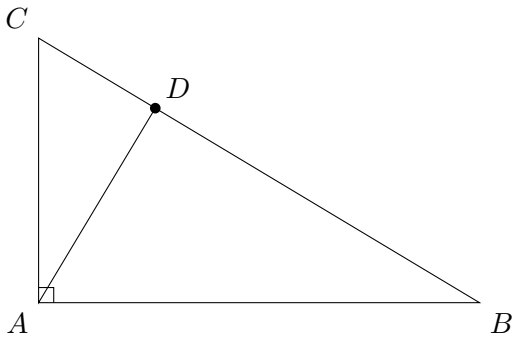
   (A) $\frac{11}{17}$ (B) $-\frac{11}{17}$ (C) $\frac{7}{13}$ (D) $-\frac{7}{13}$ (E) 1

**Short Questions**

6. Let $N$ be a 2-digit whole number. When 2692 is divided by $N$, the remainder is 13, and when 2978 is divided by $N$, the remainder is 14. Find the sum of all the possible values of $N$.

7. If $x$ is a positive integer such that $2^x + 2^{89} = 512^{10}$, find the value of $x$.

8. The diagram shows a right-angled triangle $ABC$. The sides $AC$ and $AB$ are in the ratio $3 : 5$. The point $D$ lies on $BC$ such that $AD$ is perpendicular to $BC$. Furthermore, $DB$ is 8 cm longer than $CD$. What is the length of $BC$ in cm?




9. Let $n$ be a positive integer. Suppose that $a_1, a_2, a_3, \ldots$ is a sequence of numbers defined by

$$a_1 = \sqrt{(n+3)(n-1)+4}, \quad a_k = \sqrt{(n+2k+1)a_{k-1}+4} \quad \text{for } k \geq 2.$$

If $a_{100} = 2024$, find the value of $n$.

10. Let $N$ be the smallest positive integer such that the sum of its digits is 2024. What is the sum of the digits of the number $N + 2$.

11. If $a$ and $b$ are non-zero real numbers such that $\dfrac{1}{b} - \dfrac{1}{a} = 4$, find the value of $\dfrac{3a + 7ab - 3b}{a - 3ab - b}$.

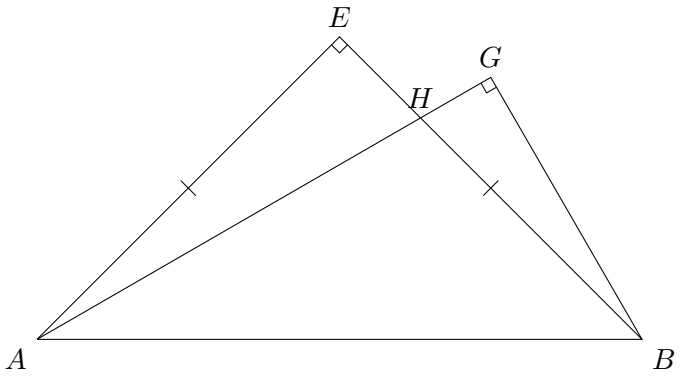
12. If the 5-digit whole number $\overline{11ab6}$ is a perfect square, find the value of $a + b$.

13. Let $N = 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + 30 \times 31$. Find the value of $N$.

14. Let $x$ and $y$ be non-zero real numbers where $x \neq y$. If $x^2 + \sqrt{3}y = 4$ and $y^2 + \sqrt{3}x = 4$, find the value of $\left(\dfrac{y}{x}\right)^2 + \left(\dfrac{x}{y}\right)^2$.

15. Find the sum of all 2-digit even numbers $N$ with the following property: $N$ is a multiple of the product of its two digits.

16. In the diagram below, $AEB$ is an isosceles right-angled triangle and $ABG$ is a 30°-60°-90° right-angled triangle with $\angle GAB = 30°$. The sides $AG$ and $BE$ intersect at $H$. If the area of triangle $AHE$ is 50 cm$^2$, find the area of triangle $BGH$ in cm$^2$.




17. Find the smallest positive integer $n$ such that $\sqrt{n} - \sqrt{n-1} < \dfrac{1}{99}$.

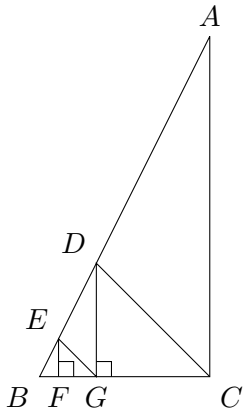
18. Let $a$, $b$ and $c$ be real numbers such that $a + b + c = 8$ and $ab + bc + ca = 0$. Find the maximum value of $3(a + b)$.

19. In the table below, every row and column is an infinite arithmetic progression.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | $\cdots$ |
| 3 | 6 | 9 | 12 | 15 | $\cdots$ |
| 5 | 9 | 13 | 17 | 21 | $\cdots$ |
| 7 | 12 | 17 | 22 | 27 | $\cdots$ |
| 9 | 15 | 21 | 27 | 33 | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ |

How many times does the number 2025 appear in the table?

20. In the diagram below, $ABC$ is a right-angled triangle. Points $D$ and $E$ lie on $AB$ while points $F$ and $G$ lie on $BC$ such that $\triangle EFG$ and $\triangle DGC$ are right-angled isosceles triangles. It is given that $DC = 3EG$ and the area of $\triangle DGC = 1$ cm$^2$. What is the area of $\triangle ADC$ in cm$^2$?

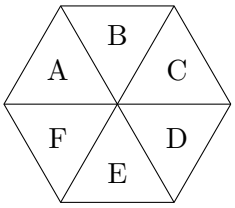



21. How many different 4-tuples $(a, b, c, d)$ are there, where $a$, $b$, $c$ and $d$ are positive integers, such that

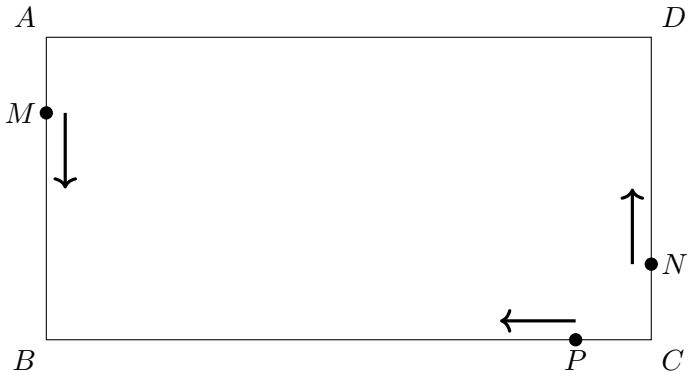
$$a > b > c > d, \quad a + b + c + d = 2024 \quad \text{and} \quad a^2 - b^2 + c^2 - d^2 = 2024?$$

22. Points $A$ and $B$ lie on the graph of $y = x^2 + 5x - 8$ such that $A$, $B$ and the origin $O$ are collinear and $|OB| = 2\,|OA|$. It is given that $A$ lies in the first quadrant. Find $|AB|^2$.

23. The diagram below shows a regular hexagon that is divided into six congruent triangular regions $A$, $B$, $C$, $D$, $E$ and $F$. Two triangular regions are adjacent if they share a common side. For example, $A$ and $B$ are adjacent but $A$ and $C$ are not adjacent. In how many ways can we colour these regions $A$, $B$, $C$, $D$, $E$ and $F$ using six different colours such that the adjacent regions do not receive the same colour? (Note that not all six colours need to be used in colouring the six regions and non-adjacent regions can receive the same colour).

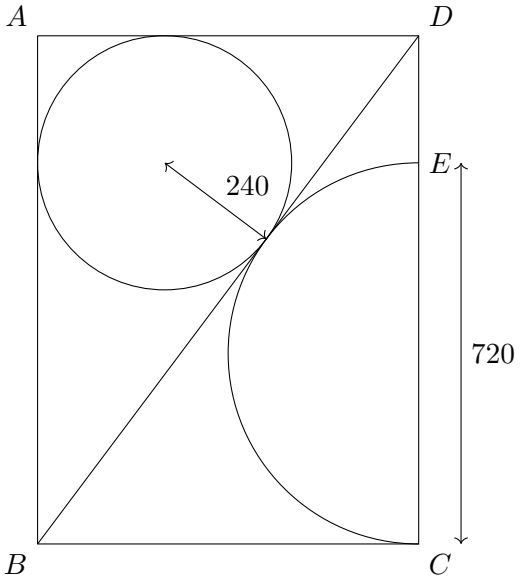



24. The diagram below shows three toy cars moving in a rectangular circuit $ABCD$ where $AB = 10$m and $BC = 20$m. Toy cars $M$ and $N$ start from the vertices $A$ and $C$ respectively and move in an anti-clockwise direction with constant speeds 10 m/min and 4 m/min respectively. Toy car $P$ starts from $C$ and moves in a clockwise direction with a constant speed of 8 m/min.




The three toy cars start their motion at the same time. Assume that when any two toy cars meet, there is no collision and the cars will continue with their motion. Find the total time elapsed in minutes when all the three toy cars meet simultaneously for the fifth time.

25. In the diagram below, $ABCD$ is a rectangle. A circle of radius 240 mm is inscribed in $\triangle ABD$ and $BD$ is a common tangent to both the circle and a semicircle whose diameter $CE$ lies on $CD$. It is given that $CE = 720$ mm. Find the perimeter of rectangle $ABCD$ in mm.



### 5.1.2. Round 2 Problems

Solutions can be found in Section 10.1.2.

1. Let $ABC$ be an isosceles right-angled triangle of area 1. Find the length of the shortest segment that divides the triangle into two parts of equal area.

2. Let $ABCD$ be a parallelogram and points $E$, $F$ be on its exterior. If triangles $BCF$ and $DEC$ are similar, i.e. $\triangle BCF \sim \triangle DEC$, prove that triangle $AEF$ is similar to these two triangles.

3. Seven triangles of area 7 lie in a square of area 27. Prove that among the 7 triangles there are 2 that intersect in a region of area not less than 1.

4. Suppose for some positive integer $n$, the numbers $2^n$ and $5^n$ have equal first digit. What are the possible values of this first digit?

5. Find all integer solutions of the equation

$$y^2 + 2y = x^4 + 20x^3 + 104x^2 + 40x + 2003.$$

## 5.2. Senior Section

### 5.2.1. Round 1 Problems

Solutions can be found in Section 10.2.1.

#### Multiple Choice Questions

1. Find the largest positive integer $A$ such that $2^x + \dfrac{2025}{2^x} - A > 0$ for all real numbers $x$

   (A) 59 (B) 69 (C) 79 (D) 89 (E) 99

2. If $x = \dfrac{1}{\log_{\frac{2024}{2023}} 7} + \dfrac{1}{\log_{\frac{2023}{2022}} 7} + \dfrac{1}{\log_{\frac{2022}{2021}} 7}$, find $7^x$.

   (A) $\dfrac{2021}{2024}$ (B) $\dfrac{2024}{2021}$ (C) $\dfrac{2022}{2024}$ (D) $\dfrac{2024}{2022}$ (E) 2024

3. Simplify $\dfrac{2024}{\sqrt{4+\sqrt{12}}} + \dfrac{2024}{\sqrt{4-\sqrt{12}}}$.

   (A) 1012 (B) $1012\sqrt{3}$ (C) 2024 (D) $2024\sqrt{3}$
   (E) $1012 + 1012\sqrt{3}$

4. Suppose $x^{1/3} + 12 = y^{1/3}$ for some real numbers $x$ and $y$. Find the minimum possible value of $y - x$.

   (A) 432 (B) 532 (C) 632 (D) 732
   (E) None of the above

5. Find the largest possible value of $\dfrac{\sqrt{2}\cos(2x)}{\sin(x) + \cos(x)}$.

   (A) $\dfrac{1}{2}$ (B) 1 (C) $\dfrac{3}{2}$ (D) 2 (E) $\dfrac{5}{2}$

#### Short Questions

6. If $\sqrt{x+\sqrt{x}} + \sqrt{x-\sqrt{x}} = 4$, find the value of $15x$.

7. Find the smallest positive integer $K$ such that

   $$x^2 - 200x + y^2 = 0 \text{ and } x + y \leq K.$$

8. Given that $\dfrac{\cos(x)}{\sin(3x)} - \dfrac{\sin(x)}{\cos(3x)} - 2 \cdot \dfrac{\sin(4x)}{\cos(6x)} = 2024$, find the value of $\dfrac{\cos(10x)}{\sin(12x)}$.

9. Find the smallest positive integer $k$ such that the coefficient of $x^k$ in the expansion of $\left(5x^3 + \frac{1}{\sqrt{x}}\right)^{2024}$ is **not** zero.

10. Let

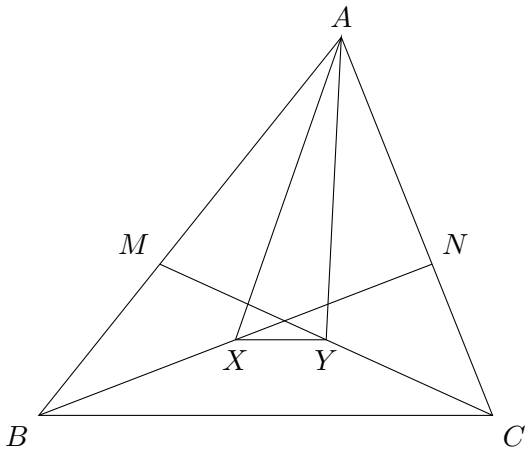
    $$P = \left(2024^2 + 1\right)\left(2024^{2^2} + 1\right)\left(2024^{2^3} + 1\right)\cdots\left(2024^{2^{10}} + 1\right) \times 2025 + \frac{1}{2023}.$$

    Find the smallest positive integer $N$ such that $N > \log_{2024} P$.

11. Let $\triangle ABC$ be a triangle with area 1000. Let $M$ and $N$ be points on $AB$ and $AC$ respectively such that

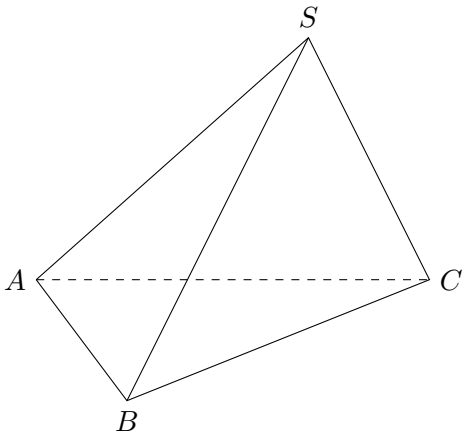
$$AM : MB = 3 : 2 \text{ and } AN : NC = 7 : 3.$$

Let $X$ and $Y$ be the midpoints of $BN$ and $CM$ respectively. Find the area of $\triangle AXY$.




12. Find the largest positive integer $n \leq 10000$ such that $1 + 2024n^2$ is a perfect square.

13. In a tetrahedron $SABC$, the faces $SBC$ and $ABC$ are perpendicular to each other. The angles $\angle ASB$, $\angle BSC$, $\angle ASC$ are all $60°$, and $SB = SC = 4$. Find the square of the volume of the tetrahedron.




14. Let $a$, $b$, $c$ be the three real roots of the cubic equation

$$2x^3 - 4x^2 - 21x - 8 = 0.$$

Given that

$$S = \frac{1}{ab + c - 1} + \frac{1}{bc + a - 1} + \frac{1}{ca + b - 1}$$

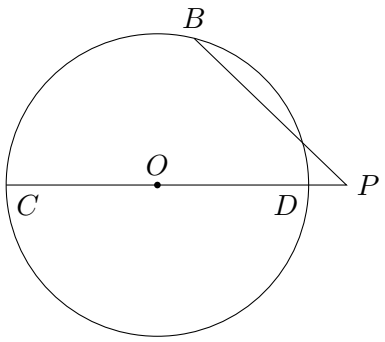
is a rational number that can be expressed as a fraction in the lowest form $\frac{m}{n}$, find the value of $m^2 + n^2$.

15. Consider the equation

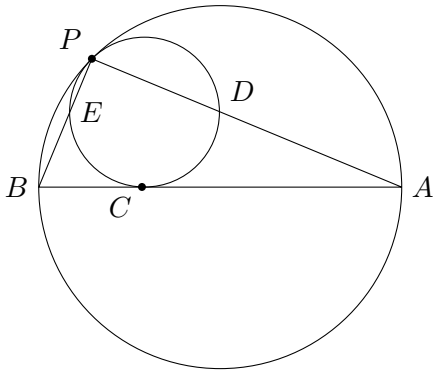
$$\frac{\sqrt{3} - 1}{\sin x} + \frac{\sqrt{3} + 1}{\cos x} = 4\sqrt{2}.$$

For the range $0 < x < \pi/2$, the sum of the solutions of the equation can be expressed in the form $\frac{m\pi}{n}$, where $\frac{m}{n}$ is a fraction in the lowest form. Find $m + n$.

16. An engineer constructs a circle with centre $O$ and diameter $CD$ on level ground, and builds a vertical tower of height 20 at the centre. $B$ is another point on the circumference and $P$ is on $CD$ produced such that $PB$ is a secant line of the circle. Given that $PB = 33$, $PC = 77$ and $CD = 74$, find the minimum possible distance of any point on $PB$ to the top of the tower.




17. $P$ is a common point of tangency of two circles. $BA$ is a chord of the larger circle which is tangent to the smaller circle at a point $C$. $PB$ and $PA$ intersect the smaller circle at points $E$ and $D$ respectively. If $BA = 15$, $PE = 2$, and $PD = 3$, find the length $CA$.




18. On each face of a cube, an integer greater than 2 is written. Each vertex of the cube is the intersection of three unique faces, and each edge is the intersection of two unique faces. Assign to each vertex the product of the numbers written on the faces intersecting the vertex, and assign to each edge the product of the numbers written on the faces intersecting the edge. The sum of the numbers assigned to the eight vertices is equal to 2024. Find the maximum possible value of an edge.

19. Find the sum of the squares of each of the roots of the equation

$$x^2 - 4\lfloor x \rfloor - 12 = 0,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

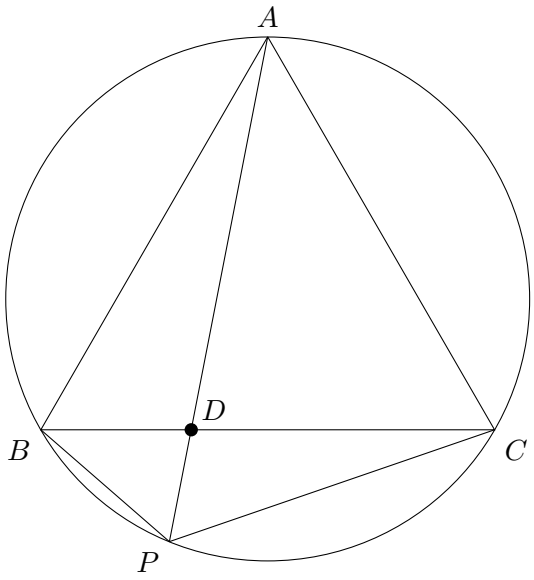
20. Calculate the remainder when $1901^{2024}$ is divided by 1216.

21. Let $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial with non-negative integer coefficients satisfying $0 \leq a_i \leq 17$ for all $i$. If $P(18) = 367616$, find the value of $P(3)$.

22. Evaluate the sum

$$\frac{2}{1 + \tan\left(\frac{\pi}{260}\right)} + \frac{2}{1 + \tan\left(\frac{2\pi}{260}\right)} + \frac{2}{1 + \tan\left(\frac{3\pi}{260}\right)} + \cdots + \frac{2}{1 + \tan\left(\frac{129\pi}{260}\right)}.$$

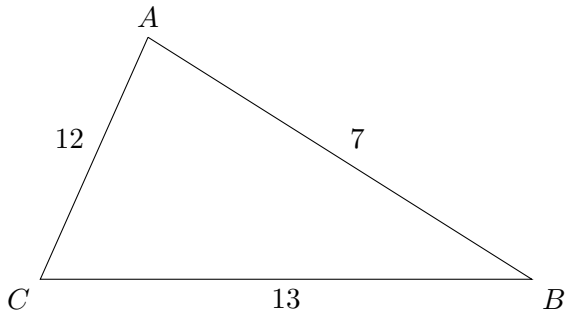
23. An equilateral triangle $ABC$ is inscribed in a circle and $P$ is a point on the minor arc $BC$. Point $D$ is the intersection of $AP$ and $BC$.




Suppose that $BP = 5$, $CP = 20$. Find the length of $AD$.

24. Find the number of positive integers $x < 9000$ such that $x^3 + 95$ is divisible by 96.

25. A scalene triangle $\triangle ABC$ has sides $AB = 7$, $AC = 12$ and $BC = 13$. Write $\tan\frac{A-B}{2}\tan\frac{C}{2}$ as a fraction $\frac{m}{n}$ in its lowest form and find $m + n$.



### 5.2.2. Round 2 Problems

Solutions can be found in Section 10.2.2.

1. In an acute triangle $ABC$, $AC > AB$, $D$ is the point on $BC$ such that $AD = AB$. Let $\omega_1$ be the circle through $C$ tangent to $AD$ at $D$, and $\omega_2$ the circle through $C$ tangent to $AB$ at $B$. Let $F$ ($\neq C$) be the second intersection of $\omega_1$ and $\omega_2$. Prove that $F$ lies on $AC$.

2. Find all integer solutions of the equation

$$y^2 + 2y = x^4 + 20x^3 + 104x^2 + 40x + 2003.$$

3. Find the smallest positive integer $n$ for which there exist integers $x_1 < x_2 < \cdots < x_n$ such that every integer from 1000 to 2000 can be written as a sum of some of the integers from $x_1, x_2, \ldots, x_n$ without repetition.

4. Suppose $p$ is a prime number and $x$, $y$, $z$ are integers satisfying $0 < x < y < z < p$. If $x^3$, $y^3$, $z^3$ have equal remainders when divided by $p$, prove that $x^2 + y^2 + z^2$ is divisible by $x + y + z$.

5. Let $a_1, a_2, \ldots$ be a sequence of positive numbers satisfying, for any positive integers $k$, $l$, $m$, $n$ such that $k + n = m + l$,

$$\frac{a_k + a_n}{1 + a_k a_n} = \frac{a_m + a_l}{1 + a_m a_l}.$$

   Show that there exist positive numbers $b$, $c$ so that $b \leq a_n \leq c$ for any positive integer $n$.

## 5.3. Open Section

### 5.3.1. Round 1 Problems

Solutions can be found in Section 10.3.1.

1. Let $S_k = 1+2+3+\cdots+k$ for any positive integer $k$. Find $S_1+S_2+S_3+\cdots+S_{20}$.
2. Let $S = \sum_{r=1}^{64} r\binom{64}{r}$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $0! = 1$. Find $\log_2 S$.
3. Let $x$ be the largest number in the interval $[0, 2\pi]$ such that $(\sin x)^{2024} - (\cos x)^{2024} = 1$. Find $\lfloor x \rfloor$.

   *(Note: If you think that such a number $x$ does not exist, enter your answer "99999".)*
4. Find the number of real numbers $x$ that satisfies the equation $|x-2| + |x-3| = |2x-5|$.

   *(Note: If you think that there are no such numbers, enter "0"; if you think that there are infinitely many such numbers, enter "99999".)*
5. Among all the real numbers that satisfies the inequality $e^x \geq 1 + 2e^{-x}$, find the minimum value of $\lceil e^x + e^{-x} \rceil$.
6. Find the smallest positive integer $C$ greater than 2024 such that the sets $A = \{2x^2 + 2x + C : x \in \mathbb{Z}\}$ and $B = \{x^2 + 2024x + 2 : x \in \mathbb{Z}\}$ are disjoint.
7. Let $ABCD$ be a convex quadrilateral inscribed in a circle $\omega$. The bisector of $\angle BAC$ meets $\omega$ at $E$ ($\neq A$), the bisector of $\angle ABD$ meets $\omega$ at $F$ ($\neq B$), $AE$ intersects $BF$ at $P$ and $CF$ intersects $DE$ at $Q$. Suppose $EF = 20$, $PQ = 11$. Find the area of the quadrilateral $PEQF$.
8. Let $f(x) = \sqrt{x^2+1} + \sqrt{(4-x)^2+4}$. Find the minimum value of $f(x)$.
9. It is known that $a \geq 0$ satisfies $\sqrt{4+\sqrt{4+\sqrt{4+\sqrt{4+a}}}} = a$. Find the value of $(2a-1)^2$.
10. A rectangle with sides parallel to the horizontal and vertical axes is inscribed in the region bounded by the graph of $y = 60 - x^2$ and the $x$-axis. If the area of the largest such rectangle has area $k\sqrt{5}$, find the value of $k$.
11. Let $x$ be a real number satisfying the equation $x^{x^5} = 100$. Find the value of $\lfloor x^5 \rfloor$.
12. Let $a$, $b$, $c$, $d$, $e$ be distinct integers with $a+b+c+d+e = 9$. If $m$ is an integer such that
    $$(m-a)(m-b)(m-c)(m-d)(m-e) = 2009,$$
    determine the value of $m$.
13. Let $\{x\}$ be the fractional part of the number $x$, i.e., $\{x\} = x - \lfloor x \rfloor$. If $S = \int_0^9 \{x\}^2 \, dx$, find $\lfloor S \rfloor$.
14. The solution of the inequality $|(x+1)(x-6)| > |(x+4)(x-2)|$ can be expressed as $x < a$ or $b < x < c$. If $S = |a| + |b| + |c|$, find $\lfloor 14S \rfloor$.
15. Given that $x, y > 0$ and $x\sqrt{2-y^2} + y\sqrt{2-x^2} = 2$, find the value of $x^2 + y^2$.
16. A convex polygon has $n$ sides such that no three diagonals are concurrent. It is known that all its diagonals divide the polygon into 2500 regions. Determine $n$.

17. Find the number of integers $n$ between $-2029$ and 2029 inclusive such that $(n+2)^2+n^2$ is divisible by 2029.

18. Let $f$ be a function such that for any real number $x$, we have $f(x)+2f(2-x)=x+x^2$. Find the value of $f(1)+f(2)+f(3)+\cdots+f(34)$.

19. Find the largest possible positive prime integer $p$ such that $p$ divides

$$S(p)=1^{p-2}+2^{p-2}+3^{p-2}+4^{p-2}+5^{p-2}+6^{p-2}+7^{p-2}+8^{p-2}.$$

20. Let $f$ be a function such that $f(x)+f(\frac{1}{1-x})=1+\frac{1}{x}$ for all $x\notin\{0,1\}$. Find the value of $\lfloor 180\cdot f(10)\rfloor$.

21. Let $C$ be a circle with equation $(x-a)^2+(y-b)^2=r^2$, where at least one of the $a$ and $b$ are irrational numbers. Find the maximum possible numbers of points $(p,q)$ on $C$ where both $p$ and $q$ are rational numbers.

22. On the plane there are 2024 points coloured either red or blue such that each red point is the centre of a circle passing through 3 blue points. Determine the least number of blue points.

23. It is given that the positive real numbers $x_1,\ldots,x_{2026}$ satisfy $\dfrac{x_1^2}{x_1^2+1}+\cdots+\dfrac{x_{2026}^2}{x_{2026}^2+1}=2025$. Find the maximum value of $\dfrac{x_1}{x_1^2+1}+\cdots+\dfrac{x_{2026}}{x_{2026}^2+1}$.

24. Let $n$ denote the number of ways of arranging all the letters of the word MATHEMATICS in one row such that

    - both M's precede both T's; and
    - neither the two M's nor the two T's are next to each other.

    Determine the value of $\frac{n}{6!}$.

25. The incircle of the triangle $ABC$ centred at $I$ touches the sides $BC$, $CA$, $AB$ at $D$, $E$, $F$ respectively. Let $D'$ be the intersection of the extension of $ID$ with the circle through $B$, $I$, $C$; $E'$ be the intersection of the extension of $IE$ with the circle through $A$, $I$, $C$; and $F'$ the intersection of the extension of $IF$ with the circle through $A$, $I$, $B$. Suppose $AB=52$, $BC=56$, $CA=60$. Find $DD'+EE'+FF'$.

### 5.3.2. Round 2 Problems

Solutions can be found in Section 10.3.2.

1. In triangle $ABC$, $\angle B = 90°$, $AB > BC$, and $P$ is the point such that $BP = BC$ and $\angle APB = 90°$, where $P$ and $C$ lie on the same side of $AB$. Let $Q$ be the point on $AB$ such that $AP = AQ$, and let $M$ be the midpoint of $QC$. Prove that the line through $M$ parallel to $AP$ passes through the midpoint of $AB$.

2. Let $n$ be a fixed positive integer. Find the minimum value of

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n}$$

where $x_1, x_2, \ldots, x_n$ are distinct positive integers.

3. Prove that for every positive integer $n$ there exists an $n$-digit number divisible by $5^n$ all of whose digits are odd.

4. Alice and Bob play a game. Bob starts by picking a set $S$ consisting of $M$ vectors of length $n$ with entries either 0 or 1. Alice picks a sequence of numbers $y_1 \leq y_2 \leq \cdots \leq y_n$ from the interval $[0, 1]$, and a choice of real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Bob wins if he can pick a vector $(z_1, z_2, \ldots, z_n) \in S$ such that

$$\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} x_i z_i,$$

otherwise Alice wins. Determine the minimum value of $M$ so that Bob can guarantee a win.

5. Let $p$ be a prime number. Determine the largest possible $n$ such that the following holds: it is possible to fill an $n \times n$ table with integers $a_{ik}$ in the $i$th row and $k$th column, for $1 \leq i, k \leq n$, such that for any quadruple $i$, $j$, $k$, $l$ with $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, the number $a_{ik}a_{jl} - a_{il}a_{jk}$ is not divisible by $p$.

# Part II.

# Solutions

# 6. SMO 2020

## 6.1. Open Section

### 6.1.1. Round 1 Solutions

**Resources:** Review by *Way Tan*

**Question 1** [Ans: 0]

If $S$ is the sum of all the *real* roots of the equation $x^2 + \dfrac{1}{x^2} = 2020^2 + \dfrac{1}{2020^2}$ find $\lfloor S \rfloor$.

**Solution.** Observe that $x^2 + \frac{1}{x^2}$ is even. Hence, the sum of roots is 0.

**Question 2** [Ans: 2030]

Find the largest positive integer $x$ that satisfies the equation

$$(\lfloor x \rfloor - 2020)^2 + (\lceil x \rceil - 2030)^2 = (\lfloor x \rfloor - \lceil x \rceil + 10)^2.$$

*(Note: If you think that the above equation has no solution in the positive integers, enter your answer as "0".)*

**Solution.** Since $x$ is an integer, we obviously have $x = \lfloor x \rfloor = \lceil x \rceil$. We are hence left with the equation $(x-2020)^2 + (x-2030)^2 = 10^2$, of which 2020 and 2030 are clearly solutions to. Thus, $x = 2030$.

**Question 3** [Ans: 12]

Let $S_n = \dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \cdots + \dfrac{1}{(2n-1) \times (2n+1)}$. Find the value of $n$ such that $S_n$ takes the value of 0.48.

**Solution.** Using partial fraction decomposition, we see that

$$S_n = \sum_{i=1}^{n} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \sum_{i=1}^{n} \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right).$$

This sum clearly telescopes, giving $S_n = \frac{1}{2}(1 - \frac{1}{2n+1})$ Setting $S_n = 0.48$ yields $n = 12$.

**Question 4** [Ans: 27]

Given that the three planes in the Cartesian space with equations $2x + 4y + 6z = 5$, $3x + 5y + 2z = 6$ and $8x + 14y + az = b$ have a common line of intersection, find the value of $a + b$.

**Solution.** Solving $2x + 4y + 6z = 5$ and $3x + 5y + 2z = 6$ simultaneously, we see that the line of intersection has equation $x = 11t - \frac{1}{2}$, $y = \frac{3}{2} - 7t$ and $z = t$. Substituting this into $8x + 14y + az = b$, we get $(17 - b) + t(a - 10) = 0$. Since this must hold for all real $t$, we immediately have $a = 10$ and $b = 17$, whence $a + b = 27$.

**Question 5** [Ans: 70]

Let $i$ be the complex number $\sqrt{-1}$, and $n$ be the smallest positive integer such that $(\sqrt{3} + i)^n = a$, where $a$ is a real number. Find the value of $\lfloor n - a \rfloor$.

**Solution.** Observe that $(\sqrt{3}+i)^n = 2^n e^{in\pi/6}$. Since we want this to be real, its argument must be an integer multiple of $\pi$. Thus, $\frac{n}{6} \in \mathbb{Z}$. This immediately gives us $n = 6$, whence $a = 2^6 e^{i\pi} = -64$. Hence, $n - a = 70$.

**Question 6** [Ans: 945]

In the three-dimensional Cartesian space, let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors along three mutually perpendicular $x$, $y$ and $z$-axes respectively. Three straight lines $l_1$, $l_2$ and $l_3$ have equations defined by

$$\begin{aligned} l_1\colon \mathbf{r} &= (4 + \lambda)\mathbf{i} + (5 + \lambda)\mathbf{j} + (6 + \lambda)\mathbf{k}, \\ l_2\colon \mathbf{r} &= (4 + 3\mu)\mathbf{i} + (5 - \mu)\mathbf{j} + (6 - 2\mu)\mathbf{k}, \\ l_3\colon \mathbf{r} &= (1 + 6\nu)\mathbf{i} + (2 + 2\nu)\mathbf{j} + (3 + \nu)\mathbf{k}, \end{aligned}$$

where $\mu$, $\lambda$ and $\nu$ are real numbers. If the area of the triangle enclosed by the three lines $l_1$, $l_2$ and $l_3$ is denoted by $S$, find the value of $10S^2$.

**Solution.** Rewriting the equations of the three lines in vector form, we get

$$l_1 : \mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad l_2 : \mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}, \quad l_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \nu \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}.$$

It is simple to find the pairwise intersections of the planes: $l_1$ and $l_2$ intersect at $A(4, 5, 6)$, $l_2$ and $l_3$ intersect at $B(7, 4, 4)$, while $l_3$ and $l_1$ intersect at $C(1, 2, 3)$. $S$, the area of $\triangle ABC$, can hence be calculated as

$$S = [ABC] = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{3\sqrt{42}}{2}.$$

Thus, $10S^2 = 945$.

**Question 7** [Ans: 50]

Given that $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^2 - b^2) = (a - b)(f(a) + f(b))$$

for all real numbers $a$ and $b$, and that $f(1) = \dfrac{1}{101}$, find the value of $\displaystyle\sum_{k=1}^{100} f(k)$.

**Solution.** By inspection, we have $f(x) = kx$. Since $f(1) = \frac{1}{101}$, we get $k = \frac{1}{101}$. Thus, the desired sum evaluates to $\frac{1}{101} \cdot \frac{100(101)}{2} = 50$.

**Question 8** [Ans: 4]

Find the sum of all the positive integers $n$ such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.
*(Note: If you think that there are infinitely many such positive integers $n$ that satisfy that above conditions, enter your answer as "9999".)*

**Solution.** Observe that $n^4 - 4n^3 + 22n^2 - 36n + 18 = (n^2 - 2n + 9)^2 - 63$. Let this be $m^2$, where $m$ is some integer. By the difference of two squares identity, we get

$$63 = (n^2 - 2n + 9 + m)(n^2 - 2n + 9 - m).$$

Let $A = n^2 - 2n + 9 + m$ and $B = n^2 - 2n + 9 - m$. We clearly have that $AB = 63$ and $\frac{1}{2}(A + B) - 9 = (n - 1)^2$, a perfect square. Going through all factors of 63, we see that the only pairs of factors satisfying this condition are $(A, B) = (9, 7)$ and $(21, 3)$, which give $0^2$ and $2^2$ respectively. Hence, $n = 1$ or 3, thus the sum desired is 4.

**Question 9** [Ans: 43]

Assume that

$$(x + 2 + m)^{2019} = a_0 + a_1(x + 1) + a_2(x + 1)^2 + \cdots + a_{2019}(x + 1)^{2019}.$$

Find the largest possible integer $m$ such that

$$(a_0 + a_2 + a_4 + \cdots + a_{2018})^2 - (a_1 + a_3 + a_5 + \cdots + a_{2019})^2 \leq 2020^{2019}.$$

**Solution.** Substituting $x = 0$ yields $a_0 + a_1 + \cdots + a_{2018} + a_{2019} = (m + 2)^{2019}$, while substituting $x = -2$ yields $a_0 - a_1 + \cdots + a_{2018} - a_{2019} = m^{2019}$. Hence,

$$a_0 + a_2 + \cdots + a_{2018} = \frac{(m + 2)^{2019} + m^{2019}}{2},$$

whence

$$a_1 + a_3 + \cdots + a_{2019} = (m + 2)^{2019} - \frac{(m + 2)^{2019} + m^{2019}}{2}.$$

We thus want

$$\left(\frac{(m + 2)^{2019} + m^{2019}}{2}\right)^2 - \left((m + 2)^{2019} - \frac{(m + 2)^{2019} + m^{2019}}{2}\right)^2 \leq 2020^{2019}.$$

The LHS simplifies to $(m(m + 2))^{2019}$. Thus, $m(m + 2) \leq 2020$, whence $m \leq 43$.

**Question 10** [Ans: 8]

Given that $S = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{1}{\sqrt{n(n+k)}}$, find the value of $\lfloor (S+2)^2 \rfloor$.

**Solution.** Dividing through by $n$ yields

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{1}{\sqrt{1+k/n}},$$

which is very clearly a Riemann sum. In the limit, we get

$$S = \int_0^1 \frac{1}{\sqrt{1+x}}\,\mathrm{d}x = 2\sqrt{2} - 2.$$

Thus, $(S+2)^2 = 8$.

**Question 11** [Ans: 4097]

Let $A = \{1, 2, \cdots, 10\}$. Count the number of ordered pairs $(S_1, S_2)$, where $S_1$ and $S_2$ are non-intersecting and non-empty subsets of $A$ such that the largest number in $S_1$ is smaller than the smallest number in $S_2$. For example, if $S_1 = \{1, 4\}$ and $S_2 = \{5, 6, 7\}$, then $(S_1, S_2)$ is such an ordered pair.

**Solution.** Let the largest element of $S_1$ be $k$. There are $2^{k-1}$ ways to choose $S_1$, and there are $2^{10-k} - 1$ ways to choose $S_2$ (note that we subtract 1 since $S_2 \neq \varnothing$). There are hence $2^{k-1}\left(2^{10-k} - 1\right)$ possibilities. Summing over $k = 1, \ldots, 9$, we get a total of

$$\sum_{k=1}^{9} 2^{k-1}\left(2^{10-k} - 1\right) = \sum_{k=1}^{9} \left(2^9 - 2^{k-1}\right) = 9 \cdot 2^9 - \frac{1-2^9}{1-2} = 4097$$

ordered pairs.

**Question 12** [Ans: 1010]

Each cell of a $2020 \times 2020$ table is filled with a number which is either 1 or $-1$. For $u = 1, \ldots, 2020$, let $R_i$ be the product of all the numbers in the $i$th row and let $C_i$ be the product of all the numbers in the $i$th column. Suppose $R_i + C_i = 0$ for all $i = 1, \ldots, 2020$. What is the least number of $-1$'s in the table?

**Solution.** Let $a_{ij}$ represent the number in the cell in the $i$th row and $j$th column. Observe that $R_i = \prod_{n=1}^{2020} a_{in}$ and $C_i = \prod_{n=1}^{2020} a_{ni}$. Then

$$\prod_{n=1}^{2020} a_{in} = -\prod_{n=1}^{2020} a_{ni}.$$

The number of $-1$s in the two products hence differ by an odd number. In the optimal case, one product has no $-1$s, while the other has one $-1$. We now construct a grid with such a property. Let $a_{ij} = -1$ if $(i, j) \in \{(1,2), (3,4), \ldots, (2019, 2020)\}$, and $a_{ij} = 1$ otherwise. It is quite clear that

$$\prod_{n=1}^{2020} a_{in} = \begin{cases} 1, & i \text{ odd} \\ -1, & i \text{ even} \end{cases}, \qquad \prod_{n=1}^{2020} a_{ni} = \begin{cases} -1, & i \text{ odd} \\ 1, & i \text{ even} \end{cases},$$

whence $R_i + C_i = 0$ for $i = 1, \ldots, 2020$ as desired. Thus, the least number of $-1$s is $|\{(1,2), (3,4), \ldots, (2019, 2020)\}| = 1010$.

**Question 13** [Ans: 5120]

Assume that the sequence $\{a_k\}_{k=1}^{\infty}$ follows an arithmetic progression with $a_2+a_4+a_9 = 24$. Find the maximum value of $S_8 \times S_{10}$, where $S_k$ denotes the sum $a_1+a_2+\cdots+a_k$.

**Solution.** Let $a_k = a_1 + (k-1)d$. From the given equation, we immediately have $3a_1 + 12d = 24$, whence $2a_1 + 8a_1 = 16$. Since $S_k = ka_1 + \frac{(k-1)k}{2} \cdot d$, we have

$$S_8 \cdot S_{10} = (8a_1 + 28d)(10a_1 + 45d) = 20(16-d)(16+d) = 20\left(16^2 - d^2\right).$$

Thus, the maximum value of $S_8 \cdot S_{10}$ is $20 \cdot 16^2 = 5120$, when $d = 0$.

**Question 14** [Ans: 0]

Consider all functions $g : \mathbb{R} \to \mathbb{R}$ satisfying the conditions that

1. $|g(a) - g(b)| \leq |a - b|$ for any $a, b \in \mathbb{R}$;
2. $g(g(g(0))) = 0$.

Find the *largest* possible value of $g(0)$.

**Solution.** Let $a = x + h$ and $b = x$. From the first condition, we get

$$\frac{|g(x+h) - g(x)|}{h} \leq 1 \implies \left|g'(x)\right| \leq 1.$$

Now consider the fixed point iteration $x_{n+1} = g(x_n)$, which must converge to the root of $g(x) = x$ since $|g'(x)| \leq 1$. The second condition states that if $x_n = 0$, then $x_{n+3} = x_n = 0$. This immediately implies that $x = 0$ is a root of $g(x) = x$, whence $g(0) = 0$.

**Question 15** [Ans: 4039]

A sequence $\{a_i\}_{i=1}^{\infty}$ is called a *good* sequence if $\dfrac{S_{2n}}{S_n}$ is a constant for all $n \geq 1$, where $S_k$ denotes the sum $a_1 + a_2 + \cdots + a_k$. Suppose it is known that the sequence $\{a_i\}_{i=1}^{\infty}$ is a *good* sequence that follows an arithmetic progression. Determine $a_{2020}$ if $a_1 = 1 \neq a_2$.

**Solution.** Let $a_i = 1 + (i-1)k$. Then $S_n = n + \frac{k(n-1)n}{2}$. Since $\frac{S_{2n}}{S_n}$ is constant, we have $\frac{S_2}{S_1} = \frac{S_4}{S_2}$. This gives $S_2^2 = S_1 S_4$, whence $(2+k)^2 = 4 + 6k$. This gives us $k = 2$ (note that $k \neq 0$ since $a_1 \neq a_2$). Thus, $a_{2020} = 1 + 2(2020 - 1) = 4039$.

**Question 16** [Ans: 12]

Determine the smallest positive integer $p$ such that the system

$$\begin{cases} 6x + 4y + 3z = 0 \\ 4xy + 2yz + pxz = 0 \end{cases}$$

has more the one set of real solutions in $x$, $y$, $z$.

**Solution.** From the first equation, we have $6x = -4y - 3z$. Multiplying the second equation by 6 and substituting $6x$, we get

$$16y^2 + 4pyz + 3pz^2 = 0. \tag{1}$$

Taking the discriminant with respect to $4y$, we have $z^2(p^2 - 12p)$, which must be greater than or equal to 0 to admit multiple solutions. Hence, $p^2 - 12p \geq 0$, whence $p \geq 12$. Thus, $\min p = 12$. Indeed, when $p = 12$, we get $4y + 6z = 0$ from (1), whence $(x, y, z) = (t, -3t, 2t)$ for all real $t$.

**Question 17** [Ans: 8800]

Let $ABC$ be a triangle with $a = BC$, $b = AC$ and $c = AB$. It is given that $c = 100$ and

$$\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{4}{3}.$$

Let $P$ be a point on the inscribed circle of $\triangle ABC$. Find the maximum value of

$$PA^2 + PB^2 + PC^2.$$

**Solution.** Since $\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{4}{3}$, $\triangle ABC$ is congruent to a 3-4-5 right triangle, where $\angle C = 90°$. Since $c = 100 = 20 \cdot 5$, we have $a = 20 \cdot 3 = 60$ and $b = 20 \cdot 4 = 80$. Let $C(0, 0)$. Then $A(80, 0)$ and $B(0, 60)$. Note that the incircle of $\triangle ABC$ has radius $20 \cdot 1 = 20$ and centre $(20, 20)$. Let $(x, y)$ be a point on the incircle, i.e.

$$(x - 20)^2 + (y - 20)^2 = 20^2. \implies x^2 + y^2 = 40x + 40y - 20^2. \tag{1}$$

We thus aim to maximize

$$PA^2 + PB^2 + PC^2 = \left[(x - 80)^2 + y^2\right] + \left[x^2 + (y - 60)^2\right] + \left[x^2 + y^2\right].$$

Expanding and using (1), we get

$$PA^2 + PB^2 + PC^2 = -40x + 8800.$$

Since $x \geq 0$, the maximum value of $PA^2 + PB^2 + PC^2$ is 8800.

**Question 18** [Ans: 2017]

Find the largest positive integer $n$ less than 2020 such that $\binom{n-1}{k} - (-1)^k$ is divisible by $n$ for $k = 0, 1, \ldots, n - 1$.

**Solution.** We begin by showing that all primes satisfy the given condition.

**Claim 1.** *If $n$ is prime, then $\binom{n-1}{k} - (-1)^k \equiv 0 \pmod{n}$.*

*Proof.* Observe that

$$\binom{n-1}{k} = \frac{(n-1)(n-2)\cdots(n-k)}{k!}.$$

Since $n$ is prime, $k!$ has a multiplicative inverse in $\mathbb{F}_n$. We can thus take congruences in the numerator without any problem:

$$\frac{(n-1)(n-2)\cdots(n-k)}{k!} \equiv \frac{(-1)(-2)\cdots(-k)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k \pmod{n}.$$

Thus,

$$\binom{n-1}{k} - (-1)^k \equiv (-1)^k - (-1)^k = 0 \pmod{n}.$$

□

Observe that the largest prime less than 2020 is 2017. To finish, we show that $n = 2018$ and $n = 2019$ do not work.

*Case 1*. Suppose $n = 2018$. When $k = 2$, we have

$$\binom{2017}{2} - 1 = 2017 \cdot 1008 - 1 \equiv -1 \cdot 1008 - 1 \equiv 1009 \neq 0 \pmod{2018}.$$

*Case 2*. Suppose $n = 2019$. When $k = 3$, we have

$$\binom{2018}{3} - 1 = 2018 \cdot 2017 \cdot 336 - 1 \equiv -1 \cdot -2 \cdot 336 - 1 \equiv 672 \neq 0 \pmod{2019}.$$

Thus, the largest $n$ is 2017.

**Question 19** [Ans: 41]

Assume that $\{a_k\}_{k=1}^{\infty}$ is a sequence with the property that for any distinct positive integers $m$, $n$, $p$, $q$ with $m + n = p + q$, the following equality always holds:

$$\frac{a_m + a_n}{(a_m + 1)(a_n + 1)} = \frac{a_p + a_q}{(a_p + 1)(a_q + 1)}.$$

Given $a_1 = 0$ and $a_2 = \frac{1}{2}$, determine $\frac{1}{1 - a_5}$.

*(Hint: Consider* $c_k = \frac{1}{a_k + 1} - \frac{1}{2}$ *for all positive integer* $k$*.)*

**Solution.** Let $m = 1$, $n = 3$, and $p = q = 2$. Using the given equation and conditions, we get

$$\frac{a_1 + a_3}{(a_1 + 1)(a_3 + 1)} = \frac{a_2 + a_2}{(a_2 + 1)(a_2 + 1)} \implies a_3 = \frac{4}{5}.$$

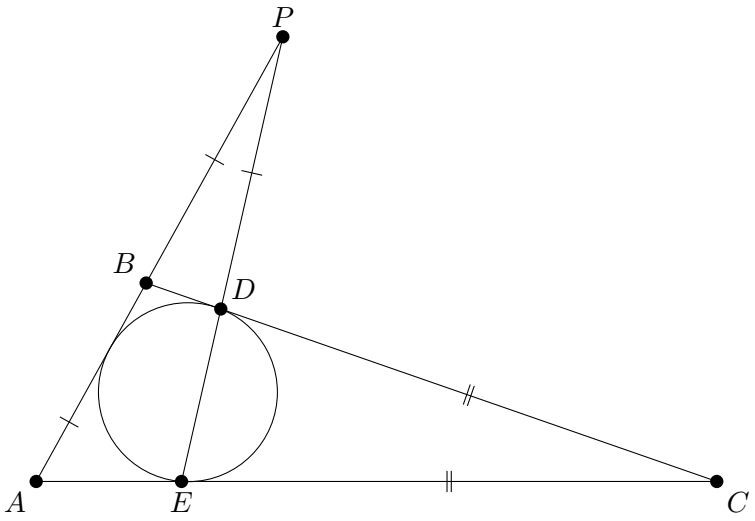
Let $m = 1$, $n = 5$, $p = q = 3$. Once again, using the given equation and conditions, we get

$$\frac{a_1 + a_5}{(a_1 + 1)(a_5 + 1)} + \frac{a_3 + a_3}{(a_3 + 1)(a_3 + 1)} \implies a_5 = \frac{40}{41}.$$

Thus, $\frac{1}{1 - a_5} = 41$.

**Question 20** [Ans: 140]

In the triangle $ABC$, the incircle touches the sides $BC$, $CA$, $AB$ at $D$, $E$, $F$ respectively. The line segments $ED$ and $AB$ are extended to intersect at $P$ such that $AB = BP = PD$. Suppose $CA = 9$. Find the value of $[ABC]^2$, where $[ABC]$ is the area of the triangle $ABC$.




**Solution.** Let $a = BC$, $b = CA = 9$, $c = AB$ and $s = \frac{1}{2}(a + b + c)$. We clearly have $AF = AE = s - a$, $BF = BD = s - 9$ and $CD = CE = s - c$.

Using Menalaus' theorem with respect to $\triangle ABC$, we have

$$\frac{AP}{PB}\frac{BD}{DC}\frac{CE}{EA} = 1 \implies 2 \cdot \frac{s-9}{s-c} \cdot \frac{s-c}{s-a} = 1.$$

Rearranging, we get $3a + c = 27$.

Using Menalaus' theorem with respect to $\triangle APE$, we have

$$\frac{AC}{CE}\frac{ED}{DP}\frac{PB}{BA} = 1 \implies \frac{9}{s-c} \cdot \frac{ED}{c} \cdot 1 = 1.$$

Rearranging, we get $ED = \frac{c(a+9-c)}{18}$. Since $CE = \frac{1}{2}(a + 9 - c)$, we have

$$\frac{ED}{CE} = \frac{c}{9}.$$

Since $\triangle BPD$ and $\triangle ECD$ are isosceles, and $\angle BDP = \angle ECD$, it follows that $\angle BDP = \angle C$. Using the cosine rule on $\angle BDP$ in $\triangle BPD$, we have

$$1 - \cos C = \frac{BD^2}{2 \cdot BP^2} = \frac{(s-b)^2}{2c^2}.$$

Using the cosine rule on $\angle C$ in $\triangle CED$, we have

$$1 - \cos C = \frac{ED^2}{2 \cdot CE^2} = \frac{c^2}{2 \cdot 81}.$$

Hence,

$$\frac{(s-b)^2}{2c^2} = \frac{c^2}{2 \cdot 81} \implies \frac{(a+c-9)^2}{4c^2} = \frac{c^2}{81}.$$

Using the substitution $3a + c = 27$ yields

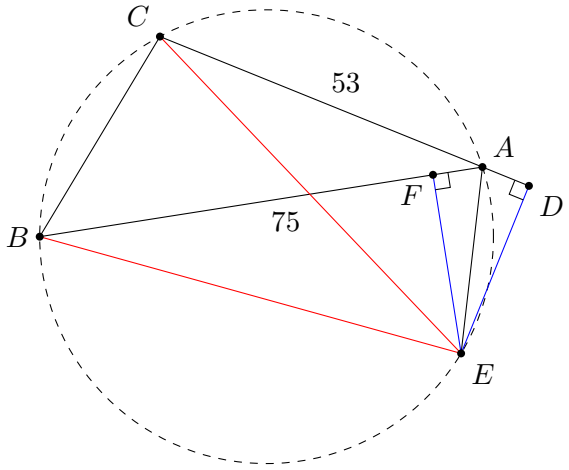
$$\frac{4(9-a)^2}{4 \cdot 9(9-a)^2} = \frac{9(9-a)^2}{81},$$

whence $a = 8$ and $c = 3$. Finally, by Heron's formula, we get

$$[ABC]^2 = s(s-a)(s-b)(s-c) = 10(10-8)(10-9)(10-8) = 140.$$

**Question 21** [Ans: 704]

In an acute-angled triangle $ABC$, $AB = 75$, $AC = 53$, the external bisector of $\angle A$ on $CA$ produced meets the circumcircle of triangle $ABC$ at $E$, and $F$ is the foot of the perpendicular from $E$ onto $AB$. Find the value of $AF \times FB$.




**Solution.** Let $D$ be a point on $CA$ produced such that $\angle ADE = 90°$. Clearly $\triangle AFE \equiv \triangle ADE$ by AAS, hence $FE = DE$ and $AF = AD$. Additionally, it is a well-known fact that $E$ is equidistant from $B$ and $C$. Hence, by RHS, $\triangle CDE \equiv \triangle BFE$. Thus, $CD = BF$. Since $CD = 53 + AD$ and $BF = 75 - AF$, we get $AF = 11$. Thus, $AF \times FB = 11 \times (75 - 11) = 704$.

**Question 22** [Ans: 165]

Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence with $a_k < a_{k+1}$ for all $k = 1, 2, 3, \cdots$ formed by arranging all the terms in the set $\left\{2^r + 2^s + 2^t : 0 \le r < s < t\right\}$ in increasing order. Find the largest value of the integer $n$ such that $a_n \le 2020$.

**Solution.** Observe that an integer is in the set if and only if its binary expansion has exactly 3 ones. Since $2020_{10} = 11111100100_2$, the largest $a_n$ less than 2020 is $11100000000_2$. Notice that this is also the largest $a_n$ where $a_n$ has 11 digits in binary. Thus, there are a total of $\binom{11}{3} = 165$ integers in the sequence before this $a_n$, thus $\max n = 165$.

**Question 23** [Ans: 838]

Let $n$ be a positive integer and $S$ be the set of all numbers that can be written in the form $\sum_{i=2}^{k} a_{i-1}a_i$ with $a_1, \ldots, a_k$ being positive integers that sum to $n$. Suppose the average value of all the numbers in $S$ is 88199. Determine $n$.

**Solution.** Testing small values of $n$, we see that

$$S = \left\{ s \in \mathbb{Z} \mid n - 1 \leq \left(\frac{n}{2}\right)^2 \right\}$$

when $n$ is even, and

$$S = \left\{ s \in \mathbb{Z} \mid n - 1 \leq \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \right\}$$

when $n$ is odd.

*Case 1*. If $n$ is even, we get $\frac{1}{2}\left[(n-1) + \left(\frac{n}{2}\right)^2\right] = 88199$, whence $n = 838$.

*Case 2*. If $n$ is odd, we get $\frac{1}{2}\left[(n-1) + \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\right] = 88199$, which has no integer solutions.

Thus, $n = 838$.

**Question 24** [Ans: 1600]

Let $x$, $y$, $z$ and $w$ be real numbers such that $x + y + z + w = 5$. Find the minimum value of $(x+5)^2 + (y+10)^2 + (z+20)^2 + (w+40)^2$.

**Solution.** By the Cauchy-Schwarz inequality, one has

$$\begin{aligned}&[(x+5) + (y+10) + (z+20) + (w+40)]^2 \\ &\qquad \leq 4\left[(x+5)^2 + (y+10)^2 + (z+20)^2 + (w+40)^2\right]\end{aligned}$$

Hence, the minimum value of $(x+5)^2 + (y+10)^2 + (z+20)^2 + (w+40)^2$ is $\frac{80^2}{4} = 1600$.

**Question 25** [Ans: 290]

Let $p$ and $q$ be positive integers satisfying the equation $p^2 + q^2 = 3994(p - q)$. Determine the largest possible value of $q$.

**Solution.** Completing the square gives

$$(p - 1997)^2 + (q + 1997)^2 = 2 \cdot 1997^2. \tag{1}$$

Let $P = p - 1997$ and $Q = q + 1997$. Multiplying (1) by 2 gives

$$2P^2 + 2Q^2 = 3994^2.$$

We now recognize the LHS to be a sum of two squares:

$$(P + Q)^2 + (P - Q)^2 = 3994^2.$$

We hence get a Pythagorean triple. Using the standard parameterization of such triples, we have $P + Q = m^2 - n^2$, $P - Q = 2mn$ and $3994 = m^2 + n^2$ for some positive integers $m$ and $n$. Since $m > n$, we see that $(m, n) = (63, \pm 5)$ are the only two pairs that give $m^2 + n^2 = 3994$. Hence, $Q = \frac{1}{2}(m^2 - n^2 - 2mn) = 2287$ (note we reject $Q = 1657$ since $Q \geq 1997$), whence $q = 2287 - 1997 = 290$.

# 7. SMO 2021

## 7.1. Open Section

### 7.1.1. Round 1 Solutions

**Resources:** Review by *Way Tan*

**Question 1** [Ans: 1741]

It is given that $\frac{\pi}{2} < \beta < \alpha < \frac{3\pi}{4}$, $\cos(\alpha-\beta) = \frac{12}{13}$ and $\sin(\alpha+\beta) = -\frac{3}{5}$. Find $\lfloor|2021\sin(2\alpha)|\rfloor$.

**Solution.** Note that $\alpha-\beta$ is in the first quadrant, while $\alpha+\beta$ is in the third quadrant. Hence, $\sin(\alpha-\beta) = \frac{5}{13}$, while $\cos(\alpha+\beta) = -\frac{4}{5}$. Thus,

$$\sin(2\alpha) = \sin(\alpha+\beta)\cos(\alpha-\beta) + \cos(\alpha+\beta)\sin(\alpha-\beta) = -\frac{56}{65}.$$

The required answer is hence 1741.

**Question 2** [Ans: 99999]

Find the number of solutions of the equation $|x-3|+|x-5| = 2$.
*(Note: If you think that there are infinitely many solutions, enter your answer as "99999".)*

**Solution.** Observe that for all $x \in [3,5]$, we have $|x-3|+|x-5| = 2$. There are hence infinitely many solutions.

**Question 3** [Ans: 4290]

Evaluate $1\times2\times3+2\times3\times4+3\times4\times5+\cdots+10\times11\times12$.

**Solution.** We are tasked with evaluating $\sum_{k=1}^{10} k(k+1)(k+2)$. Expanding, we have $\sum_{k=1}^{10} k^3 + 3k^2 + 2k$. Using the standard formulae

$$\sum_{k=1}^{n} k = \frac{k(k+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{k(k+1)(2k+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \left(\frac{k(k+1)}{2}\right)^2,$$

we arrive at 4290.

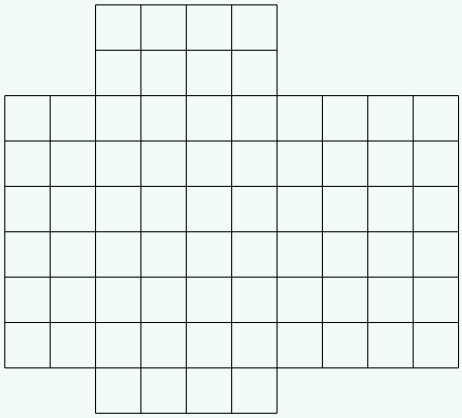
**Question 4** [Ans: 7]

It is given that the solution of the inequality $\sqrt{81-x^4} \leq kx+1$ is $a \leq x \leq b$ with $b-a=2$, where $k>0$. Determine $\lfloor k \rfloor$.

**Solution.** Observe that $\sqrt{81-x^4}$ is even, is defined only on $[-3,3]$, and is decreasing for $x>0$. It follows that $b=3$, whence $a=1$. However, at $x=a$, we have equality. Hence, $\sqrt{81-1^2}=k+1$, immediately implying $\lfloor k \rfloor = \lfloor \sqrt{80}-1 \rfloor = 7$.

**Question 5** [Ans: 1395]

The figure below shows a cross that is cut out from a $10 \times 9$ rectangular board.



Find the total number of rectangles in the above figure.
*(Note: A square is a rectangle.)*

**Solution.** Consider an $m \times n$ rectangular grid. Choosing a rectangle is equivalent to choosing 2 horizontal lines and two vertical lines (the four lines uniquely outline a rectangle). Since there are a total of $m+1$ horizontal lines and $n+1$ vertical lines, the number of rectangles in such a grid can be calculated as $\binom{m+1}{2}\binom{n+1}{2}$.

Returning to our problem, the total number of rectangles in the figure is hence $\binom{11}{2}\binom{7}{2} + \binom{5}{2}\binom{10}{2} - \binom{5}{2}\binom{7}{2} = 1395$. Note that we subtracted $\binom{5}{2}\binom{7}{2}$ to account for the double-counting in the middle of the grid.

**Question 6** [Ans: 2020]

Consider all polynomials $P(x,y)$ in two variables such that $P(0,0)=2020$ and for all $x$ and $y$, $P(x,y)=P(x+y,y-x)$. Find the largest possible value of $P(1,1)$.

**Solution 1.** Setting $x=0$ and relabelling $y$ as $x$, we get

$$P(0,x)=P(x,x). \tag{1}$$

Setting $y=x$, we get

$$P(x,x)=P(2x,0).$$

Setting $y=0$, we get

$$P(x,0)=P(x,-x).$$

Setting $y=-x$, we get

$$P(x,-x)=P(0,-2x).$$

We hence have

$$P(0,x) = P(x,x) = P(2x,0) = P(2x,-2x) = P(0,-4x),$$

implying

$$P(0,x) = P(0,-\tfrac{1}{4}x). \tag{2}$$

From (1), we have $P(1,1) = P(0,1)$. Using (2) repeatedly, we have

$$P(0,1) = P(0,-\tfrac{1}{4}) = P(0,\tfrac{1}{16}) = \cdots = P(0,0) = 2020.$$

Thus, $P(1,1) = 2020$.

**Solution 2** (Abusing uniqueness)**.** Suppose $k > 2020$ is the largest possible value of $P(1,1)$. Then $k$ can be as big as we wish it to be. However, by the nature of the problem, $k$ should be unique. Hence, $k > 2020$ is impossible, implying that 2020 is indeed the largest possible value of $P(1,1)$ (occurring when $P(x,y) = 2020$ for all $x$, $y$).

**Question 7** [Ans: 16]

In the three-dimensional Cartesian space with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denoting the unit vectors along three perpendicular directions in a clockwise manner, the line $l$ with equation given by $\mathbf{r} \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 5\mathbf{i} - 13\mathbf{j} + 7\mathbf{k}$ intersects the plane $\Pi$ with equation $x + y + z = 16$ at the point $(a,b,c)$. Find the value of $a + b + c$.

**Solution.** $(a,b,c)$ lies on $\Pi$ and hence satisfies the equation $x + y + z = 16$. Hence, $a + b + c = 16$.

**Question 8** [Ans: 169]

Find the minimum value of $(x+7)^2 + (y+2)^2$ subject to the constraint $(x-5)^2 + (y-7)^2 = 4$.

**Solution.** Let $(x+7)^2 + (y+2)^2 = r^2$, which describes a circle with centre $(-7,-2)$ and radius $r$. Meanwhile, $(x-5)^2 + (y-7)^2 = 4$ describes a circle with centre $(5,7)$ and radius 2. The smallest $r$ occurs when the two circles are externally tangent. This implies that the sum of the radii is equal to the distance between their centres: $r + 2 = \sqrt{12^2 + 9^2}$. Hence, the minimum value is $r^2 = 169$.

**Question 9** [Ans: 98]

Find the largest possible value $\alpha^4+\beta^4+\gamma^4$ among all possible sets of numbers $(\alpha,\beta,\gamma)$ that satisfy the equations

$$\alpha + \beta + \gamma = 2$$
$$\alpha^2 + \beta^2 + \gamma^2 = 14$$
$$\alpha^3 + \beta^3 + \gamma^3 = 20.$$

**Solution.** Newton's identities state that

$$n \geq k \geq 1: \qquad ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i$$

$$k > n \geq 1: \qquad 0 = \sum_{i=k-n}^{k} (-1)^{i-1} e_{k-i} p_i$$

where $e_k$ is the $k$th elementary symmetric polynomial of $n$ variables, and $p_k$ is the $k$th power sum of $n$ variables.

In our case, we have $n = 3$, along with

$$e_0 = 1, \quad p_0 = 3, \quad p_1 = e_1 = 2, \quad p_2 = 14, \quad p_3 = 20,$$

and we wish to find $p_4$. Evaluating the above sums at $k = 2, 3, 4$, we get

$$\begin{aligned} k = 2: \quad & 2e_2 = e_1p_1 - e_0p_2 \implies e_2 = -5 \\ k = 3: \quad & 3e_3 = e_2p_1 - e_1p_2 + e_0p_3 \implies e_3 = -6 \\ k = 4: \quad & 0 = e_3p_1 - e_2p_2 + e_1p_3 - e_0p_4 \implies p_4 = 98 \end{aligned}$$

Hence, $\alpha^4 + \beta^4 + \gamma^4 = 98$.

**Question 10** [Ans: 6]

If $p$ is the product of all the non-zero real roots of the equation

$$\sqrt[9]{x^7 + 30x^5} = \sqrt[7]{x^9 - 30x^5},$$

find $\lfloor |p| \rfloor$.

**Solution.** Dividing through by $x$ and substituting $y = x^2$ yields

$$\sqrt[9]{y^{-1} + 30y^{-2}} = \sqrt[7]{y - 30y^{-1}}.$$

Observe that the LHS is decreasing while the RHS is increasing. There is hence at most one real root. Indeed, by inspection, $y = 6$ satisfies the equation. We hence have $x = \pm\sqrt{6}$, whence $\lfloor |p| \rfloor = 6$.

**Question 11** [Ans: 6]

Let $S$ be the sum of a convergent geometric series with first term 1. If the third term of the series is the arithmetic mean of the first two terms, find $\lfloor 3S + 4 \rfloor$.

**Solution.** Let $r$ be the common ratio of the geometric series. Then we have $r^2 = (1+r)/2$, whence $r = -1/2$ (note that we reject $r = 1$ since the series is convergent). Hence,

$$\lfloor 3S + 4 \rfloor = \left\lfloor 3 \cdot \frac{1}{1 - (-1/2)} + 4 \right\rfloor = 6.$$

**Question 12** [Ans: 89]

Given that $\sin\alpha + \sin\beta = \dfrac{1}{10}$, and $\cos\alpha + \cos\beta = \dfrac{1}{9}$, find $\lfloor \tan^2(\alpha + \beta) \rfloor$.

**Solution.** By the sum-to-product identities, we have

$$\begin{aligned} \sin\alpha + \sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \\ \cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right). \end{aligned}$$

Hence, $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{1/10}{1/9} = \frac{9}{10}$. By the tangent double-angle identity, we finally get

$$\lfloor \tan^2(\alpha + \beta) \rfloor = \left\lfloor \left( \frac{2 \cdot 9/10}{1 - (9/10)^2} \right)^2 \right\rfloor = 89.$$

**Question 13** [Ans: 2]

Determine the number of positive integers that are divisible by 2021 and has exactly 2021 divisors (including 1 and itself).

**Solution.** Let $n = p_1^{d_1} \cdot p_2^{d_2} \cdots$, where $p_i$ are primes and $d_i$ are non-negative integers. The number of divisors of $n$ can be calculated as $(d_1+1)(d_2+1)\cdots$. Since 2021 only has two prime factors (43 and 47), there are only two possibilities, namely $(d_1, d_2) = (42, 46)$ or $(46, 42)$. Hence, $n$ has only two prime factors, which must be 43 and 47 (since $2021 \mid n$). Thus, there are two possibilities for $n$, namely $n = 43^{42} \cdot 47^{46}$ or $43^{46} \cdot 47^{42}$.

**Question 14** [Ans: 2]

Let $S = \sum_{k=0}^{25} \binom{100}{4k} - 2^{98}$. Find $\left\lfloor \left| \frac{S}{2^{48}} \right| \right\rfloor$.

**Solution.** Let

$$S_0 = \sum_{k=0}^{25} \binom{100}{4k}, \quad S_1 = \sum_{k=0}^{24} \binom{100}{4k+1}, \quad S_2 = \sum_{k=0}^{24} \binom{100}{4k+2}, \quad S_3 = \sum_{k=0}^{24} \binom{100}{4k+3}.$$

Consider $(1+x)^{100} = \sum_{k=0}^{100} \binom{100}{k} x^k$. Evaluating the binomial at $x = 1, i, -1, -i$, we have the following system of equations:

$$\begin{cases} (1+1)^{100} = S_0 + S_1 + S_2 + S_3 \\ (1+i)^{100} = S_0 + iS_1 - S_2 - iS_3 \\ (1-1)^{100} = S_0 - S_1 + S_2 - S_3 \\ (1-i)^{100} = S_0 - iS_1 - S_2 + iS_3 \end{cases}$$

Solving, one gets

$$4S_0 = 2^{100} + (1+i)^{100} + (1-i)^{100} = 2^{100} + 2 \cdot 2^{100/2} \cos\left(100 \cdot \tfrac{\pi}{4}\right) = 2^{100} - 2^{51}.$$

Hence,

$$\left\lfloor \left| \frac{S}{2^{48}} \right| \right\rfloor = \left\lfloor \left| \frac{(2^{100} - 2^{51})/4 - 2^{98}}{2^{48}} \right| \right\rfloor = 2.$$

**Question 15** [Ans: 1348]

Assume that $ABC$ is an acute triangle with $\sin(A+B) = \frac{3}{5}$ and $\sin(A-B) = \frac{1}{5}$. If $AB = 2022(\sqrt{6}-2)$, determine $\lfloor h \rfloor$, where $h$ is the height of the triangle from $C$ on $AB$.

**Solution.** It is trivial to see that $AB = h(\cot A + \cot B)$. Since both $A$ and $B$ are acute, we have $\cos(A+B) = \frac{4}{5}$ and $\cos(A-B) = \frac{2\sqrt{6}}{5}$. We thus have

$$\begin{aligned} 2\cos A \cos B &= \cos(A+B) + \cos(A-B) = \frac{4+2\sqrt{6}}{5} \\ 2\sin A \cos B &= \sin(A+B) + \sin(A-B) = \frac{4}{5} \\ 2\sin A \sin B &= \cos(A-B) - \cos(A+B) = \frac{2\sqrt{6}-4}{5} \end{aligned}$$

It follows that $\cot A = \frac{2+\sqrt{6}}{2}$ and $\cot B = \frac{2}{\sqrt{6}-2}$. Hence,

$$2022(\sqrt{6}-2) = h\left(\frac{2+\sqrt{6}}{2} + \frac{2}{\sqrt{6}-2}\right) \implies h = 1348.$$

**Question 16** [Ans: 10]

Let $a_1, a_2, \cdots$ be a sequence with $a_1 = 1$ and $a_{n+1} = \dfrac{n+2}{n} S_n$ for all $n = 1, 2, \cdots$, where $S_n = a_1 + a_2 + \cdots + a_n$. Determine the minimum integer $n$ such that $a_n \geq 2021$.

**Solution.** We claim that $a_n = 2^{n-2}(n+1)$. We prove this via induction. Consider $a_2$ as the base case: from the given equation, $a_2 = \frac{1+2}{1}(a_1) = 3$, which clearly satisfies our claim. Suppose that $a_k = 2^{k-2}(k+1)$ for some positive integer $k$. We first show that $S_k = k2^{k-1}$. Observe that

$$\sum_{i=1}^{k} x^i = \frac{x(1-x^k)}{1-x} \overset{\mathrm{d}/\mathrm{d}x}{\implies} \sum_{i=1}^{k} ix^{i-1} = \frac{kx^{k+1} - (k+1)x^k + 1}{(1-x)^2}.$$

Evaluating at $x = 2$, we have

$$\sum_{i=1}^{k} i2^{i-1} = k2^{k+1} - (k+1)2^k + 1.$$

Thus,

$$S_k = \sum_{i=1}^{k}\left(\frac{1}{2}\cdot i2^{i-1} + 2^{i-2}\right) = \frac{1}{2}\left(k2^{k+1} - (k+1)2^k + 1\right) + \frac{1}{2}\left(2^k - 1\right) = k2^{k-1}.$$

This immediately implies

$$a_{k+1} = \frac{k+2}{k}\cdot k2^{k-1} = (k+2)2^{k-1},$$

closing the induction. It is hence easy to see that $\min n = 10$.

**Question 17** [Ans: 101]

Each card of a stack of 101 cards has one side coloured red and the other coloured blue. Initially all cards have the red side facing up and stacked together in a deck. On each turn, Ah Meng takes 8 cards on the top, flip them over, and place them to the bottom deck. Determine the minimum number of turns required so that all the cards have the red sides facing up again.

**Solution.** Notice that the stack of cards can be represented by an array of numbers. Initially, all entries are 0. When a card is flipped over, the entry associated with that card is incremented by 1. The "incrementer" starts at the first entry and makes its way sequentially across the array, jumping back to the start once it reaches the end.

The figure below shows the array after one turn:

$$\underbrace{1,1,1,1,1,1,1,1}_{\text{the first 8 entries}},\underbrace{0,0,0,0,0,0,0,0,\ldots,0}_{\text{the remaining 93 entries}}$$

Let $k$ be the sum of the entries. At any given turn, we clearly have $8 \mid k$ (since eight cards are flipped over in a single turn). Furthermore, by the construction of the "incrementer", there are at most two distinct numbers in the array, and their difference must be 1. However, when all the cards have the red sides facing up again, each entry must be even. Hence, all 101 entries have the same number, whence $101 \mid k$. The smallest $k$ is thus $8 \cdot 101 = 808$, which occurs after $808/8 = 101$ turns.

**Question 18** [Ans: 88]

Let $ABC$ be a triangle with $AB = 10$ and $\dfrac{\cos A}{\cos B} = \dfrac{AC}{BC} = \dfrac{4}{3}$. Let $P$ be a point on the inscribed circle of triangle $ABC$. Find the largest possible value of $PA^2+PB^2+PC^2$.

**Solution.** By inspection, it is obvious that $\triangle ABC$ is a $6-8-10$ right triangle, with $a = 6$, $b = 8$ and $c = 10$ (where $a$, $b$ and $c$ denote the sides opposite $A$, $B$ and $C$ respectively). From $[ABC] = rs$, we see that the inradius of $\triangle ABC$ is 2. Let $A(8, 0)$, $B(0, 6)$ and $C(0, 0)$. Then the incircle has equation

$$(x-2)^2 + (y-2)^2 = 2^2. \tag{1}$$

Our goal now is to maximize $PA^2 + PB^2 + PC^2$, which can be expressed as

$$\left[(x-8)^2 + y^2\right] + \left[x^2 + (y-6)^2\right] + \left[x^2 + y^2\right],$$

where $(x, y)$ are subject to (1). We can rewrite the above expression as

$$3\left[(x-2)^2 + (y-2)^2\right] + 76 - 4x.$$

Using (1), this simplifies to $88 - 4x$. The largest possible value of $PA^2 + PB^2 + PC^2$ is hence 88, where $P(0, 2)$.

**Question 19** [Ans: 65]

A basket contains 19 apples labelled by the numbers $2, 3, \ldots, 20$, and 19 bananas labelled by the numbers $2, 3, \ldots, 20$. Ah Beng picks $m$ apples and $n$ bananas from the basket. However, he needs to ensure that for any apple labelled $a$ and any banana labelled $b$ that he picks, $a$ and $b$ are relatively prime. Determine the largest possible value of $mn$.

**Solution.** Let $[n]$ be the set of prime factors of $n$. For instance, $[18] = \{2, 3\}$. The following table arranges all integers $n \in [2, 20]$ according to $[n]$:

| $[n]$ | $n$ | $[n]$ | $n$ | $[n]$ | $n$ |
|---|---|---|---|---|---|
| (**2**) | 2, 4, 8, 16 | (**11**) | 11 | (**2,3**) | 6, 12, 18 |
| (**3**) | 3, 9 | (**13**) | 13 | (**2,5**) | 10, 20 |
| (**5**) | 5 | (**17**) | 17 | (**2,7**) | 14 |
| (**7**) | 7 | (**19**) | 19 | (**3,5**) | 15 |

We now create two sets by combining the above sets. It seems clear that the largest set can be formed by grouping (2), (3), (5), (2, 3), (2, 5) and (3, 5) together. This gives a set of size 13. By grouping the remaining sets (except for (2, 7), since it would be in conflict with (2)), we get another set of size 5. Hence, $\max mn = 13 \cdot 5 = 65$.

**Question 20** [Ans: 25]

Let $p(x) = ax^2 - bx + c$ be a polynomial where $a$, $b$, $c$ are positive integers and $p(x)$ has two distinct roots in $(0, 1)$. Determine the least possible value of $abc$.

**Solution.** Since $p(x)$ has two distinct roots, its discriminant must be positive:

$$b^2 - 4ac > 0. \tag{1}$$

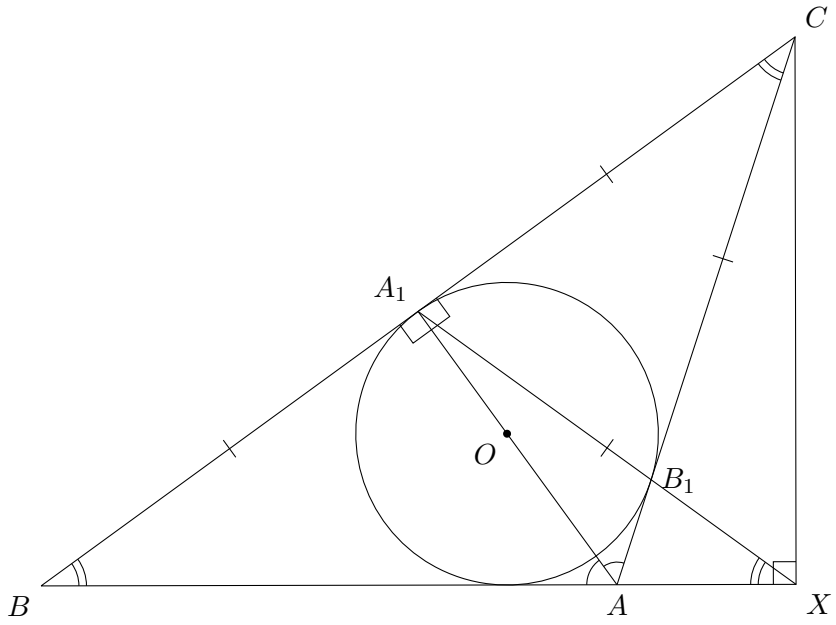
Furthermore, the two roots are in $(0, 1)$. By the quadratic formula, we have $\frac{b \pm \sqrt{b^2-4ac}}{2a} \in (0, 1)$, implying $\sqrt{b^2 - 4ac} \in [0, 2a - b)$. Squaring, we get $b^2 - 4ac < (2a - b)^2$, whence

$$a > b - c. \tag{2}$$

Since we wish to find the smallest value of $abc$, we fix $c = 1$. From (2), we see that $a = b$. (1) thus implies that $a = b = 5$, whence $abc = 25$.

**Question 21** [Ans: 108]

In the triangle $ABC$, $\angle A > 90°$, the incircle touches the side $BC$ and $AC$ at $A_1$ and $B_1$ respectively. The line $A_1B_1$ meets the extension of $BA$ at $X$ such that $CXB = 90°$. Suppose $BC^2 = AB^2 + BC \cdot AC$. Find the size of $\angle A$ in degrees.




**Solution.** Let $\theta = \frac{1}{2}\angle A$, and let $O$ be the incentre of $\triangle ABC$. Since $OA$ bisects $\angle A$, we have $\angle BAO = \angle CAO = \theta$. Since $BC$ and $AC$ are tangent to the circle, we get several equalities: $CA_1 = CB_1$, and $\angle BA_1O = \angle CA_1O = 90°$. This immediately implies that $AOA_1$ is a straight line, thus $\triangle BA_1A \equiv \triangle CA_1A$, whence $BA_1 = CA_1 = CB_1$. Since $A_1$ is the midpoint of $BC$ and $\angle BXC = 90°$, it follows that $XA_1 = CA_1$. We thus have two isosceles triangles, namely $\triangle BA_1X$ and $\triangle A_1CB_1$.

We now find two different expressions for $\angle A_1B_1C$. Firstly, we know that $\angle B_1AX = 180° - 2\theta$, while $\angle A_1XB = \angle A_1BX = 90° - \theta$ (using $\triangle BA_1X$ isosceles), thus $\angle A_1B_1C = \angle AB_1X = 3\theta - 90°$.

Secondly, we have $\angle ACB = \angle ABC = 90° - \theta$ (using $\triangle BA_1A \equiv \triangle CA_1A$). Since $\triangle A_1CB_1$ is isosceles, we have that $\angle A_1B_1C = 45° + \frac{1}{2}\theta$.

Thus, $3\theta - 90° = 45° + \frac{1}{2}\theta$, whence $\angle A = 2\theta = 108°$.

**Question 22** [Ans: 2021]

Find the number of positive integers $n$ such that $7n-16$ divides $n \cdot 13^{2019}$.

**Solution.** Observe that $7 \nmid 7n-16$. Hence, $7n-16 \mid 7n \cdot 13^{2019} = (7n-16) \cdot 13^{2019} + 16 \cdot 13^{2019}$. This gives $7n-16 \mid 2^4 \cdot 13^{2019}$.

*Case 1*. Suppose $7n-16 \leq 0$. Then $n = 1, 2$. Testing, we see that $n = 1$ fails ($-9 \nmid 13^{2019}$) while $n = 2$ works ($-2 \mid 2 \cdot 13^{2019}$).

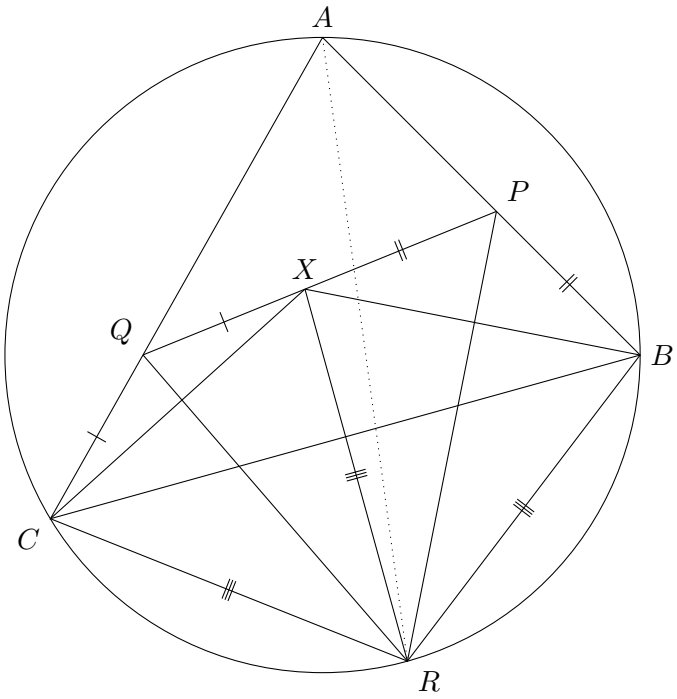
*Case 2*. Suppose $7n-16 > 0$. Then $7n-16$ is a factor of $2^4 \cdot 13^{2019}$. However, observe that $7n-16 \equiv 5 \pmod 7$. It hence suffices to find the number of factors of $2^4 \cdot 13^{2019}$ that have a residue of 5 modulo 7.

Let $7n-16 = 2^a \cdot 13^b$, where $a \leq 4$ and $b \leq 2019$ are non-negative integers. Note that $2^a \cdot 13^b \equiv 5 \pmod 7$ if and only if $a = 1, 4$ and $b$ is odd. This gives $2 \cdot 2020/2 = 2020$ solutions in this case.

Thus, there are a total of $2020 + 1 = 2021$ positive integers $n$ that satisfy the given condition.

**Question 23** [Ans: 75]

In the acute triangle $ABC$, $P$ is a point on $AB$, $Q$ is a point on $AC$ such that $BP + CQ = PQ$. The bisector of $\angle A$ meets the circumcircle of the triangle $ABC$ at the point $R$ distinct from $A$. Suppose $\angle PRQ = 52.5°$. Find the size of $\angle BAC$ in degrees.




**Solution.** By the incenter-excenter lemma, we have $CR = RB$. Let $X$ be the point on $PQ$ such that $CQ = QX$ and $XP = PB$. Note that $\angle XQC + \angle XPB = 180° + \angle A$. Since $\triangle CXQ$ and $\triangle XBP$ are both isosceles, it follows that $\angle QXC + \angle PXB = 90° - \frac{1}{2}\angle A$. Hence, $\angle CXB = 90° + \frac{1}{2}\angle A$. Also, since $ABRC$ is cyclic, $\angle CRB = 180° - \angle A$, whence reflex $\angle CRB = 180° + \angle A = 2\angle CXB$. Hence, $R$ is the circumcentre of $\triangle CXB$, implying that $RX = CR = RB$. Thus, $QCXR$ and $PXBR$ are kites, hence $\angle QRX = \angle QRC$, and $\angle PRX = \angle PRB$. Thus, $\angle CRB = 2\angle PRQ = 105°$, whence $\angle A = 180° - 105° = 75°$.

**Question 24** [Ans: 6]

Let $S = \displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x$. Determine the value of $\lfloor S^2 \rfloor$.

**Solution.** The generalized Gaussian integral $\int_{-\infty}^{\infty} e^{-kx^2}\,\mathrm{d}x$ evaluates to $\sqrt{\pi/k}$. Hence, $\lfloor S^2 \rfloor = \lfloor 2\pi \rfloor = 6$.

**Question 25** [Ans: 5]

Let $p$, $q$, $r$ be positive numbers with $p - r = 4q$ and $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be two sequences defined by $a_1 = p$, $b_1 = q$ and for $n \geq 2$,

$$a_n = pa_{n-1}, \quad b_n = qa_{n-1} + rb_{n-1}.$$

Find the value of $\displaystyle\lim_{n\to\infty} \frac{\sqrt{a_n^2 + (3b_n)^2}}{b_n}$.

**Solution 1.** Observe that $a_n$ is simply a geometric series, with $a_n = p^n$. Hence, $b_n = qp^{n-1} + rb_{n-1}$. We now claim that $b_n = \frac{1}{4}\left(p^n - r^n\right)$. We prove this via induction. Observe that the base case $b_1 = q = \frac{1}{4}(4q) = \frac{1}{4}\left(p^1 - r^1\right)$ holds. Suppose that $b_k = \frac{1}{4}\left(p^k - r^k\right)$ for some positive integer $k$. Then

$$\begin{aligned} b_{k+1} &= qp^k + r\left[\frac{1}{4}\left(p^k - r^k\right)\right] = \frac{1}{4}\left(p - r\right)p^k + \frac{1}{4}rp^k - \frac{1}{4}r^{k+1} \\ &= \frac{1}{4}\left(p^{k+1} - rp^k + rp^k - r^{k+1}\right) = \frac{1}{4}\left(p^{k+1} - r^{k+1}\right), \end{aligned}$$

closing the induction. The limit hence evaluates to

$$\lim_{n\to\infty} \frac{\sqrt{a_n^2 + (3b_n)^2}}{b_n} = \lim_{n\to\infty} \sqrt{\left(\frac{a_n}{b_n}\right)^2 + 9} = \lim_{n\to\infty} \sqrt{\left(\frac{p^n}{\left(p^n - r^n\right)/4}\right)^2 + 9} = \sqrt{4^2 + 9} = 5.$$

**Solution 2.** Observe that

$$\frac{b_n}{a_n} = \frac{qa_{n-1} + rb_{n-1}}{pa_{n-1}} = \frac{q}{p} + \frac{r}{p}\frac{b_{n-1}}{a_{n-1}}.$$

Assuming that $b_n/a_n$ converges, define $L = \lim_{n\to\infty} b_n/a_n$. Then

$$L = \frac{q}{p} + \frac{r}{p}L \implies (p - r)\,L = q \implies L = \frac{q}{p - r} = \frac{q}{4q} = \frac{1}{4}.$$

Hence, the desired limit is

$$\lim_{n\to\infty} \frac{\sqrt{a_n^2 + (3b_n)^2}}{b_n} = \lim_{n\to\infty} \sqrt{\left(\frac{a_n}{b_n}\right)^2 + 9} = \lim_{n\to\infty} \sqrt{\frac{1}{L^2} + 9} = \sqrt{16 + 9} = 5.$$

# 8. SMO 2022

## 8.1. Open Section

### 8.1.1. Round 1 Solutions

**Resources:** Review by *Way Tan*

**Question 1** [Ans: 4043]

If $S = \sum_{k=-2021}^{2021} \frac{1}{10^k + 1}$, find $\lfloor 2S \rfloor$.

**Solution.** Observe that $\frac{1}{10^{-k}+1} = \frac{10^k}{10^k+1}$. The $-k$th term can thus be paired with the $k$th term to form $\frac{10^k+1}{10^k+1} = 1$, with the 0th term being the only unpaired term. The sum hence evaluates to

$$S = 2021 + \frac{1}{10^0 + 1} \implies \lfloor 2S \rfloor = 4043.$$

**Question 2** [Ans: 1303]

All the positive integers $1, 2, 3, 4, \cdots$, are grouped in the following way: $G_1 = \{1, 2\}$, $G_2 = \{3, 4, 5, 6\}$, $G_3 = \{7, 8, 9, 10, 11, 12, 13, 14\}$, that is, the set $G_n$ contains the next $2^n$ positive integers listed in ascending order after the set $G_{n-1}$, $n > 1$. If $S$ is the sum of all the positive integers from $G_1$ to $G_8$, find $\lfloor \frac{S}{100} \rfloor$.

**Solution.** It is not too hard to show that the last term in $G_n$ is $2^{n+1} - 2$. The last term of $G_8$ is hence $2^9 - 2 = 510$, whence $S$ evaluates to $510(511)/2$. Thus, $\lfloor S/100 \rfloor = 1303$.

**Question 3** [Ans: 100]

A sequence of one hundred positive integers $x_1, x_2, x_3, \cdots, x_{100}$ are such that

$$(x_1)^2 + (2x_2)^2 + (3x_3)^2 + (4x_4)^2 + \cdots + (100x_{100})^2 = 338350.$$

Find the largest possible value of $x_1 + x_2 + x_3 + \cdots + x_{100}$.

**Solution.** Observe that $1^2 + 2^2 + 3^2 + \cdots + 100^2 = 338350$. Since $x_i$ are positive integers, they must all be 1 (to prevent the sum from exceeding 338350). The desired value is hence 100.

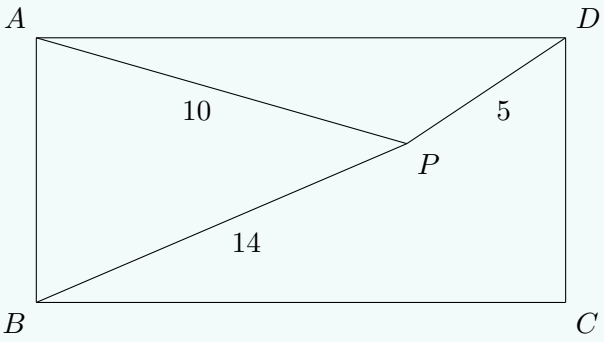
**Question 4** [Ans: 9]

Let $a$ and $b$ be two real numbers satisfying $a < b$, and such that for each real number $m$ satisfying $a < m < b$, the circle $x^2 + (y - m)^2 = 25$ meets the parabola $4y = x^2$ at four distinct points in the Cartesian plane. Let $S$ be the maximum possible value of $b - a$. Find $\lfloor 4S \rfloor$.

**Solution.** Note that $a$ is clearly 5. Now consider the extreme case where $m > 5$ and the circle is tangent to the parabola. The discriminant of the quadratic $4y + (y - m)^2 = 25$ must be 0, whence $m = 29/4$. Hence, $S = b - a = 29/4 - 5 = 9/4$. Thus, $\lfloor 4S \rfloor = 9$.

**Question 5** [Ans: 11]

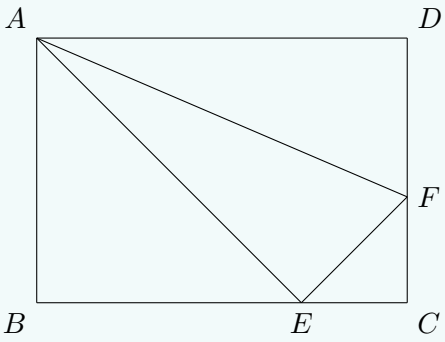
Let $P$ be a point within a rectangle $ABCD$ such that $PA = 10$, $PB = 14$ and $PD = 5$, as shown below. Find $\lfloor PC \rfloor$.




**Solution.** By the British flag theorem, one has $10^2 + PC^2 = 5^2 + 14^2$, whence $\lfloor PC \rfloor = 11$.

**Question 6** [Ans: 50]

In the diagram below, the rectangle $ABCD$ has area 180 and both triangles $ABE$ and $ADF$ have areas 60. Find the area of triangle $AEF$.




**Solution.** Let $BE = x$ and $DF = y$. Observe that $x/(x + EC) = [ABE]/[ABC] = 60/90 = 2/3$. Hence, $EC = x/2$. Similarly, $y/(y + FC) = [ADF]/[ADC] = 60/90 = 2/3$, whence $FC = y/2$. Thus, $[CEF] = [CDE]/3 = [ABE]/6 = 10$. Finally, $[AEF] = 180 - 60 - 60 - 10 = 50$.

**Question 7** [Ans: 144]

A tetrahedron in $\mathbb{R}^3$ has one vertex at the origin $O$ and other vertices at the points $A(6,0,0)$, $B(4,2,4)$ and $C(3,2,6)$. If $x$ is the height of the tetrahedron from $O$ to the plane $ABC$, find $\lfloor 5x^2 \rfloor$.

**Solution.** The plane $ABC$ is given by the vector equation $\mathbf{r} \cdot (2,0,1) = 12$. Since $x$ is the perpendicular distance from $O$ to $ABC$, we have $x = 12/\sqrt{2^2+0^2+1^2} = 12/\sqrt{5}$. Thus, $\lfloor 5x^2 \rfloor = 144$.

**Question 8** [Ans: 208]

Let $x$ and $y$ be real numbers such that $(x-2)^2 + (y-3)^2 = 4$. If $S$ is the largest possible value of $x^2 + y^2$, find $\lfloor (S-17)^2 \rfloor$.

**Solution.** Observe that $(x-2)^2 + (y-3)^2 = 4$ describes a circle with centre $(2,3)$ and radius 2, while $S$ is the square of the distance from some point $P$ on the circle to the origin. The largest distance clearly occurs when the origin, the centre $(2,3)$, and $P$ are collinear. This distance can be calculated as $\sqrt{2^2+3^2} + 2 = \sqrt{13} + 2$. Hence,

$$\lfloor (S-17)^2 \rfloor = \left\lfloor \left( (\sqrt{13}+2)^2 - 17 \right)^2 \right\rfloor = 208.$$

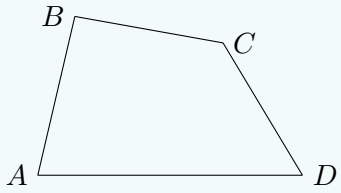
**Question 9** [Ans: 18]

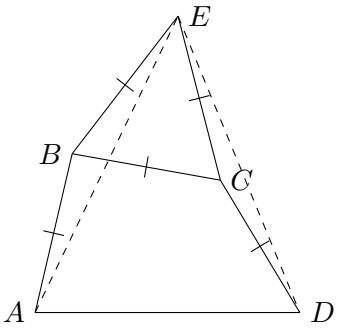
Let $S$ be the maximum value of $w^3 - 3w$ subject to the condition that $w^4 + 9 \leq 10w^2$. Find $\lfloor S \rfloor$.

**Solution.** Consider $w^4 + 9 \leq 10w^2$. Solving, we have $(w^2-9)(w^2-1) \leq 0$, whence $w \in [-3,-1] \cup [1,3]$. Now notice that $w^3 - 3w$ is odd and is increasing when $w < -1$ or $w > 1$. The maximum value thus occurs either at $w = -1$ or 3. Comparing values, we see that $\lfloor S \rfloor = \lfloor 3^3 - 3 \cdot 3 \rfloor = 18$.

**Question 10** [Ans: 40]

In the quadrilateral $ABCD$ below, it is given that $AB = BC = CD$ and $\angle ABC = 80°$ and $\angle BCD = 160°$. Suppose $\angle ADC = x°$. Find the value of $x$.




**Solution.** Let $E$ be such that $\triangle BCE$ is equilateral, as shown above. Note that $\triangle ABE$ is isosceles. Since $\angle ABE = 80° + 60° = 140°$, we have $\angle BAE = \angle BEA = 20°$. Hence, $\angle AED = 60° - 20° = 40°$. However, notice that $\triangle DCE$ is also isosceles, and reflex $\angle CDE = 360° - 60° - 160° = 140°$. Thus, $\angle CED = \angle CDE = 20°$, whence $\triangle ABE \equiv \triangle DCE$, implying $AE = DE$. Because $\angle AED = 40° + 20° = 60°$, it follows that $\triangle AED$ is equilateral, thus giving $x° = 60° - 20° = 40°$.

**Question 11** [Ans: 544]

Let $a$, $b$, $c$ be integers with $ab + c = 49$ and $a + bc = 50$. Find the largest possible value of $abc$.

**Solution.** Subtracting the two equations, we obtain $a + bc - ab - c = 1$. This can be factorized as $(b - 1)(c - a) = 1$. Since $a$, $b$ and $c$ are integers, we are left with two cases:

*Case 1*: $b - 1 = c - a = -1$. We have $b = 0$, whence $abc = 0$.

*Case 2*: $b - 1 = c - a = 1$. We have $b = 2$ and $c = 1 + a$. Substituting this back into one of the original equations, we get $a = 16$ and $c = 17$, whence $abc = 544$.

The largest possible value of $abc$ is thus 544.

**Question 12** [Ans: 17]

Find the largest possible value of $|a| + |b|$, where $a$ and $b$ are coprime integers (i.e., $a$ and $b$ are integers which have no common factors larger than 1) such that $\frac{a}{b}$ is a solution of the equation below:

$$\sqrt{4x + 5 - 4\sqrt{x+1}} + \sqrt{x + 2 - 2\sqrt{x+1}} = 1.$$

**Solution.** Note that

$$4x + 5 - 4\sqrt{x+1} = \left(2\sqrt{x+1}\right)^2 - 2 \cdot \sqrt{x+1} \cdot 1 + 1^2 = \left(2\sqrt{x+1} - 1\right)^2$$

and

$$x+2-2\sqrt{x+1}=\left(\sqrt{x+1}\right)^2-2\cdot\sqrt{x+1}\cdot 1+1^2=\left(\sqrt{x+1}-1\right)^2.$$

We hence have

$$\pm\left(2\sqrt{x+1}-1\right)\pm\left(\sqrt{x+1}-1\right)=1.$$

*Case 1*: $\left(2\sqrt{x+1}-1\right)+\left(\sqrt{x+1}-1\right)=1$. We have $3\sqrt{x+1}=3$, whence $x=0$. Hence, $a=0$ and $b=1$, whence $|a|+|b|=1$.

*Case 2*: $\left(2\sqrt{x+1}-1\right)-\left(\sqrt{x+1}-1\right)=1$. We have $\sqrt{x+1}=3$, whence $x=8$. Hence, $a=8$ and $b=1$, whence $|a|+|b|=9$.

*Case 3*: $-\left(2\sqrt{x+1}-1\right)+\left(\sqrt{x+1}-1\right)=1$. We have $\sqrt{x+1}=-1$, a contradiction.

*Case 4*: $-\left(2\sqrt{x+1}-1\right)-\left(\sqrt{x+1}-1\right)=1$. We have $3\sqrt{x+1}=1$, whence $x=-8/9$. Hence, $a=-8$ and $b=9$, whence $|a|+|b|=17$.

The maximum value of $|a|+|b|$ is 17.

**Question 13** [Ans: 3000]

Let $S$ be the set of real solutions $(x,y,z)$ of the following system of equations:

$$\begin{cases}\dfrac{4x^2}{1+4x^2}=y,\\[2ex] \dfrac{4y^2}{1+4y^2}=z,\\[2ex] \dfrac{4z^2}{1+4z^2}=x.\end{cases}$$

For each $(x,y,z)\in S$, define $m(x,y,z)=2000(|x|+|y|+|z|)$. Determine the maximum value of $m(x,y,z)$ over all $(x,y,z)\in S$.

**Solution.** Taking reciprocals, we have

$$\begin{cases}1+\dfrac{1}{4x^2}=\dfrac{1}{y},\\[2ex] 1+\dfrac{1}{4y^2}=\dfrac{1}{z},\\[2ex] 1+\dfrac{1}{4z^2}=\dfrac{1}{x}.\end{cases}$$

Summing, we obtain

$$\left[\frac{1}{(2x)^2}-\frac{1}{x}+1\right]+\left[\frac{1}{(2y)^2}-\frac{1}{y}+1\right]+\left[\frac{1}{(2z)^2}-\frac{1}{z}+1\right]=0.$$

This clearly factors as

$$\left(\frac{1}{2x}-1\right)^2+\left(\frac{1}{2y}-1\right)^2+\left(\frac{1}{2z}-1\right)^2=0,$$

whence $x=y=z=1/2$, giving $\max m(x,y,z)=3000$.

**Question 14** [Ans: 11]

Assume that $t$ is a positive solution to the equation

$$t = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+t}}}}.$$

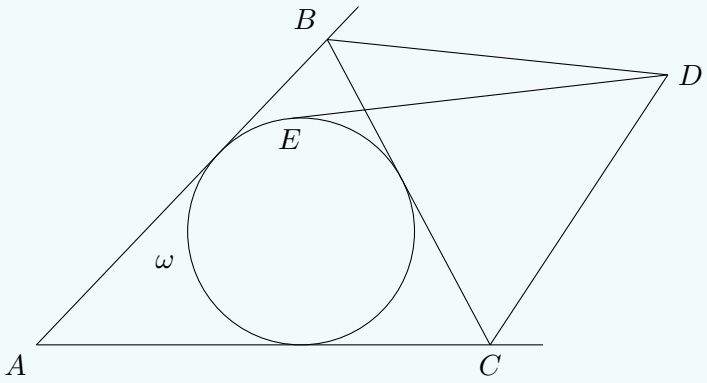
Determine the value of $t^4 - t^3 - t + 10$.

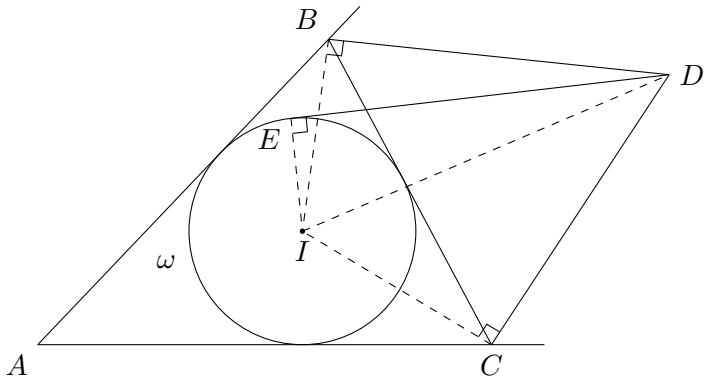
**Solution.** Observe that $t = \sqrt{1+t}$. It follows that $t^2 - t - 1 = 0$, whence $t$ is the golden ratio $\varphi$, which has the property that $\varphi^n = \varphi^{n-1} + \varphi^{n-2}$ for all integers $n$. The desired value is hence

$$t^4 - t^3 - t + 10 = t^2 - t + 10 = 1 + 10 = 11.$$

**Question 15** [Ans: 44]

In the triangle $ABC$ shown in the diagram below, the external angle bisectors of $\angle B$ and $\angle C$ meet at the point $D$. The tangent from $D$ to the incircle $\omega$ of the triangle $ABC$ touches $\omega$ at $E$, where $E$ and $B$ are on the same side of the line $AD$. Suppose $\angle BEC = 112°$. Find the size of $\angle A$ in degrees.




**Solution.** Let $I$ be the incentre of $\triangle ABC$. Observe that $BI$ and $IC$ bisect $\angle B$ and $\angle C$ respectively. Hence, $\angle IBD = \angle ICD = 90°$. Furthermore, since $ED$ is tangent to $\omega$, we have $\angle IED = 90°$. Thus, $B$, $C$, $D$, $E$ and $I$ are concyclic, with $ID$ as the diameter. Hence, $\angle BIC = \angle BEC = 112°$. Since $IB$ and $IC$ are angle bisectors, we have

$$\angle B + \angle C = 2\,(\angle IBC + \angle ICB) = 2\,(180° - 112°) = 136°.$$

It immediately follows that $\angle A = 180° - 136° = 44°$.

**Question 16** [Ans: 99]

Find the largest integer $n$ such that $n^2+5n-9486 = 10s(n)$, where $s(n)$ is the product of all digits of $n$ in the decimal representation of $n$.
(For example, $s(481) = 4 \times 8 \times 1 = 32$.)

**Solution.** Observe that $s(n) \leq n$ for all $n$. We thus have the inequality $n^2 + 5n - 9486 \leq 10n$, whence it is clear that the largest possible $n$ is 99, which does indeed satisfy the given equality. Hence, $\max n = 99$.

**Question 17** [Ans: 8]

Find the number of integer solutions to the equation $19x + 93y = 4xy$.

**Solution.** Note that $(ax+b)(cy+d) = adx+acxy+bcy+bd$, where $a$, $b$, $c$ and $d$ are real numbers. Comparing this to the given equation, we have $ad = 19$, $ac = -4$ and $bc = 93$. Taking $a = 1$, $c = -4$, $b = -93/4$ and $d = 19$, we have

$$\left(x - \frac{93}{4}\right)(-4y + 19) = -\frac{19 \cdot 93}{4}.$$

Upon simplification, one gets

$$(4x - 93)(4y - 19) = -1 \cdot 3 \cdot 19 \cdot 31.$$

Observe that both $4x - 93$ and $19 - 4y$ are congruent to 1 modulo 4. On the other hand, all four factors ($-1$, 3, 19 and 31) are also congruent to 3 modulo 4. We must hence have an even number of factors contributing to both terms. This narrows the possibilities down to a few cases.

*Case 1*: *Both terms have two factors.* The number of possibilities in this case is ${}^4C_2 = 6$.

*Case 2*: *One term has four factors, the other has none.* The number of possibilities in this case is clearly 2.

Altogether, there are $6 + 2 = 8$ possibilities, thus there are 8 integer solutions to the given equation.

**Question 18** [Ans: 121]

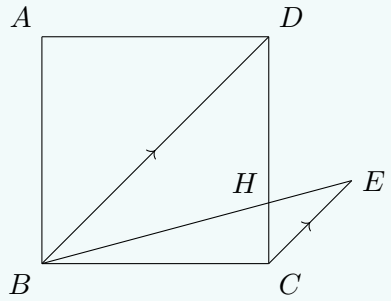
Find the number of integer solutions to the equation $x_1 + x_2 - x_3 = 20$ with $x_1 \geq x_2 \geq x_3 \geq 0$.

**Solution.** Rewriting the equation, we have $x_1 + x_2 = 20 + x_3$. Suppose $x_3$ is fixed (and admits possible values of $x_2$ and $x_1$). Observe that $x_1 \leq 20$, with equality only when $x_2 = x_3$. The only possible values of $x_1$ are hence $\{20, 19, 18, \cdots, x_2\}$. However, since $x_1 \geq x_2$, we have the condition $x_2 \leq \left\lfloor \frac{1}{2}(20 + x_3) \right\rfloor$. The number of solutions for a given $x_3$ is hence $11 - \left\lfloor \frac{1}{2}x_3 \right\rfloor$. The total number of integer solutions is thus

$$11 + 10 + 10 + 9 + 9 + \cdots + 1 + 1 = 11 + 2 \cdot \frac{10 \cdot 11}{2} = 121.$$

**Question 19** [Ans: 2]

In the diagram below, $E$ is a point outside a square $ABCD$ such that $CE$ is parallel to $BD$, $BE = BD$, and $BE$ intersects $CD$ at $H$. Given $BE = \sqrt{6} + \sqrt{2}$, find the length of $DH$.




**Solution 1.** Observe that $\triangle HDB$ is similar to $\triangle HCE$. Hence,

$$\frac{EH}{BH} = \frac{CH}{DH} \implies \frac{BE - BH}{BH} = \frac{DC - DH}{DH} \implies \frac{BE}{BH} = \frac{DC}{DH}.$$

Note that the side length of the square is $\frac{1}{\sqrt{2}}BD = \sqrt{3} + 1$. Thus,

$$\frac{\sqrt{6} + \sqrt{2}}{BH} = \frac{\sqrt{3} + 1}{DH} \implies BH^2 = 2DH^2.$$

Using Pythagoras' theorem on $\triangle BCH$, one obtains

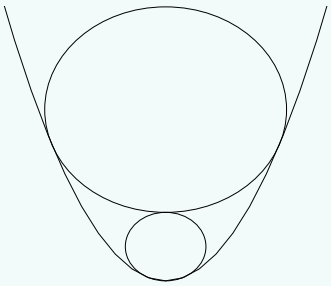
$$BC^2 + CH^2 = BH^2 \implies \left(\sqrt{3} + 1\right)^2 + \left(\sqrt{3} + 1 - DH\right)^2 = 2DH^2.$$

Solving, we have $DH = 2$.

**Solution 2** (Abusing integers)**.** Note that the side length of the square is $\frac{1}{\sqrt{2}}BD = \sqrt{3} + 1 \approx 2.73$. Since $DH$ is an integer and $CH < DH < BD$, it must be 2.

**Question 20** [Ans: 4]

The diagram below shows the region $R = \left\{(x, y) \in \mathbb{R}^2 | y \geq \frac{1}{2}x^2\right\}$ on the $xy$-plane bounded by the parabola $y = \frac{1}{2}x^2$. Let $C_1$ be the largest circle lying inside $R$ with its lowest point at the origin. Let $C_2$ be the largest circle lying inside $R$ and resting on top of $C_1$. Find the sum of radii of $C_1$ and $C_2$.



**Solution.** Let the radius of $C_1$ and $C_2$ be $r_1$ and $r_2$ respectively. The equations of $C_1$ and $C_2$ are hence

$$C_1 : x^2 + (y - r_1)^2 = r_1^2,$$
$$C_2 : x^2 + (y - (2r_1 + r_2))^2 = r_2^2.$$

Consider the intersection between $C_1$ and the parabola:

$$\begin{cases} y = \dfrac{1}{2}x^2 \\ x^2 + (y - r_1)^2 = r_1^2 \end{cases}$$

This gives $y^2 + 2y(1 - r_1) = 0$. Since the two curves only intersect at the origin, we have $r_1 = 1$.

Now consider the intersection between $C_2$ and the parabola:

$$\begin{cases} y = \dfrac{1}{2}x^2 \\ x^2 + (y - (2r_1 + r_2))^2 = r_2^2 \end{cases}$$

This gives $y^2 - 2y(1 + r_2) + 4(1 + r_2) = 0$. By symmetry, the two curves intersect at a unique $y$-value, hence the discriminant is 0. We hence obtain $4(1 + r_2)^2 - 16(1 + r_2) = 0$, whence $r_2 = 3$. The required sum is thus $1 + 3 = 4$.

**Question 21** [Ans: 30]

Find the smallest positive integer $x$ such that $3x^2 + x = 4y^2 + y$ for some positive integer $y$.

**Solution.** Completing the square, one gets

$$4(6x + 1)^2 - 4 = 3(8y + 1)^2 - 3$$

after simplification. This can be rewritten as

$$(12x + 2)^2 - 3(8y + 1)^2 = 1,$$

which one may recognize as a case of Pell's equation. We hence consider the equation $X^2 - 3Y^2 = 1$. The fundamental solution is clearly $X = 2$ and $Y = 1$. We now have the following standard recurrence relations for $X$ and $Y$:

$$X_{k+1} = 2X_k + 3Y_k, \quad Y_{k+1} = 2Y_k + X_k.$$

Keeping in mind that $X$ is of the form $12x + 2$ and $Y$ is of the form $8y + 1$, the first valid solution occurs when $k = 5$, where $X = 362$ and $Y = 209$, which corresponds to $x = 30$ and $y = 26$.

**Question 22** [Ans: 200]

A group of students participates in some sports activities among 6 different types of sports. It is known that for each sport activity there are exactly 100 students in the group participating in it; and the union of all the sports activities participated by any two students is NOT the entire set of 6 sports activities. Determine the minimum number of students in the group.

**Solution.** Let $m$ be the maximum number of sports a student can take at once. By symmetry, we only need to consider $m \geq 3$ (any $m < 3$ will lead to a less-than-optimal allocation of students). Furthermore, it is obvious that $m \geq 5$ is impossible, since it would immediately violate the given restriction. We hence analyse only the $m = 3$ and $m = 4$ case.

*Case 1*: $m = 4$. To adhere to the given restriction, there is only one possible allocation of students: 100 to Sport A, 100 to Sport B and 100 taking the other four sports. This gives a total of 300 students.

*Case 2*: $m = 3$. Note that the absolute minimum number of students is given by $100 \cdot 6/3 = 200$. We now construct an allocation that uses exactly 200 students.

Let the sports be labelled A through F. Consider the following allocation of students:

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| Student 1 | X | X | X | | | |
| Student 2 | X | | | X | X | |
| Student 3 | | X | | X | | X |
| Student 4 | | | X | | X | X |

Repeating the above allocation 50 times, we will have 100 students per sport. Hence, the minimum number of students is $4 \cdot 50 = 200$ as desired.

**Question 23** [Ans: 29]

Let $p$ and $q$ be positive prime integers such that $p^3 - 5p^2 - 18p = q^9 - 7q$. Determine the smallest value of $p$.

**Solution.** Observe that the RHS will grow incredibly fast as compared to the LHS. We hence test small values of $q$. Comparing leading terms, we also note that $p \approx q^3$. Furthermore, we require $p^3 - 5p^2 - 18p > 0$, whence $p \geq 11$.

*Case 1*: $q = 2$. We have $p\left(p^2 - 5p - 18\right) = 2 \cdot 3 \cdot 83$. It can be easily shown that there are no solutions in this case.

*Case 2*: $q = 3$. We have $p\left(p^2 - 5p - 18\right) = 2 \cdot 3 \cdot 29 \cdot 113$. Testing $p = 29$, we see that it is indeed a solution.

Hence, the smallest value of $p$ is 29.

**Question 24** [Ans: 27648]

Given that $a$, $b$, $c$ are positive real numbers such that $a+b+c=9$, find the maximum value of $a^2b^3c^4$.

**Solution.** Note that

$$9 = a+b+c = \frac{a}{2}+\frac{a}{2}+\frac{b}{3}+\frac{b}{3}+\frac{b}{3}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4}.$$

By the Cauchy-Schwarz inequality, we obtain

$$\frac{1}{9}\cdot 9 \geq \sqrt[9]{\frac{a^2b^3c^4}{2^2 3^3 4^4}}.$$

Hence, the maximum value of $a^2b^3c^4$ is $2^2 3^3 4^4 = 27648$.

**Question 25** [Ans: 2023]

Let $\mathbb{R}^+$ be the set of all positive real numbers. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying

$$xyf(x)\big(f(y) - f(yf(x))\big) = 1$$

for all $x, y \in \mathbb{R}^+$. Find $f(\frac{1}{2022})$.

**Solution.** Letting $y = 1$, we get the following expression for $f^2(x)$:

$$f^2(x) = f(1) - \frac{1}{xf(x)}. \tag{1}$$

Replacing $y$ with $f(y)$ in the original equation gives

$$f^2(y) = \frac{1}{xf(y)f(x)} + f(f(y)f(x)). \tag{2}$$

Substituting (1) into (2) yields

$$f(1) - \frac{1}{yf(y)} = \frac{1}{xf(y)f(x)} + f(f(y)f(x)). \tag{3}$$

Swapping $x$ and $y$ gives a similar equation:

$$f(1) - \frac{1}{xf(x)} = \frac{1}{yf(x)f(y)} + f(f(x)f(y)). \tag{4}$$

Subtracting (4) from (3) and simplifying, we obtain

$$yf(y) - y = xf(x) - x,$$

from which it is clear that for all $x \in \mathbb{R}^+$, we have $xf(x) - x = c$ for some constant $c$. This immediately gives $f(x) = 1 + \frac{c}{x}$. Substituting this into (1), we have

$$1 + \frac{c}{1+\frac{c}{x}} = \left(1+\frac{c}{1}\right) - \frac{1}{x\left(1+\frac{c}{x}\right)},$$

whence $c^2 = 1$. Since the range of $f$ is $\mathbb{R}^+$, we take $c = 1$, thus $f(x) = 1 + \frac{1}{x}$ and $f(\frac{1}{2022}) = 2023$.

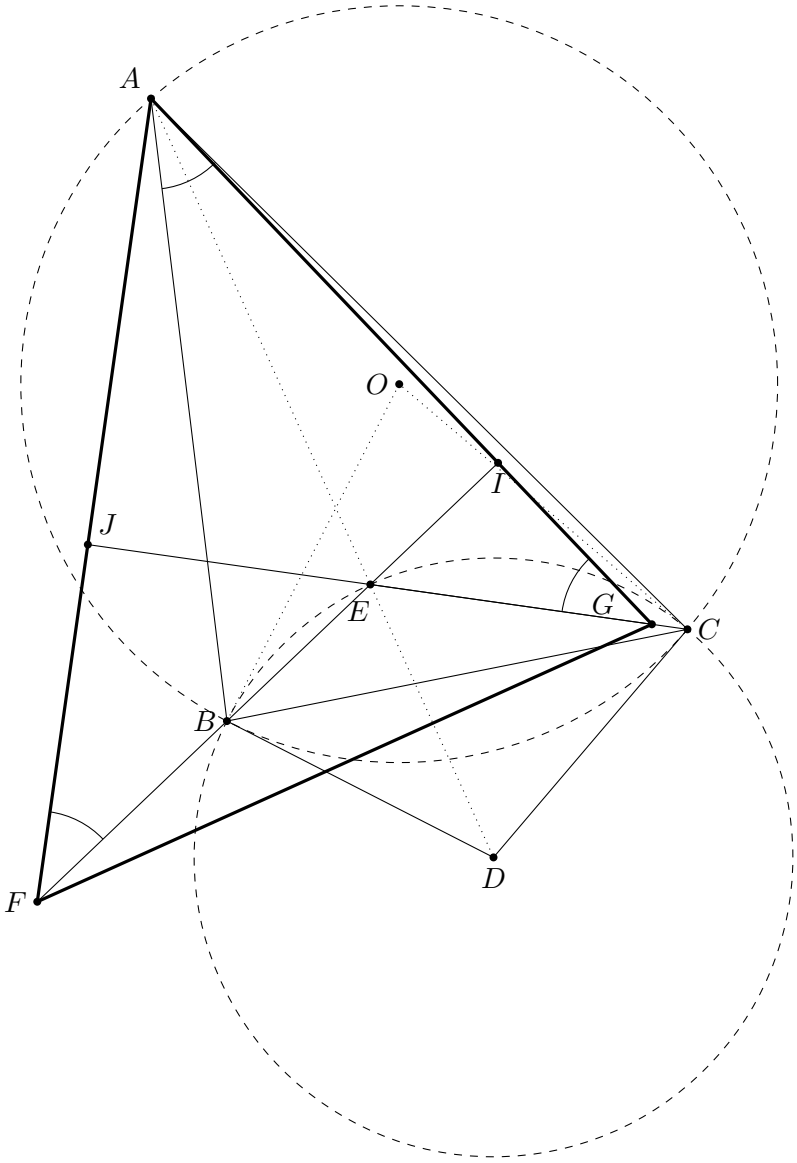
### 8.1.2. Round 2 Solutions

**Resources:** AoPS threads

**Question 1**

For $\triangle ABC$ and its circumcircle $\omega$, draw the tangents at $B$, $C$ to $\omega$ meeting at $D$. let the line $AD$ meet the circle with centre $D$ and radius $DB$ at $E$ inside $\triangle ABC$. Let $F$ be the point on the extension of $EB$ and $G$ be the point on the segment $EC$ such that $\angle AFB = \angle AGE = \angle A$. Prove that the tangent at $A$ to the circumcircle of $\triangle AFG$ is parallel to $BC$.

**Solution** (gghx).




**Claim 1.** *$E$ is the orthocentre of $\triangle AFG$.*

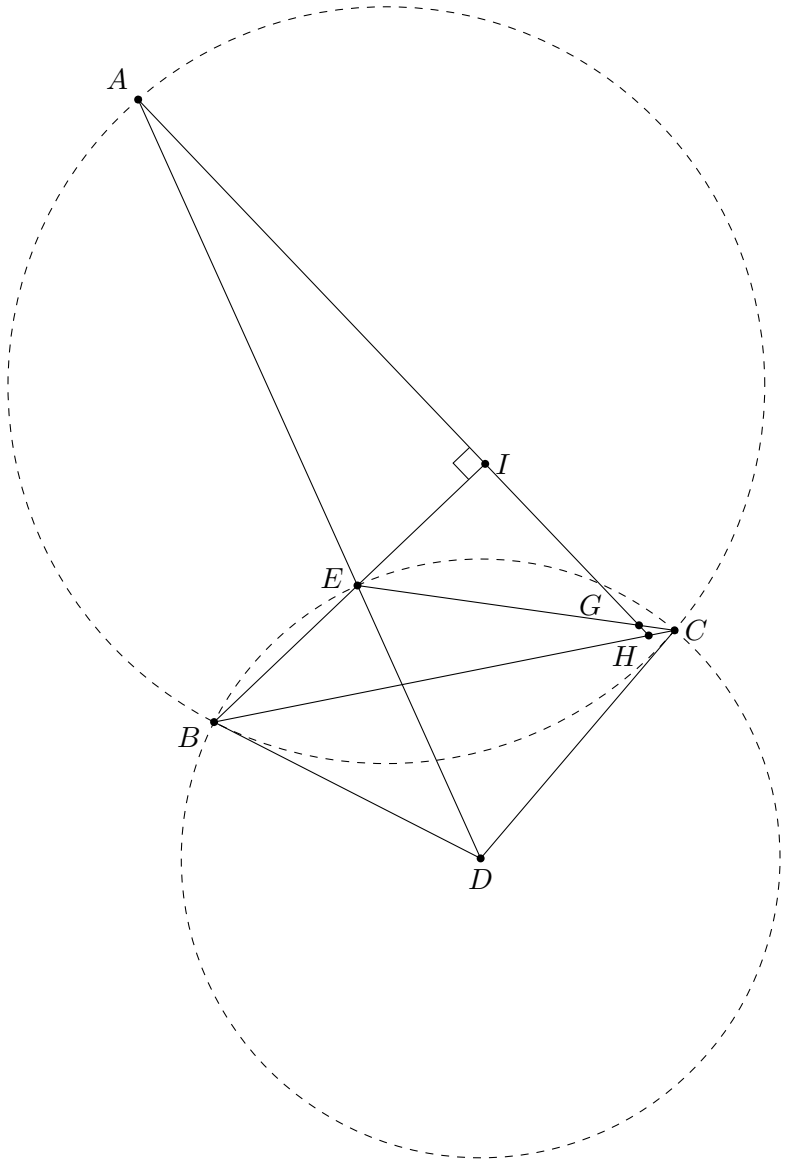
*Proof.* Let $O$ be the circumcentre of $\triangle ABC$. Let $I = AG \cap FE$ and $J = AF \cap GE$. Since $\angle BOC = 2\angle A$ and $\angle OBD = \angle OCD = 90°$, we have $\angle BDC = 180° - 2\angle A$, whence reflex $\angle BDC = 180° + 2\angle A$. Hence, $\angle BEC = 90° + \angle A$. This immediately gives

$$\angle EIC + \angle AGE = 90° + \angle A \implies \angle EIC = 90°$$

and

$$\angle EJF + \angle AFB = 90^\circ + \angle A \implies \angle EJF = 90^\circ.$$

Thus, $E$ is the orthocentre of $\triangle AFG$. □



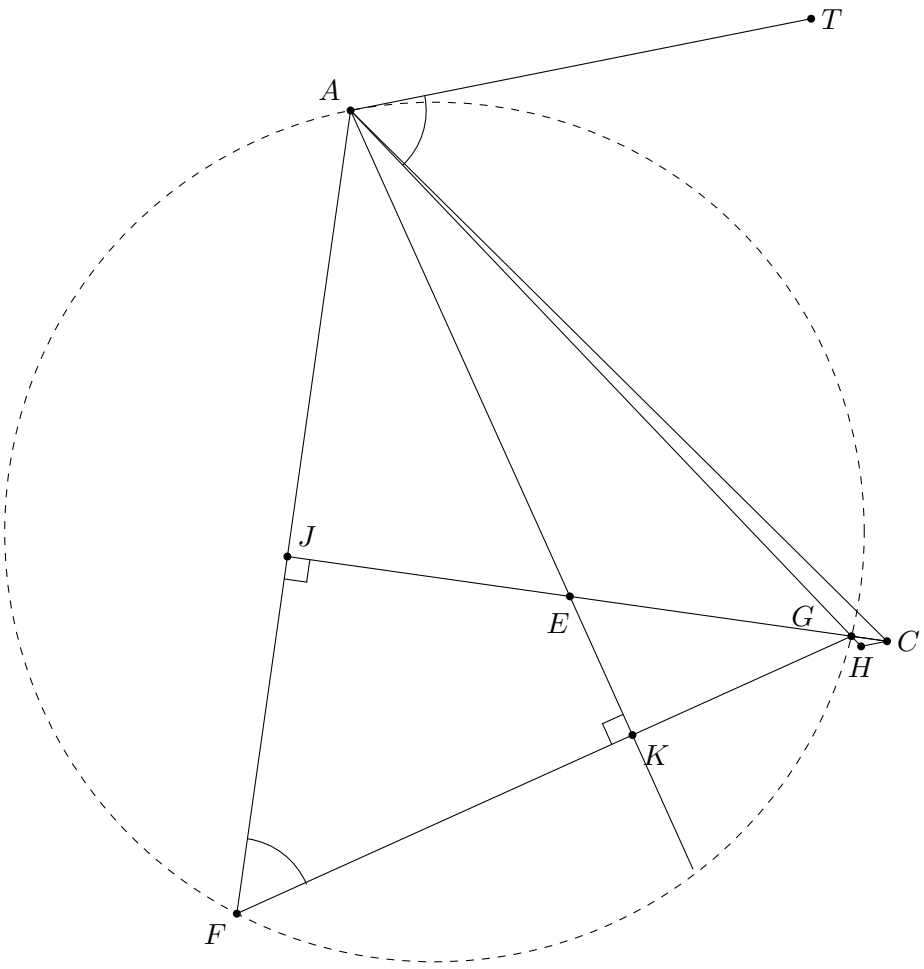

Let $H = AG \cap BC$.

**Claim 2.** *$AEHC$ is cyclic.*

*Proof.* Note that $\triangle BDE$ is isosceles, with $\angle EBD = \angle BED$. Hence, $\angle BDE = 180^\circ - 2\angle BED$. Thus,

$$\angle GCH = \frac{1}{2}\angle BDE = 90^\circ - \angle BED = 90^\circ - \angle AEI = \angle GAE.$$

Thus, by the converse of angles in same segment, $AEHC$ is cyclic. □




**Claim 3.** *The tangent at $A$ to $(AFG)$ is parallel to $HC$ (i.e. $BC$).*

*Proof.* Let $K = AE \cap GF$. Let $T$ be a point on the line parallel to $HC$ through $A$, such that $G$ and $T$ are on the same side of $AE$. It suffices to show that $AT$ is tangent to $(AFG)$ at $A$.

Observe that

$$\angle TAG = 180^\circ - \angle GHC = 180^\circ - \angle GEA = 180^\circ - \angle KEJ = \angle GFA.$$

Thus, by the converse of the alternate segment theorem, $AT$ is tangent to $(AFG)$ at $A$. □

**Question 2**

Prove that if the length and breadth of a rectangle are both odd integers, then there does not exist a point $P$ inside the rectangle such that each of the distances from $P$ to the 4 corners of the rectangle is an integer.

**Solution** (sarjinius)**.** Let $m$ and $n$ be positive odd integers. Let $A(0,0)$, $B(m,0)$, $C(m,n)$ and $D(0,n)$. Let the perpendicular distance from $P$ to $AB$, $BC$, $CD$ and $DA$ be $a$, $b$, $c$ and $d$ respectively, whence $P(d,a)$. Also, we have $b+d=m$ and $a+c=n$. Seeking a contradiction, suppose $PA$, $PB$, $PC$ and $PD$ are all integers.

By Pythagoras' theorem, we have $PA^2 = a^2+d^2$ and $PB^2 = a^2+(m-d)^2$. Thus, both $a^2+d^2$ and $a^2+d^2-2md+m^2$ are perfect squares. This immediately gives

$$d = \frac{m^2+PA^2-PB^2}{2m},$$

whence $d$ is rational with a denominator not divisible by 4 (when expressed in lowest terms). Similarly, we also get

$$a = \frac{n^2 + PB^2 - PC^2}{2n}, \quad b = \frac{m^2 + PC^2 - PD^2}{2m}, \quad c = \frac{n^2 + PD^2 - PA^2}{2n},$$

thus $a$, $b$ and $c$ are also rational with denominator not divisible by 4.

Let $k$ be the denominator of $a$ and $c$, and $l$ be the denominator of $b$ and $d$. Note that $k \mid 2n$ and $l \mid 2m$. We now scale $ABCD$ with respect to $A$ by a factor of $\operatorname{lcm}(k, l)$ (note that this is not divisible by 4). Let $X_*$ represent the point $X$ after scaling and $z_*$ represent the distance $z$ after scaling.

*Case 1*. Suppose $\operatorname{lcm}(k, l)$ is odd. Thus, both $m_*$ and $n_*$ are odd. Since $a_*, b_*, c_*, d_* \in \mathbb{Z}$, with $b_* + d_* = m_*$ and $a_* + c_* = n_*$, it follows that one of $a$ and $c$ is odd, and one of $b$ and $d$ is odd. Hence, one of $PA_*^2 = a_*^2 + d_*^2$, $PB_*^2 = a_*^2 + b_*^2$, $PC_*^2 = c_*^2 + b_*^2$ and $PD_*^2 = c_*^2 + d_*^2$ must be 2 mod 4, which cannot be perfect square, a contradiction.

*Case 2*. Suppose $\operatorname{lcm}(k, l)$ is even. Then at least one of $k$ and $l$ is even (and also 2 mod 4).

*Subcase 2A*. Suppose both $k$ and $l$ are even. This implies that $a_*$, $b_*$, $c_*$ and $d_*$ are all odd. This is a clear contradiction since the sum of two odd squares will always be 2 mod 4, which cannot be a perfect square.

*Subcase 2B*. Without loss of generality, suppose $k$ is odd and $l$ is even. Hence, $a_*$ and $c_*$ are even, while $b_*$ and $d_*$ are odd. However, because $a_* + c_* = n_* \equiv 2 \pmod 4$, without loss of generality, we must have $a_* \equiv 0 \pmod 4$ and $c_* \equiv 2 \pmod 4$. Thus, $a_*^2 \equiv 0 \pmod 8$ while $c_*^2 \equiv 4 \pmod 8$. We also have $b_*^2 \equiv d_*^2 \equiv 1 \pmod 8$. Thus, $PC_*^2 = c_*^2 + b_*^2$ and $PD_*^2 = c_*^2 + d_*^2$ must be 5 mod 8, which cannot be a perfect square, a contradiction. □

**Question 3** [Ans: $f(m) \equiv m$]

Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying

$$m!! + n!! \mid f(m)!! + f(n)!!$$

for each $m, n \in \mathbb{Z}^+$, where $n!! = (n!)!$ for all $n \in \mathbb{Z}^+$.

**Solution.** DVDthe1st When $m = n$, we see that $m!! \mid f(m)!!$. This immediately gives us $f(m) \geq m$. Repeatedly chaining the given relation gives

$$m!! + n!! \mid f(m)!! + f(n)!! \mid f^2(m)!! + f^2(n!!) \mid \cdots \mid f^k(m)!! + f^{k+1}(n)!!, \tag{1}$$

where $k$ is a non-negative integer. We now consider the sequence $\{m, f(m), f^2(m), \ldots\}$.

*Case 1*. Suppose the sequence is unbounded. Let $k > 0$ be the smallest integer such that $m!! + f(m)!! \mid f^{k+1}(m)!!$ ($k$ must exist due to the unboundedness of the sequence). To not violate the minimality of $k$, we must have $m!! + f(m)!! \nmid f^k(m)!!$. Thus, $m!! + f(m)!! \nmid f^k(m)!! + f^{k+1}(m)!!$, contradicting (1).

*Case 2*. Suppose the sequence is bounded. Let $k \geq 0$ be the smallest integer such that $f^k(m) = \max\{m, f(m), f^2(m), \ldots\}$. If $k > 0$, we have

$$f^k(m)!! + f^{k+1}(m)!! \leq 2f^k(m)!! < 2\left(f^{k-1}(m)!! + f^k(m)!!\right).$$

However, from (1), we see that

$$f^{k-1}(m)!! + f^k(m)!! \mid f^k(m)!! + f^{k+1}(m).$$

This immediately gives $f^{k-1}(m)!! + f^k(m)!! = f^k(m)!! + f^{k+1}(m)$, whence

$$f^{k-1}(m) = f^{k+1}(m) \geq f^k(m),$$

which violates the minimality of $k$. Thus, $k = 0$, whence $f(m) \equiv m$ for any initial choice of $m \in \mathbb{Z}^+$, which clearly satisfies the given relation.

**Question 4** [Ans: $n - \gcd(n, k)$]

Let $n$, $k$, $1 \leq k \leq n$ be fixed integers. Alice has $n$ cards in a row, where the card in position $i$ has the label $i + k$ (or $i + k - n$ if $i + k > n$). Alice starts by colouring each card either red or blue. Afterwards, she is allowed to make several moves, where each move consists of choosing two cards of different colours and swapping them. Find the minimum number of moves she has to make (given that she chooses the colouring optimally) to put the cards in order (i.e. card $i$ is at position $i$).

**Solution** (gghx)**.** We claim that Alice requires a minimum of $n - \gcd(n, k)$ moves.

Consider a graph over vertices labelled 1 through $n$. If the card at position $i$ has label $j$, draw a directed edge from $i$ to $j$. Since each vertex has indegree 1 and outdegree 1, the graph is composed of disjoint cycles. We now consider the effect of swapping two cards (say, at positions $i$ and $j$) on our graph.

**Claim 1.** *If the cards at $i$ and $j$ were initially in the same cycle, then the cycle will split into two cycles upon swapping the two cards.*

*Proof.* Without loss of generality, let the cycle that $i$ and $j$ are in be

$$(i', i, i'', \underbrace{\ldots}_{I}, j', j, j'', \underbrace{\ldots}_{J}).$$

Consider the effect of swapping $i$ and $j$ on the cycle. Since the label on the card at position $i'$ is still $i$, we see that $i'$ still maps to $i$. Similarly, $j' \to j$. However, the card at position $j$ (originally at position $i$) now has the label $i''$, hence we now have $j \to i''$. Similarly, $i \to j''$. It is hence easy to see that the cycle now splits as

$$(i', i, j'', \underbrace{\ldots}_{J})(j', j, i'', \underbrace{\ldots}_{I}).$$

□

**Claim 2.** *If the cards at $i$ and $j$ were initially in different cycles, then the two cycles will merge upon swapping the two cards.*

*Proof.* Using a similar argument as Claim 1, the cycles

$$(i', i, i'', .\underbrace{\ldots}_{I})$$

and

$$(j', j, j'', \underbrace{\ldots}_{J})$$

will merge into a single cycle

$$(i', i, j'', \underbrace{\ldots}_{J}, j', j, i'', \underbrace{\ldots}_{I})$$

upon swapping positions $i$ and $j$. □

From Claims 1 and 2, it follows that every move, the number of cycles increases by at most one. We now show that the initial number of cycles is $\gcd(n, k)$.

**Claim 3.** *The number of cycles is initially* $\gcd(n, k)$.

*Proof.* Let $D = \{d \in \mathbb{Z}_n \mid 1 \leq d \leq \gcd(n, k)\}$. Let $C(m)$ be the cycle starting from some integer $m$. Then $C(m)$ is clearly of the form

$$(m, m + k, m + 2k, m + 3k, \ldots, m + (l - 1)k),$$

where $l = \frac{n}{\gcd(n,k)}$ is the smallest positive integer such that $lk \equiv 0 \pmod{n}$.

Consider $C(m)$ for any choice of $m$. Since $k \equiv 0 \pmod{\gcd(n, k)}$, it follows that each member of $C(m)$ is congruent to $m \pmod{\gcd(n, k)}$. In addition, from the minimality of $l$, all elements of $C(m)$ are distinct. Thus, there is exactly one $d \in D$ that is also in $C(m)$ (namely, $d \equiv m \pmod{\gcd(n, k)}$). Conversely, each $d \in D$ has a unique cycle $m$ that it is a member of. Hence, the number of cycles is $\gcd(n, k)$ as desired. □

Since there are $n$ cycles when all cards are in their correct position, Alice must make at least $n - \gcd(n, k)$ moves. We now construct a strategy that guarantees Alice can indeed win in $n - \gcd(n, k)$ moves.

**Claim 4.** *Alice can win in* $n - \gcd(n, k)$ *moves.*

*Proof.* Let all cards with labels $1, 2, \ldots, \gcd(n, k)$ be red, and all other cards be blue. By Claim 3, each cycle initially contains exactly one red card. For each cycle, keep swapping the red card with the card that is pointing towards it. Doing so removes one blue card every move. Since the cycle has length $\frac{n}{\gcd(n,k)}$, each cycle requires $\frac{n}{\gcd(n,k)} - 1$ moves to completely sort it. Since there are $\gcd(n, k)$ cycles to sort, Alice can sort all cycles within $n - \gcd(n, k)$ moves, as desired. □

As an example, consider the case where $n = 6$ and $k = 4$. We start with the following deck, which has been coloured using the strategy in Claim 4:

| 5 | 6 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|

The current cycles are $(1, 5, 3)(2, 6, 4)$. We first focus on the $(1, 5, 3)$ cycle. Since card 3 is pointing to card 1, we swap them:

| 5 | 6 | 3 | 2 | 1 | 4 |
|---|---|---|---|---|---|

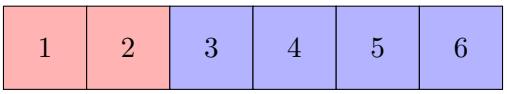
The cycles are now $(1, 5)(2, 6, 4)(3)$. Since card 5 is pointing to card 1, we swap them:

| 1 | 6 | 3 | 2 | 5 | 4 |
|---|---|---|---|---|---|

At this point, the $(1, 5, 3)$ cycle has been completely sorted, taking $\frac{6}{\gcd(6,4)} - 1 = 2$ moves as expected. We now focus on the $(2, 6, 4)$ cycle. Since card 4 is pointing at card 2, we swap them:

| 1 | 6 | 3 | 4 | 5 | 2 |
|---|---|---|---|---|---|

Finally, we swap cards 2 and 6, completing the game in $6 - \gcd(6, 4) = 4$ moves as expected.




**Question 5**

Let $n \geq 2$ be a positive integer. For any integer $a$, let $P_a(x)$ denote the polynomial $x^n + ax$. Let $p$ be a prime number and define the set $S_a$ as the set of residues mod $p$ that $P_a(x)$ attains. That is,

$$S_a = \{b \mid 0 \leq b \leq p-1, \text{ and there is } c \text{ such that } P_a(x) \equiv p \pmod{p}\}.$$

Show that the expression $\frac{1}{p-1} \sum_{a=1}^{p-1} |S_a|$ is an integer.

*Remark.* This question is identical to 2023/Open/R2/Q3.

**Solution** (Evan Chen)**.** Note that $0 \in S_a$ for all integers $a \in [1, p-1]$. We thus consider only non-zero elements of $S_a$. Observe that

$$|S_1 \setminus \{0\}| + |S_2 \setminus \{0\}| + |S_3 \setminus \{0\}| + \cdots + |S_{p-1} \setminus \{0\}|$$

counts the number of elements in

$$\mathcal{T} = \left\{(y, a) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times \mid \exists x \in \mathbb{F}_p, y = x^n + ax\right\}.$$

Because $y = x^n + ax$ implies that $\lambda^n y = (\lambda x)^n + (\lambda^{n-1} a)(\lambda x)$ for any $\lambda \in \mathbb{F}_p^\times$, we have the equivalence relation

$$(y, a) \sim (\lambda^n y, \lambda^{n-1} a).$$

Now observe that the pairs $(\lambda^n y, \lambda^{n-1} a)$ are pairwise distinct:

$$(\lambda_1^n y, \lambda_1^{n-1} a) = (\lambda_2^n y, \lambda_2^{n-1} a) \implies \lambda_1^n \equiv \lambda_2^n \text{ and } \lambda_1^{n-1} \equiv \lambda_2^{n-1} \implies \lambda_1 = \lambda_2.$$

Since there are $p-1$ choices of $\lambda$, each equivalence class generated by the above equivalence relation has exactly $p-1$ elements, whence $|\mathcal{T}|$ is divisible by $p-1$. □

*Remark.* The condition $n \geq 2$ ensures that we can split $\lambda^n$ into $\lambda^{n-1} \cdot \lambda$ to form the equivalence relation.

# 9. SMO 2023

## 9.1. Senior Section

### 9.1.1. Round 1 Solutions

**Resources:** Review by *Way Tan*

**Question 1** [Ans: C]

Find the value of $m$ such that $2x^2 + 3x + m$ has a minimum value of 9.

(A) $\frac{9}{8}$ (B) $-\frac{9}{8}$ (C) $\frac{81}{8}$ (D) $-\frac{81}{8}$ (E) $\frac{63}{8}$

**Solution.** Completing the square, we obtain $2x^2+3x+m = 2(x+\frac{3}{4})^2+(m-\frac{9}{8})$, whence the minimum value is $m-\frac{9}{8}$. Hence, $m = 9+\frac{9}{8} = \frac{81}{8}$.

**Question 2** [Ans: B]

Which of the following is true?

(A) $\sin(105°) - \cos(105)° = \dfrac{\sqrt{3}}{2}$

(B) $\sin(105°) - \cos(105)° = \dfrac{\sqrt{3}}{\sqrt{2}}$

(C) $\sin(105°) + \cos(105)° = \dfrac{1}{2}$

(D) $\sin(105°) + \cos(105)° = \dfrac{1}{\sqrt{3}}$

(E) None of the above

**Solution.** Observe that $105° = 60° + 45°$. Applying the sine and cosine angle addition formulae gives

$$\sin(105°) = \frac{\sqrt{3}}{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right), \quad \cos(105°) = \frac{\sqrt{3}}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right).$$

Hence, $\sin(105°) - \cos(105)° = \frac{\sqrt{2}\sqrt{3}}{2} = \frac{\sqrt{3}}{\sqrt{2}}$.

**Question 3** [Ans: B]

If $\log_{\sqrt{2}} x = 10 - 3\log_{\sqrt{2}} 10$, find $x$.

(A) 0.32 (B) 0.032 (C) 0.0032 (D) 0.64 (E) 0.064

**Solution.** Observe that the RHS can be rewritten as

$$10 - 3\log_{\sqrt{2}} 10 = \log_{\sqrt{2}} \left( \frac{\sqrt{2}^{10}}{10^3} \right) = \log_{\sqrt{2}} 0.032.$$

Thus, $x = 0.032$.

**Question 4** [Ans: A]

If $(x-5)^2 + (y-5)^2 = 5^2$, find the smallest value of $(x+5)^2 + (y+5)^2$.

(A) $225 - 100\sqrt{2}$

(B) $225 + 100\sqrt{2}$

(C) $225\sqrt{2}$

(D) $100\sqrt{2}$

(E) None of the above

**Solution.** Observe that this is equivalent to finding the square of the shortest distance between the circles with centres $(-5,-5)$ and $(5,5)$, and radius 5. The shortest distance is attained along the line segment joining the centres of the two circles together. The shortest distance is thus

$$\left( \sqrt{(5+5)^2 + (5+5)^2} - 5 \right)^2 = 225 - 100\sqrt{2}.$$

**Question 5** [Ans: C]

Suppose $\cos(180^\circ + x) = \dfrac{4}{5}$, where $90^\circ < x < 180^\circ$. Find $\tan(2x)$.

(A) $\dfrac{24}{7}$ (B) $\dfrac{7}{24}$ (C) $-\dfrac{24}{7}$ (D) $-\dfrac{7}{24}$ (E) $-\dfrac{24}{25}$

**Solution.** Note that $\cos(180^\circ + x) = -\cos(x)$. Hence, $\cos(x) = -\frac{4}{5}$, whence $\sin x = \sqrt{1 - (-\frac{4}{5})^2} = \frac{3}{5}$ (note that $\sin x > 0$ since $x$ is in the second quadrant). Thus,

$$\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{2\sin x \cos x}{\cos^2 x - \sin^2 x} = -\frac{24}{7}.$$

**Question 6** [Ans: 7]

Suppose the roots of $x^2 + 11x + 3 = 0$ are $p$ and $q$, and the roots of $x^2 + Bx - C = 0$ are $p+1$ and $q+1$. Find $C$.

**Solution 1.** Let $f(x) = x^2 + 11x + 3$. Consider the map $M : x \mapsto x - 1$. Clearly, the roots of $Mf(x) = 0$ are $p+1$ and $q+1$. Hence, $x^2 + Bx - C = Mf(x)$. Since $-C$ is the $y$-intercept of $Mf(x) = 0$, we get $-C = Mf(0) = f(-1) = -7$. Thus, $C = 7$.

**Solution 2.** Note that $x^2 + 11x + 3 = (x - p)(x - q)$. By Vieta's formulas, we have $p + q = -11$. Hence,

$$\begin{aligned} x^2 + Bx - C = (x - p - 1)(x - q - 1) &= (x - p)(x - q) - (x - p) - (x - q) + 1 \\ &= \left(x^2 + 11x + 3\right) - (2x + 11) + 1 = x^2 + 9x - 7, \end{aligned}$$

whence $C = 7$.

**Question 7** [Ans: 25]

If the smallest possible value of $(A - x)(23 - x)(A + x)(23 + x)$ is $-(48)^2$, find the value of $A$ if $A > 0$.

**Solution.** From the difference of squares identity, we clearly have

$$(A - x)(23 - x)(A + x)(23 + x) = (x^2 - 23^2)(x^2 - A^2) = x^4 - (23^2 + A^2)x^2 + 23^2A^2.$$

Completing the square, we obtain

$$\left(x^2 - \frac{23^2 + A^2}{2}\right)^2 - \frac{1}{4}\left(23^2 + A^2\right)^2.$$

Thus,

$$-(48)^2 = -\frac{1}{4}\left(23^2 - A^2\right)^2,$$

whence $A = 25$. Note that we reject $A = \sqrt{433}$ as we require $A$ to be an integer.

**Question 8** [Ans: 3]

Find the smallest positive odd integer greater than 1 that is a factor of

$$(2023)^{2023} + (2026)^{2026} + (2029)^{2029}.$$

**Solution.** Observe that $2023 \equiv 2026 \equiv 2029 \equiv 1 \pmod 3$. Hence,

$$(2023)^{2023} + (2026)^{2026} + (2029)^{2029} \equiv 0 \pmod 3,$$

whence 3 is a factor of the above object.

**Question 9** [Ans: 48]

Find the remainder of $7^{2023} + 9^{2023}$ when divided by 64.

**Solution.** Observe that $\varphi(64) = 32$. Since $2023 \equiv 7 \pmod{32}$, by Euler's theorem, one has

$$7^{2023} + 9^{2023} \equiv 7^7 + 9^7 \equiv 7 \cdot 49^3 + 9 \cdot 81^3 \pmod{64}.$$

Since $49 \equiv -15 = -(2^4 - 1) \pmod{64}$ and $81 \equiv 17 = (2^4 + 1) \pmod{64}$, we obtain

$$7 \cdot 49^3 + 9 \cdot 81^3 \equiv 7(-15) + 9(17) \equiv 48 \pmod{64}.$$

**Question 10** [Ans: 49]

Let $x, y, z > 1$ and let $A$ be a positive number such that $\log_x A = 30$, $\log_y A = 50$ and $\log_{xy}(Az) = 150$. Find $(\log_A z)^2$.

**Solution.** Observe that $A = x^{30} = y^{50}$ and $Az = (xy)^{150}$. Hence,

$$z = \frac{(xy)^{150}}{A} = \frac{\left(A^{1/30}A^{1/50}\right)^{150}}{A} = A^7.$$

Thus, $(\log_A z)^2 = 7^2 = 49$.

**Question 11** [Ans: 1024]

Find the largest integer that is less than

$$\frac{3^{10} - 2^{10}}{10!}\left(\frac{1}{1!9!2} + \frac{1}{2!8!2^2} + \frac{1}{3!7!2^3} + \cdots + \frac{1}{9!1!2^9}\right)^{-1}.$$

Here, $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

**Solution.** Observe that

$$\sum_{i=1}^{9} \frac{10!}{i!(10-i)!} \cdot \frac{1}{2^i} = \sum_{i=1}^{9} \binom{10}{i}\left(\frac{1}{2}\right)^i = \sum_{i=0}^{10} \binom{10}{i}\left(\frac{1}{2}\right)^i - 2 = \left(1 + \frac{1}{2}\right)^{10} - 2,$$

Hence,

$$\begin{aligned}
&\frac{3^{10} - 2^{10}}{10!}\left(\frac{1}{1!9!2} + \frac{1}{2!8!2^2} + \frac{1}{3!7!2^3} + \cdots + \frac{1}{9!1!2^9}\right)^{-1} \\
&= \left(3^{10} - 2^{10}\right)\left(\sum_{i=1}^{9} \frac{10!}{i!(10-i)!} \cdot \frac{1}{2^i}\right)^{-1} = \left(3^{10} - 2^{10}\right)\left[\left(\frac{3}{2}\right)^{10} - 2\right]^{-1} \\
&= 2^{10}\left(1 + \frac{2^{10}}{3^{10} - 2^{10}}\right).
\end{aligned}$$

Clearly, $0 < \frac{2^{10}}{3^{10}-2^{10}} < 1$. Thus, the largest integer less than the above object is $2^{10} = 1024$.

**Question 12** [Ans: 84]

Consider the following simultaneous equations:

$$\begin{aligned}
xy^2 + xyz &= 91 \\
xyz - y^2z &= 72
\end{aligned}$$

where $x$, $y$, and $z$ are positive integers. Find the maximum value of $xz$.

**Solution.** Note that we have $xyz = 91 - xy^2 = 72 + y^2z$, whence $19 = y^2(x + z)$. By the AM-GM inequality, we have,

$$\frac{19}{2y^2} = \frac{x+z}{2} \geq \sqrt{xz} \implies xz \leq \left(\frac{19}{2y^2}\right)^2.$$

To maximize $xz$, we must minimize $y$. Testing $y = 1$, we get

$$\begin{cases} x + xz = 91 \\ xz - z = 72 \end{cases}$$

Observe that $x + z = 19$, and $x$ is a factor of 91. Testing the only factors of 91 less than 19, which are 1, 7 and 13, we see that $x = 7$ and $z = 12$ is the only solution to the above system. Thus, $\max xz = 7 \cdot 12 = 84$.

**Question 13** [Ans: 2]

Let $x$ be a real number such that

$$\frac{\sin^4 x + \cos^4 x}{\sin^2 x + \cos^4 x} = \frac{8}{11}.$$

Assuming $\sin^2 x > \dfrac{1}{2}$, find the value of $\sqrt{28}\left(\sin^4 x - \cos^4 x\right)$.

**Solution.** Cross-multiplying and writing $\cos^4 x = (1 - \sin^2 x)^2$, we obtain a quadratic in $\sin^2 x$:

$$14 \sin^4 x - 14 \sin^2 x + 3 = 0.$$

Since $\sin^2 x > \frac{1}{2}$, we have $\sin^2 x = \frac{7+\sqrt{7}}{14}$. This immediately gives

$$\sin^4 x = \frac{4 + \sqrt{7}}{14}, \quad \cos^4 x = \frac{4 - \sqrt{7}}{14}.$$

Thus,

$$\sqrt{28}\left(\sin^4 x - \cos^4 x\right) = \sqrt{28} \cdot \frac{2\sqrt{7}}{14} = 2.$$

**Question 14** [Ans: 160]

A sequence $a_1, a_2, \ldots$, is defined by

$$a_1 = 5, \quad a_2 = 7, \quad a_{n+1} = \frac{a_n + 1}{a_{n-1}} \text{ for } n \geq 2.$$

Find the value of $100 \times a_{2023}$.

**Solution.** Repeatedly applying the recurrence relation, we see that

$$a_1 = 5, \quad a_2 = 7, \quad a_3 = \frac{8}{5}, \quad a_4 = \frac{13}{35}, \quad a_5 = \frac{6}{7}, \quad a_6 = 5, \quad a_7 = 7.$$

We thus see that the sequence has a period of 5. Since $2023 \equiv 3 \pmod 5$, we have that $a_{2023} = a_3 = \frac{8}{5}$, whence $100 \times a_{2023} = 160$.

**Question 15** [Ans: 60]

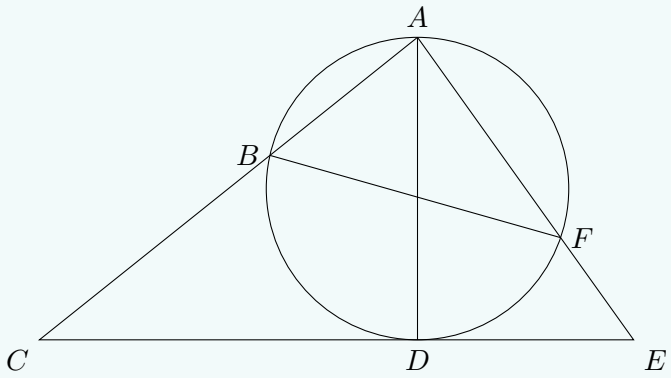
Let $C$ be a constant such that the equation $5\cos x + 6\sin x - C = 0$ have two distinct roots $a$ and $b$, where $0 < b < a < \pi$. Find the value of $61 \times \sin(a+b)$.

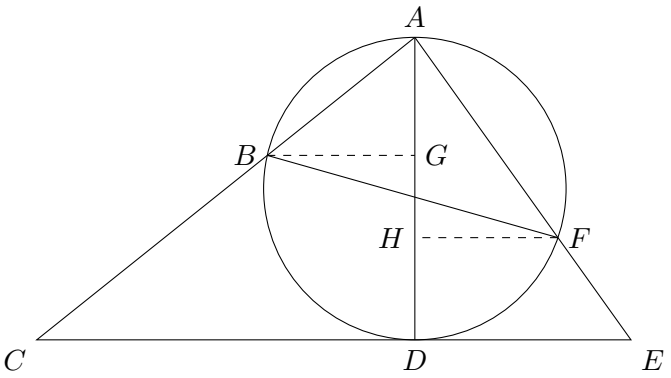
**Solution.** By the R-formula, one has $5\cos x + 6\sin x = \sqrt{61}\cos\left(x - \arctan\frac{6}{5}\right)$. Note that $\frac{\pi}{4} < \arctan\left(\frac{6}{5}\right) < \frac{\pi}{2}$. Thus, if $C = \sqrt{61}\cos\left(\arctan\frac{6}{5}\right)^+$, the equation $5\cos x + 6\sin x - C = 0$ will clearly have two solutions: namely $b = 0^+$ and $a = \left(2\arctan\frac{6}{5}\right)^-$. Hence,

$$61\sin(a+b) = 61\sin\left(2\arctan\frac{6}{5}\right) = 61\cdot 2\sin\left(\arctan\frac{6}{5}\right)\cos\left(\arctan\frac{6}{5}\right) = 60$$

**Question 16** [Ans: 58]

In the diagram below, $CE$ is tangent to the circle at point $D$, $AD$ is the diameter of the circle, and $ABC$, $AFE$ are straight lines. It is given that $\frac{AB}{AC} = \frac{16}{41}$ and $\frac{AF}{AE} = \frac{49}{74}$. Let $\tan(\angle CAE) = \frac{m}{n}$, where $m$, $n$ are positive integers and $\frac{m}{n}$ is a fraction in its lowest form. Find the sum $m+n$.




**Solution.** Let $(ABDF)$ be the unit circle. Then $AD = 2$. Let $G$ and $H$ be the foot of perpendiculars of $B$ and $F$ onto $AD$. By similar triangles, we have

$$AG = \frac{16}{41}\cdot AD = \frac{32}{41}, \quad AH = \frac{49}{74}\cdot AD = \frac{49}{37}.$$

Thus, $B$ has $y$-coordinate $1 - \frac{32}{41} = \frac{9}{41}$, while $F$ has $y$-coordinate $1 - \frac{49}{37} = -\frac{12}{37}$. Since $B$ and $F$ are on the unit circle, we have $\sin\alpha = \frac{9}{41}$ and $\sin\beta = -\frac{12}{37}$, where $\alpha$ and $\beta$ are

the arguments of $B$ and $F$ respectively. Using the Pythagorean trigonometric identity, we obtain

$$\cos\alpha = -\sqrt{1-\left(\frac{9}{41}\right)^2} = \frac{40}{41}, \quad \sin\beta = \sqrt{1-\left(-\frac{12}{37}\right)^2} = \frac{35}{37},$$

whence $B(-\frac{40}{41}, \frac{9}{41})$ and $F(\frac{35}{37}, -\frac{12}{37})$. Since $A(0,1)$, we have

$$\tan\angle ABG = \frac{BG}{AG} = \frac{40/41}{32/41} = \frac{5}{4}, \quad \tan\angle FAH = \frac{FH}{AH} = \frac{35/37}{49/37} = \frac{5}{7}.$$

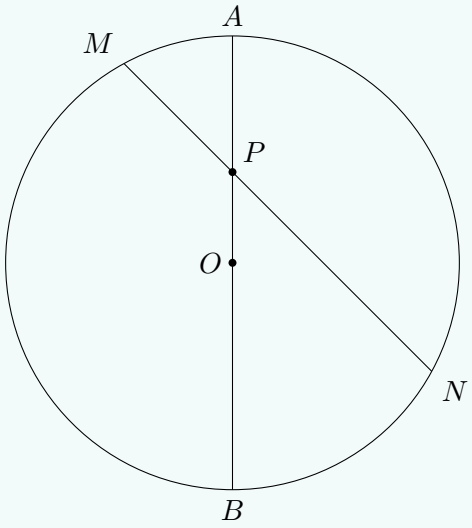
Hence,

$$\tan\angle CAE = \tan(\angle ABG + \angle FAH) = \frac{\tan\angle ABG + \tan\angle FAH}{1 - \tan\angle ABG\tan\angle FAH} = \frac{55}{3},$$

whence $m+n = 55+3 = 58$.

**Question 17** [Ans: 10]

In the diagram below, $AB$ is a diameter of the circle with centre $O$, $MN$ is a chord of the circle that intersects $AB$ at $P$, $\angle BON$ and $\angle MOA$ are acute angles, $\angle MPA = 45°$, $MP = \sqrt{56}$, and $NP = 12$. Find the radius of the circle.




**Solution.** Let $Q \in MN$ such that $OQ \perp MN$. Then $MQ = QN$, whence $PQ = 6 - \frac{1}{2}\sqrt{56} = 6 - \sqrt{14}$. Since $\angle OPQ = \angle MPA = 45°$ and $\angle PQO = 90°$, it follows that $\triangle OQP$ is a right-angled isosceles triangle, whence $OP = \sqrt{2}\left(6-\sqrt{14}\right)$. Using power of a point on $P$, we get

$$\sqrt{56}\cdot 12 = (r+OP)(r-OP) = r^2 - OP^2 \implies r^2 = OP^2 + 24\sqrt{14} = 100.$$

Thus, $r = 10$.

**Question 18** [Ans: 1011]

Let $f(x) = \cos^2\left(\frac{\pi x}{2}\right)$. Find the value of

$$f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \cdots + f\left(\frac{2021}{2023}\right) + f\left(\frac{2022}{2023}\right).$$

**Solution.** Observe that

$$f(x) + f(1-x) = \cos^2\left(\frac{\pi x}{2}\right) + \cos^2\left(\frac{\pi}{2} - \frac{\pi x}{2}\right) = \cos^2\left(\frac{\pi x}{2}\right) + \sin^2\left(\frac{\pi x}{2}\right) = 1.$$

Thus,

$$\begin{aligned}
&f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \cdots + f\left(\frac{2021}{2023}\right) + f\left(\frac{2022}{2023}\right) \\
&= \left[f\left(\frac{1}{2023}\right) + f\left(\frac{2022}{2023}\right)\right] + \left[f\left(\frac{2}{2023}\right) + f\left(\frac{2021}{2023}\right)\right] + \cdots + \left[f\left(\frac{1011}{2023}\right) + f\left(\frac{1012}{2023}\right)\right] \\
&= 1011.
\end{aligned}$$

**Question 19** [Ans: 37]

Find the remainder when $3^{2023}$ is divided by 215.

**Solution.** Note that $215 = 5 \cdot 43$. Hence, $\varphi(215) = (5-1)(43-1) = 168$. Since $2023 \equiv 7 \pmod{168}$, by Euler's theorem, we have

$$3^{2023} \equiv 3^7 \equiv 9 \cdot 243 \equiv 9 \cdot 28 \equiv 252 \equiv 37 \pmod{215}.$$

**Question 20** [Ans: 514]

Find the sum of the prime divisors of 64000027.

**Solution.** Note that $64000027 = 20^6 + 3^3 = 400^3 + 3^3$. By the sum of cubes identity, we have

$$64000027 = (400+3)(400^2 + 400 \cdot 3 + 3^2) = 403 \cdot 158809.$$

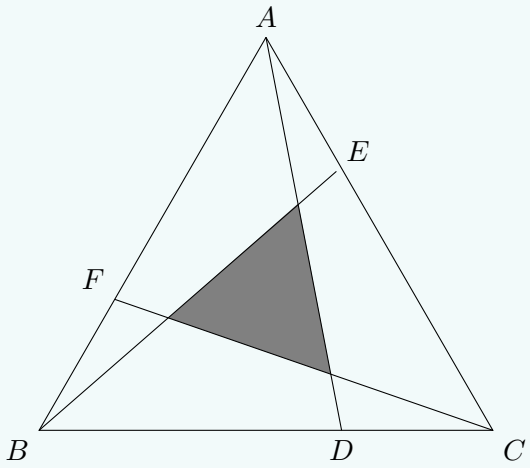
Observe that $403 = 13 \cdot 31$, while $158809 = 7^3 \cdot 463$. Hence, the prime divisors of 64000027 are 7, 13, 31 and 463, whence their sum is 514.

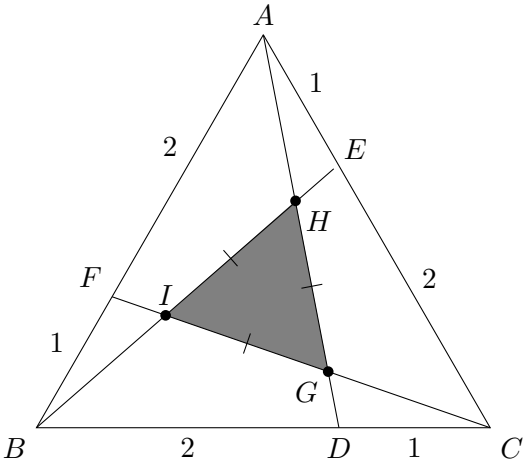
**Question 21** [Ans: 14161]

Let $\triangle ABC$ be an equilateral triangle. $D$, $E$, $F$ are points on the sides such that

$$BD : DC = CE : EA = AF : FB = 2 : 1.$$

Suppose the area of the triangle bounded by $AD$, $BE$ and $CF$ is 2023. Find the area of $\triangle ABC$.




**Solution.** Let the side lengths of $\triangle ABC$ be 3 units. Let $G = CF \cap AD$, $H = AD \cap BE$ and $I = BE \cap CF$. By symmetry, $\triangle GHI$ is similar to $\triangle ABC$ and is hence also equilateral.

Using Menalaus' theorem on $\triangle BEC$ and $\triangle ADC$, we get

$$\frac{CA}{AE}\frac{EH}{HB}\frac{BD}{DC} = 1 \implies \frac{EH}{HB} = \frac{1}{6}.$$

Using Menalaus' theorem on $\triangle EIC$ and $\triangle AGC$, we have

$$\frac{CA}{AE}\frac{EH}{HI}\frac{IG}{GC} = 1 \implies \frac{EH}{GC} = \frac{1}{3}.$$

Hence, $\frac{HI}{BE} = \frac{3}{7}$. By the cosine rule, we obtain

$$BE^2 = CE^2 + BC^2 - 2(CE)(BC)\cos\angle C \implies BE = \sqrt{7}.$$

Thus, $HI = \frac{3}{\sqrt{7}}$, whence $\frac{AB}{HI} = \sqrt{7}$. Thus,

$$[ABC] = \left(\sqrt{7}\right)^2 [GHI] = 7 \cdot 2023 = 14161.$$

**Question 22** [Ans: 47628]

Find the number of possible ways of arranging 10 ones and 11 zeros in a row such that there are in total 13 strings of ones and zeros. For example,

$$1110001001110001$$

consists of 4 strings of ones and 3 strings of zeros.

**Solution.** *Case 1*. Suppose the word starts with a 0. Then the thirteen strings must alternate as such:

$$0101010101010,$$

with seven strings of 0's and six strings of 1's. Since there are four 0's and four 1's left to place, by stars-and-bars, we have $\binom{4+7-1}{4}\binom{4+6-1}{4} = 26460$ unique combinations in this case.

*Case 2*. Suppose the word starts with 1. Then the thirteen strings must alternate as such:

$$1010101010101,$$

with seven strings of 1's and six strings of 0's. Since there are three 1's and five 0's left to place, by stars-and-bars, we have $\binom{3+7-1}{3}\binom{5+6-1}{5} = 21168$ unique combinations in this case.

Hence, there are a total oof $26460 + 21168 = 47628$ possible ways to arrange the 1's and 0's.

**Question 23** [Ans: 66795]

Suppose there exist numbers $a$, $b$, $c$ and a function $f$ such that for any real numbers $x$ and $y$,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) + ax + by + c.$$

It is given that

$$f(2) = 3, \quad f(3) = -5, \quad \text{and} \quad f(5) = 7.$$

Find the value of $f(123)$.

**Solution.** Let $P(x, y)$ be the assertion that $f(x+y)+f(x-y) = 2f(x)+2f(y)+ax+by+c$.

We first determine the values of $a$, $b$ and $c$. From $P(x, 0)$, we obtain

$$f(x) + f(x) = 2f(x) + 2f(0) + ax + c \implies f(0) = -\frac{ax+c}{2}.$$

Since $f(0)$ is a constant, it follows that $a = 0$, whence $f(0) = -\frac{c}{2}$. From $P(3, 2)$, we have

$$f(5) + f(1) = 2f(3) + 2f(2) + 2b + c \implies f(1) = -11 + 2b + c. \tag{1}$$

From $P(2, 1)$, we have

$$f(3) + f(1) = 2f(2) + 2f(1) + b + c \implies f(1) = -11 - b - c. \tag{2}$$

Equating (1) and (2) gives

$$3b + 2c = 0. \tag{3}$$

From $P(3, 1)$, we have

$$f(4) + f(2) = 2f(3) + 2f(1) + b + c \implies f(4) = -35 - b - c.$$

From $P(4,1)$, we have

$$f(5)+f(3)=2f(4)+2f(1)+b+c \implies 3b+3c=-94. \tag{4}$$

Solving (3) and (4) simultaneously gives $b=\frac{188}{3}$ and $c=-94$.

Consider $P(x,1)$. Then

$$f(x+1)+f(x-1)=2f(x)+2f(1)+b+c \implies [f(x+1)-f(x)]-[f(x)-f(x-1)]=\frac{28}{3}.$$

Let $a_n=f(n)$, where $n\in\mathbb{N}$. We thus see that $a_n-a_{n-1}$ is in arithmetic progression with common difference $\frac{28}{3}$. Hence,

$$a_n-a_{n-1}=(a_3-a_2)+(n-3)\frac{28}{3} \implies a_n=a_{n-1}+\frac{28}{3}n-36.$$

Thus,

$$f(123)=a_{123}=a_2+\sum_{k=3}^{123}\left(\frac{28}{3}k-36\right)=66795.$$

**Question 24** [Ans: 454]

Let $f$ be a function such that for any non-zero number $x$,

$$6xf(x)+5x^2f(1/x)+10=0.$$

Find the value of $f(10)$.

**Solution.** Plugging in $x=10$ and $x=\frac{1}{10}$ into the functional equation, we get the following system of equations:

$$\begin{cases} 60f(10)+500f\left(\frac{1}{10}\right)+10=0 \\ \frac{6}{10}f\left(\frac{1}{10}\right)+\frac{5}{100}f(10)+10=0 \end{cases}$$

Solving, we obtain $f(10)=454$.

**Question 25** [Ans: 5]

Find the number of triangles such that all the sides are integers and the area equals the perimeter (in number).

**Solution.** Let the side lengths of the triangle be $a$, $b$ and $c$. Let $s=\frac{a+b+c}{2}$ be the semiperimeter of the triangle. From Heron's formula, we have

$$a+b+c=\sqrt{s(s-a)(s-b)(s-c)},$$

which gives

$$16(a+b+c)=(-a+b+c)(a-b+c)(a+b-c).$$

Let $x=-a+b+c$, $y=a-b+c$ and $z=a+b-c$. Then

$$a=\frac{y+z}{2},\quad b=\frac{z+x}{2},\quad c=\frac{x+y}{2},$$

whence

$$16(x+y+z) = xyz.$$

Clearly, $x$, $y$ and $z$ are all even. Substituting $x = 2X$, $y = 2Y$ and $z = 2Z$, we get

$$4(X+Y+Z) = XYZ.$$

Without loss of generality, suppose $X \leq Y \leq Z$. Then $XYZ \leq 12Z$, whence $XY \leq 12$. Additionally, $X^2 \leq XY \leq 12$, whence $X \leq 3$. We are thus left with the following few cases:

*Case 1*. Suppose $X = 1$. Then

$$4(1+Y+Z) = YZ \implies (Y-4)(Z-4) = 20.$$

This gives $(X,Y,Z) \in \{(1,5,24),(1,6,14),(1,8,9)\}$.

*Case 2*. Suppose $X = 2$. Then

$$4(2+Y+Z) = 2YZ \implies (Y-2)(Z-2) = 8.$$

This gives $(X,Y,Z) \in \{(2,3,10),(2,4,6)\}$.

*Case 3*. Suppose $X = 3$. Then

$$4(3+Y+Z) = 3YZ \implies (3Y-4)(3Z-4) = 52.$$

Note that $3Y - 4 \geq 3X - 4 = 8$. Since $52 = 1 \cdot 52 = 2 \cdot 26 = 4 \cdot 13$, there are no possibilities for $Y$. There are hence no solutions in this case.

There are thus a total of 5 integer triangles whose perimeter is equal to its area in number.

### 9.1.2. Round 2 Solutions

**Resources:** Review by *Way Tan*, AoPS threads

**Question 1**

Let $ABCD$ be a square, $E$ be a point on the side $DC$, $F$ and $G$ be the feet of the altitudes from $B$ to $AE$ and from $A$ to $BE$ respectively. Suppose $DF$ and $CG$ intersect at $H$. Prove that $\angle AHB = 90^\circ$.

**Solution.** We prove that $\angle AHB = 90^\circ$ via coordinate bashing. Let $A(0,1)$, $B(1,1)$, $C(1,0)$, $D(0,0)$ and $E(t,0)$, where $t \in (0,1)$. The line $AE$ and $BE$ have equations

$$AE : y = -\frac{1}{t}x + 1, \qquad BE : y = \frac{1}{1-t}x - \frac{1}{1-t} + 1.$$

The lines $BF$ and $AG$ thus have equations

$$BF : y = tx - t + 1, \qquad AG : y = (t-1)x + 1.$$

Solving $AE \cap BF$ and $BE \cap AG$, we get

$$F\left(\frac{t^2}{t^2+1}, \frac{t^2-t+1}{t^2+1}\right), \qquad G\left(\frac{1}{t^2-2t+2}, \frac{t^2-t+1}{t^2-2t}\right).$$

The lines $DF$ and $CG$ thus have equations

$$DF : y = \frac{t^2-t+1}{t^2}x, \qquad CG : y = \frac{t^2-t+1}{-t^2+2t-1}x - \frac{t^2-t+1}{-t^2+2t-1}.$$

Solving $DF \cap CG$, we obtain

$$H\left(\frac{t^2}{2t^2-2t+1}, \frac{t^2-t+1}{2t^2-2t+1}\right).$$

The gradients of $AH$ and $AB$ are hence

$$m_{AH} = \frac{-t+1}{t}, \qquad m_{AB} = \frac{-t^2+t}{-t^2+2t-1}.$$

Since

$$m_{AH} \cdot m_{AB} = \frac{-t+1}{t} \cdot \frac{-t^2+t}{-t^2+2t-1} = \frac{t^3-2t^2+t}{-t^2+2t^2-t} = -1,$$

it follows that $AH \perp AB$, whence $\angle AHB = 90^\circ$.

**Question 2** [Ans: 3]

Find all positive integers $k$ such that there exists positive integers $a$, $b$ such that

$$a^2 + 4 = \left(k^2 - 4\right) b^2.$$

**Solution** (Way Tan)**.** Note that $a^2 = k^2b^2 - (4b^2+4)$, whence $a < kb$. We can hence write $a = kb - c$, where $c$ is a positive integer. Substituting this into the given equation yields

$$-2kbc + c^2 + 4 = -4b^2 \implies k = \frac{c^2+4b^2+4}{2bc}.$$

Clearly, $c$ must be even, whence we can write $c = 2x$. This gives

$$k = \frac{x^2 + b^2 + 1}{bx} \iff x^2 - kbx + \left(b^2 + 1\right).$$

Let $x$ be the smallest integer that satisfies the above quadratic, where $x \geq b$. Let $x'$ be the other solution to the quadratic. By Vieta's formulas, we have

$$x + x' = kb, \quad x \cdot x' = b^2 + 1.$$

Observe that $x' = kb - x \in \mathbb{Z}$ and $x' = \frac{b^2+1}{x} > 0$. Thus, to preserve the minimality of $x$, we require $x' \geq x$. Hence,

$$b^2 \leq x^2 \leq x \cdot x' = b^2 + 1.$$

Thus, $x = b$. Hence, $x' = b + \frac{1}{b}$. However, since $x' \in \mathbb{Z}$, we can only have $b = 1$. Substituting this back into our expression for $k$, we see that $k = 3$, which clearly works when $a = b = 1$.

**Question 3**

Let $n$ be a positive integer. There are $n$ islands with $n - 1$ bridges connecting them such that one can travel from any island to another. One afternoon, a fire breaks out in one of the islands. Every morning, it spreads to all neighbouring islands. (Two islands are neighbours if they are connected by a bridge.) To control the spread, one bridge is destroyed every night until the fire has nowhere to spread to the next day. Let $X$ be the minimum possible number of bridges one has to destroy before the fire stops spreading. Find the maximum possible value of $X$ over all possible configurations of bridges and islands where the fire starts at.

**Solution.**

**Question 4** [Ans: $\{(1,1),(2,2),(4,5)\}$]

Find all positive integers $m$, $n$ satisfying $n! + 2^{n-1} = 2^m$.

**Solution** (Way Tan)**.** Rearranging, we obtain

$$n! = 2^{n-1}\left(2^{m-n+1} - 1\right).$$

Let $\nu_2(n!)$ be the exponent of the largest power of 2 that divides $n!$. Since $2^{m-n+1} - 1$ is odd, we clearly have $\nu_2(n!) = n - 1$. However, by Legendre's formula, we also have

$$\nu_2(n!) = \frac{n - s_2(n)}{2 - 1} = n - s_2(n),$$

where $s_2(n)$ is the sum of the digits of $n$ when written in base-2. Hence, $s_2(n) = 1$, whence $n$ must be of the form $10\cdots0_2$, i.e. $n$ is a perfect power of 2. We hence write $n = 2^k$, where $k$ is a non-negative integer.

Now observe that

$$n! + 2^{n-1} = 2^{n-1}\left(1 + \frac{n!}{2^{n-1}}\right).$$

We hence require $\frac{n!}{2^{n-1}}$ to be one less than a power of 2. Define

$$S_a = \left\{1 \leq i \leq 2^k \mid \nu_2(i) = a\right\}.$$

That is, $S_a$ groups positive integers less than $2^k$ based on the exponent of the largest power of 2 that divides them. We now observe that $S_a$ has an alternative definition, namely

$$S_a = \left\{1 \leq i \leq 2^k \mid b \text{ odd} \wedge i = 2^a b\right\}.$$

For instance, $12 \in S_2$ because $2^2$ is the largest power of 2 that divides 12, but also because 12 is an odd multiple of $2^2$ ($12 = 3 \cdot 2^2$). With this, we see that

$$\frac{n!}{2^{n-1}} = \frac{1}{2^{n-1}} \prod_{1 \leq a \leq k} \prod_{i \in S_a} i = \prod_{1 \leq a \leq k-1} \prod_{\substack{b \text{ odd} \\ 1 \leq b \leq |S_a|}} b.$$

Note that we remove the $a = k$ case in the last line since $2^k$ effectively gets reduced to 1. Now, observe that the product of any four consecutive numbers is congruent to 1. We thus see that

$$\frac{n!}{2^{n-1}} \equiv (1) \cdot (1 \cdot 3) \equiv 3 \pmod 8,$$

where the products corresponding to $S_a$ for $a \leq k - 3$ disappear since $|S_a| = 2^{k-a-1}$ is a multiple of 4 in those cases. We thus have

$$1 + \frac{n!}{2^{n-1}} \equiv 4 \pmod 8.$$

However, as previously mentioned, we require this object to be a power of 2. Thus,

$$1 + \frac{n!}{2^{n-1}} = 4 \implies n! = 3 \cdot 2^{n-1}.$$

Since $n!$ does not contain any odd factors larger than 5, we clearly have $n < 5$.

*Case 1*. Suppose $n = 1$. Then $1! + 2^{1-1} = 2^m \implies m = 1$.

*Case 2*. Suppose $n = 2$. Then $2! + 2^{2-1} = 2^m \implies m = 2$.

*Case 3*. Suppose $n = 4$. Then $4! + 2^{4-1} = 2^m \implies m = 5$.

Thus, the only solutions are

$$(m, n) \in \{(1, 1), (2, 2), (4, 5)\}.$$

### Question 5

Colour a $20000 \times 20000$ square grid using 2000 different colours with 1 colour in each square. Two squares are neighbours if they share a vertex. A path is a sequence of squares so that 2 successive squares are neighbours. Mark $k$ of the squares. For each unmarked square $x$, there is exactly 1 marked square $y$ of the same colour so that $x$ and $y$ are connected by a path of squares of the same colour. For any 2 marked squares of the same colour, any path connecting them must pass through squares of all the colours. Find the maximum value of $k$.

**Solution.**

## 9.2. Open Section

### 9.2.1. Round 1 Solutions

**Resources:** Review by *Way Tan*

**Question 1** [Ans: 10]

The graph $C$ with equation $y = \dfrac{ax^2+bx+c}{x+2}$ has an oblique asymptote with equation $y = 4x - 6$ and one of the stationary points at $x = -4$. Find the value of $a + b + c$.

**Solution.** Since $C$ has an oblique asymptote with equation $y = 4x - 6$, its equation can be written as

$$C : y = 4x - 6 + \frac{d}{x+2}$$

for some $d \in \mathbb{R}$. Multiplying throughout by $x+2$ and comparing coefficients, we get $a = 4$, $b = 2$ and $c = -12+d$. Differentiating and using the fact that $\mathrm{d}y/\mathrm{d}x = 0$ at $x = -4$ yields $d = 16$. Thus, $a + b + c = 10$.

**Question 2** [Ans: 2000]

If $x = \dfrac{1}{1} + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \dfrac{1}{1+2+3+4} + \cdots + \dfrac{1}{1+2+3+\cdots+100}$, find the value of $\lfloor 1010x \rfloor$.

**Solution.** Observe that the denominators are the triangular numbers. It is well known that the $n$th triangular number is given by $n(n+1)/2$. Thus, each term of $x$ is of the form $2/[n(n+1)]$, which can be written as $2/n - 2/(n+1)$ via partial fraction decomposition. $x$ is hence a telescoping sum, which evaluates to $2 - 2/101$, whence $1010x = 2020 - 20 = 2000$.

**Question 3** [Ans: 7]

The set of all possible values of $x$ for which the sum of the infinite series

$$1 + \frac{1}{6}\left(x^2-5x\right) + \frac{1}{6^2}\left(x^2-5x\right)^2 + \frac{1}{6^3}\left(x^2-5x\right)^3 + \cdots$$

exists can be expressed as $(a, b) \cup (c, d)$, where $a < b < c < d$. Find $d - a$.

**Solution.** Observe that the common ratio of the given infinite series is $r = (x^2 - 5x)/6$. For the sum to exist, $|r| < 1$, whence $-6 < x^2 - 5x < 6$. Since we are interested in the extreme values of $x$, we consider only the equality case. We thus obtain $x^2 - 5x = -6$ or $x^2 - 5x = 6$, which clearly has solutions $x = -1, 2, 3, 6$, giving $d - a = 6 - (-1) = 7$.

**Question 4** [Ans: 2]

Find the value of $\lfloor y \rfloor$, where $y = \displaystyle\sum_{k=0}^{\infty} (2k+1)(0.5)^{2k}$.

*(Hint: Consider the series expansion of* $(1-x)^{-2}$*)*

**Solution.** Note that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Differentiating yields

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1} = x^{-1} \sum_{k=0}^{\infty} kx^k.$$

Now observe $y$ can be rewritten in terms of the above series:

$$y = 2\sum_{k=0}^{\infty} k(1/4)^k + \sum_{k=0}^{\infty} (1/4)^k,$$

from which it clearly follows that

$$\lfloor y \rfloor = \left\lfloor \frac{2 \cdot 1/4}{(1-1/4)^2} + \frac{1}{1-1/4} \right\rfloor = 2.$$

**Question 5** [Ans: 3]

The solution of the inequality $|x-1|+|x+1| < ax+b$ is $-1 < x < 2$. Find the value of $\lfloor a+b \rfloor$.

**Solution.** At the extreme ends of the solution interval, equality is achieved. This yields $2 = -a+b$ and $4 = 2a+b$ upon substituting $x = -1$ and $x = 2$ into the two expressions. Solving, we get $a = 2/3$ and $b = 8/3$, whence $\lfloor a+b \rfloor = 3$.

**Question 6** [Ans: 32]

The equation $x^4 - 4x^2 + qx - r = 0$ has three equal roots. Find the value of $\left\lfloor \frac{3q^2}{r^2} \right\rfloor$.

**Solution.** Let $\alpha$ be the root of multiplicity 3 and $\beta$ be the remaining root. By Vieta's formulas, we have the following system of equations:

$$\begin{cases} -r = \alpha^3\beta \\ -q = \alpha^3 + 3\alpha^2\beta \\ -4 = 3\alpha^2 + 3\alpha\beta \\ 0 = 3\alpha + \beta \end{cases}$$

From the third and fourth equations, we have $\alpha\beta = -2$. From the second and third equations, we have $-4\alpha + q = 2\alpha^3$. However, $q = -\alpha^3 - 3\alpha(\alpha\beta) = -\alpha^3 + 6\alpha$. Combining equations gives $3\alpha^3 - 2\alpha = 0$, whence $\alpha = \sqrt{2/3}$, since $\alpha$ is clearly non-zero. Thus,

$$\begin{aligned} \left\lfloor \frac{3q^2}{r^2} \right\rfloor &= \left\lfloor \frac{3(-\alpha^3+6\alpha)^2}{(-2\alpha^2)^2} \right\rfloor = \left\lfloor \frac{3(\alpha^6 - 12\alpha^4 + 36\alpha^2)}{4\alpha^4} \right\rfloor \\ &= \left\lfloor \frac{3((2/3)^3 - 12(2/3)^2 + 36(2/3))}{4(2/3)^2} \right\rfloor = 32 \end{aligned}$$

**Question 7** [Ans: 8]

The parabolas $y = x^2 - 16x + 50$ and $x = y^2$ intersect at 4 distinct points which lie on a circle centred at $(a, b)$. Find $|a - b|$.

**Solution.** From the first equation, we have $x^2 - 17x + x - y = -50$. By the second equation, this is equivalent to $x^2 - 17x + y^2 - y = -50$. Completing the square, we obtain

$$\left(x - \frac{17}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = -50 + \left(\frac{17}{2}\right)^2 + \left(\frac{1}{2}\right)^2.$$

It is thus clear that $a = 17/2$ and $b = 1/2$, giving $|a - b| = 8$.

**Question 8** [Ans: 33]

In the 3-dimensional Euclidean space with origin $O$ and three mutually perpendicular $x$-, $y$- and $z$-axes, two planes $x + y + 3z = 4$ and $2x - z = 6$ intersect at the line $\mathbf{r} \times \begin{pmatrix} -1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ c \\ d \end{pmatrix}$. Find the value of $|a + b + c + d|$.

**Solution.** Solving the Cartesian equations of the two planes simultaneously, we get

$$\mathbf{r} = \frac{1}{7}\begin{pmatrix} 22 - \lambda \\ 7\lambda \\ 2 - 2\lambda \end{pmatrix},$$

where $\lambda \in \mathbb{R}$. Taking the cross product yields

$$\frac{1}{7}\begin{pmatrix} -2a + \lambda(2a + 7b) \\ -2 - 22b + \lambda(2 + b) \\ 22a + \lambda(7 - a) \end{pmatrix} = \begin{pmatrix} -2 \\ c \\ d \end{pmatrix}$$

Since the above equation must hold for all real $\lambda$, we immediately get $a = 7$ and $b = -2$. It quickly follows by equating the $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ components of both vectors that $c = 6$ and $d = 22$, giving $|a + b + c + d| = 33$.

**Question 9** [Ans: 200]

Let $x$, $y$, $z$ be real numbers with $3x + 4y + 5z = 100$. Find the minimum value of $x^2 + y^2 + z^2$.

**Solution.** Observe that $3x + 4y + 5z = 100$ describes a plane $\pi$ in 3-dimensional Euclidean space with vector equation

$$\pi : \mathbf{r} \cdot \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 100.$$

Observe also that $\min(x^2 + y^2 + z^2)$ is the square of the perpendicular distance between the origin and $\pi$. Applying the standard formula for perpendicular distance between a plane and a point, one gets

$$\min(x^2 + y^2 + z^2) = \left(\frac{100}{\sqrt{3^2 + 4^2 + 5^2}}\right)^2 = 200.$$

**Question 10** [Ans: 2]

Find the area of the region represented by the equation $\lfloor x \rfloor + \lfloor y \rfloor = 1$ in the region $0 \leq x < 2$.
*(Note: If you think that there is no area defined by the graph, enter "0"; if you think that the area is infinite, enter "9999".)*

**Solution.** When $x \in [0, 1)$, we have $\lfloor x \rfloor = 0$. Thus, $\lfloor y \rfloor = 1$, giving $y \in [1, 2)$. This is a square of area 1. Similarly, when $x \in [1, 2)$, we have $\lfloor x \rfloor = 1$. Thus, $\lfloor y \rfloor = 0$, giving $y \in [0, 1)$. This is another square of area 1. Hence, the total area of the region is 2.

**Question 11** [Ans: 1011]

Let $ABC$ be a triangle satisfying the following conditions that $\angle A + \angle C = 2\angle B$, and $\dfrac{1}{\cos A} + \dfrac{1}{\cos C} = \dfrac{-\sqrt{2}}{\cos B}$. Determine the value of $\dfrac{2022 \cos\left(\frac{A-C}{2}\right)}{\sqrt{2}}$.

**Solution 1.** Note that $\angle A + \angle B + \angle C = 180^\circ$, whence $\angle B = 60^\circ$. Clearing denominators in the given equation, we have

$$\cos A + \cos B = -2\sqrt{2} \cos A \cos C.$$

Without loss of generality, let $\angle A = 60^\circ + \theta$ and $\angle C = 60^\circ - \theta$. We now aim to find $\cos\left(\frac{A-C}{2}\right) = \cos\theta$. We have

$$\cos(60^\circ + \theta) + \cos(60^\circ - \theta) = -2\sqrt{2} \cos(60^\circ + \theta) \cos(60^\circ - \theta).$$

Expanding using cosine identities yields

$$4\sqrt{2} \cos^2\theta + 2\cos\theta - 3\sqrt{2} = 0,$$

which has the unique solution $\cos\theta = 1/\sqrt{2}$ (keeping in mind $|\cos\theta| \leq 1$). The desired expression thus evaluates to

$$\frac{2022 \cos\left(\frac{A-C}{2}\right)}{\sqrt{2}} = \frac{2022/\sqrt{2}}{\sqrt{2}} = 1011.$$

**Solution 2** (Abusing integers)**.** In order for $\frac{2022 \cos\left(\frac{A-C}{2}\right)}{\sqrt{2}}$ to be an integer, we need $\cos\left(\frac{A-C}{2}\right)$ to be of the form $k\sqrt{2}$, where $k$ is a positive rational such that $2022k$ is an integer. It is exceedingly likely that $\frac{2022 \cos\left(\frac{A-C}{2}\right)}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, as it is a special value of the cosine function. The required answer is thus $\frac{2022 \cdot \sqrt{2}/2}{\sqrt{2}} = 1011$.

**Question 12** [Ans: 2020]

Find $x$ which satisfies the following equation

$$\frac{x - 2019}{1} + \frac{x - 2018}{2} + \frac{x - 2017}{3} + \cdots + \frac{x + 2}{2022} + \frac{x + 3}{2023} = 2023.$$

**Solution.** By inspection, 2020 is clearly a solution, as there are 2023 terms and each term evaluates to 1, giving a sum of 2023 as desired.

**Question 13** [Ans: 229]

Assume that $x$ is a positive number such that $x - \frac{1}{x} = 3$ and

$$\frac{x^{10} + x^8 + x^2 + 1}{x^{10} + x^6 + x^4 + 1} = \frac{m}{n},$$

where $m$ and $n$ are positive integers without common factors larger than 1. Determine the value of $m + n$.

**Solution.** Observe that

$$\frac{x^{10} + x^8 + x^2 + 1}{x^{10} + x^6 + x^4 + 1} = \frac{x^8 + 1}{x^4 + 1} \cdot \frac{x^2 + 1}{x^6 + 1} = \frac{x^4\left(x^4 + x^{-4}\right)}{x^2\left(x^2 + x^{-2}\right)} \cdot \frac{x\left(x + x^{-1}\right)}{x^3\left(x^3 + x^{-3}\right)} = \frac{x^4 + x^{-4}}{x^2 + x^{-2}} \cdot \frac{x + x^{-1}}{x^3 + x^{-3}}$$

Repeatedly squaring $x - x^{-1} = 3$ yields

$$x^2 + x^{-2} = 3^2 + 2 = 11,$$
$$x^4 + x^{-4} = 11^2 - 2 = 119.$$

Now observe that

$$x^3 + x^{-3} = \left(x + x^{-1}\right)\left(x^2 + x^{-2} - 1\right) = 10\left(x + x^{-1}\right).$$

The given expression thus evaluates to

$$\frac{x^4 + x^{-4}}{x^2 + x^{-2}} \cdot \frac{x + x^{-1}}{x^3 + x^{-3}} = \frac{119}{11} \cdot \frac{1}{10} = \frac{119}{110},$$

whence $m + n = 119 + 110 = 229$.

**Question 14** [Ans: 24]

Consider the set of all possible pairs $(x, y)$ of real numbers that satisfy $(x-4)^2 + (y-3)^2 = 9$. If $S$ is the largest possible value of $\frac{y}{x}$, find the value of $\lfloor 7S \rfloor$.

**Solution.** Observe that $(x-4)^2 + (y-3)^2 = 9$ describes a circle with centre $(4, 3)$ and radius 3. Also observe that $y/x$ is the gradient of the line passing through the origin and some point on the circle with coordinates $(x, y)$. The largest possible value of $y/x$ hence occurs when the line in question is tangent to the circle. Consider the simultaneous equations $(x-4)^2+(y-3)^2 = 9$ and $S = y/x$. This combines to give $(x-4)^2+(Sx-3)^2 = 9$. Expanding, we have $(1+S^2)x^2-(8+6S)x+16 = 0$. Since the line is tangent to the circle, there is only one solution. The discriminant of the above quadratic is hence 0, giving $(8+6S)^2 - 4(1+S^2)(16) = 0$. Solving, we get $S = 24/7$, whence $\lfloor 7S \rfloor = 24$.

**Question 15** [Ans: 46]

Let $x$, $y$ be positive integers with $16x^2 + y^2 + 7xy \leq 2023$. Find the maximum value of $4x + y$.

**Solution.** Let $k$ be the maximum value of $4x + y$ without the restriction of $x$ and $y$ being integers. Then the line $4x + y = k$ is tangent to the elliptical region given by $16x^2 + y^2 + 7xy \leq 2023$. Equating the two gives

$$16x^2 + (k - 4x)^2 + 7x(k - 4x) = 2023 \implies 4x^2 - kx + (k^2 - 2023) = 0.$$

Setting the discriminant to 0, we get $k^2 - 16(k^2 - 2023) = 0$, whence $k^2 = 2023 \cdot 16/15$. Reinstating the integral restriction on $x$ and $y$, we get $k = 46$, which can indeed be achieved (e.g. $x = 5$, $y = 26$).

**Question 16** [Ans: 25]

Let $x$ be the largest real number such that

$$\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}=x.$$

Determine the value of $(2x-1)^4$.

**Solution.** Clearing denominators, we have

$$\sqrt{x^2-1}+\sqrt{x-1}=x^{3/2} \implies \sqrt{x^2-1}=x^{3/2}-\sqrt{x-1}.$$

Squaring both sides yields

$$x^2-1=x^3+x-1-2x^{3/2}\sqrt{x-1} \implies 2x^{1/2}\sqrt{x-1}=x^2-x+1.$$

Squaring once again yields

$$4x(x-1)=x^4-2x^3+3x^2-2x+1 \implies x^4-2x^3-x^2+2x+1=0.$$

Now note that

$$\begin{aligned}(2x-1)^4&=16x^4-32x^3+24x^2-8x+1\\&=16(x^4-2x^3-x^2+2x+1)+10(4x^2-4x+1)+25\\&=10(2x-1)^2-25\end{aligned}$$

whence $(2x-1)^2=5$ and thus $(2x-1)^4=25$.

**Question 17** [Ans: 64]

Two positive integers $m$ and $n$ differ by 10 and the digits in the decimal representation of $mn$ are all equal to 9. Determine $m+n$.

**Solution.** By inspection, $999=27\cdot 37$. Thus, $m+n=27+37=64$.

**Question 18** [Ans: 1]

Let $\{a_n\}$ be a sequence of positive numbers, and let $S_n=a_1+a_2+a_3+\cdots+a_n$. For any positive integer $n$, let $b_n=\frac{1}{2}\left(\frac{a_{n+1}}{a_n}+\frac{a_n}{a_{n+1}}\right)$. Given that $\frac{a_n+2}{2}=\sqrt{2S_n}$ holds for all positive integers $n$, determine the limit $\lim_{n\to\infty}(b_1+b_2+\cdots+b_n-n)$.

**Solution.** We claim that $a_n=4n-2$. When $n=1$, $S_1=a_1$, whence it is clear that

$$\frac{a_1+2}{2}=\sqrt{2a_1} \implies a_1=2,$$

satisfying our claim. Now assume that $a_k=4k-2$ for some $k\in\mathbb{N}$. We have

$$S_k=\sum_{n=1}^{k}(4n-2)=2k^2.$$

From the given condition,

$$\frac{a_{k+1}+2}{2}=\sqrt{2\left(2k^2+a_{k+1}\right)} \implies a_{k+1}^2-4a_{k+1}+4-16k^2=0,$$

which has the unique positive solution $a_{k+1} = 4k + 2 = 4(k + 1) - 2$. This closes the induction.

$b_n$ thus simplifies to

$$b_n = \frac{1}{2}\left(\frac{a_{n+1}}{a_n} + \frac{a_n}{a_{n+1}}\right) = \frac{1}{2} \cdot \frac{a_{n+1}^2 + a_n^2}{a_n a_{n+1}} = \frac{4n^2+1}{4n^2-1} = 1 + \frac{1}{2n-1} - \frac{1}{2n+1}.$$

The limit in question hence telescopes to 1.

**Question 19** [Ans: 20]

Let $ABC$ be a triangle with $AB = c$, $AC = b$ and $BC = a$, and satisfies the conditions $\tan C = \dfrac{\sin A + \sin B}{\cos A + \cos B}$, $\sin(B - A) = \cos C$ and that the area of triangle $ABC = 3 + \sqrt{3}$. Determine the value of $a^2 + c^2$.

**Solution.** By the sum-to-product formulae, we have

$$\tan C = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)} = \tan\left(\tfrac{A+B}{2}\right).$$

Hence, $C = (A + B)/2$, implying that $\angle C = 60°$. Let $\angle A = 60° - \theta$ and $\angle B = 60° + \theta$. Then $\sin 2\theta = \cos C = 1/2$, whence $\theta = 15°$, thus $\angle A = 45°$ and $\angle C = 75°$. We now express the area of the triangle in terms of the side lengths:

$$3 + \sqrt{3} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B.$$

It quickly follows that

$$\frac{a}{c} = \sqrt{\frac{2}{3}}, \quad ac = \frac{2(3+\sqrt{3})}{\sin B}.$$

This immediately gives us $a^2$ and $c^2$ in terms of $\sin B$:

$$a^2 = \sqrt{\frac{2}{3}} \cdot \frac{2(3+\sqrt{3})}{\sin B}, \quad c^2 = \sqrt{\frac{3}{2}} \cdot \frac{2(3+\sqrt{3})}{\sin B}.$$

Since $\sin B = \sin(30° + 45°) = \sqrt{2}(1 + \sqrt{3})/4$, we finally get

$$a^2 + c^2 = \frac{2(3+\sqrt{3})}{\sqrt{2}(1+\sqrt{3})/4}\left(\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}}\right) = 20.$$

**Question 20**

**[VOID]** Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(0) = 4$ and that

$$g(xy + 1) = g(x)g(y) - g(y) - x + 2023.$$

Find the value of $g(2023)$.

**Solution.** We show that there are two different expressions for $g(x)$ that result in different answers to $g(2023)$.

Firstly, let $x = 0$. Then

$$g(1) = g(0)g(y) - g(y) - 0 + 2023 \implies g(y) = \frac{g(1) - 2023}{3},$$

which is constant. Letting $y = 1$, we get $g(1) = -2023/2$, whence $g(2023) = -2023/2$.

Secondly, let $y = 0$. Then

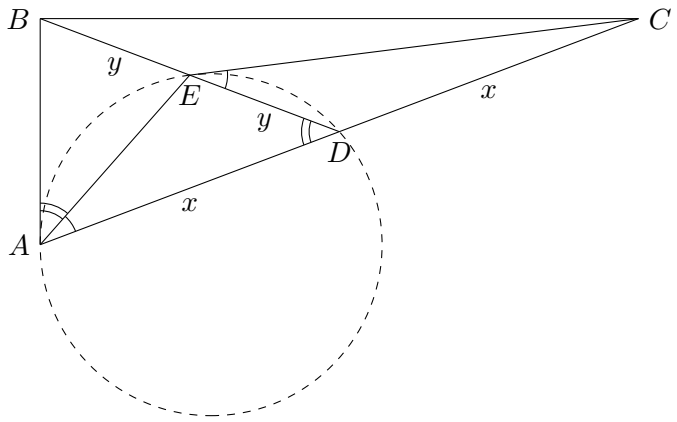
$$g(1) = g(x)g(0) - g(0) - x + 2023 \implies g(x) = \frac{g(1) + x - 2019}{4}.$$

Letting $x = 1$, we get $g(1) = -2018/3$, whence $g(2023) = -1003/6$, a contradiction.

Hence, $g$ does not exist, and the question is void.

**Question 21** [Ans: 7]

In the triangle $ABC$, $D$ is the midpoint of $AC$, $E$ is the midpoint of $BD$, and the lines $BA$ and $CE$ are tangent to the circumcircle of the triangle $ADE$ at $A$ and $E$ respectively. Suppose the circumradius of the triangle $AED$ is $(\frac{64}{7})^{1/4}$. Find the area of the triangle $ABC$.




**Solution.** Let $AD = DC = x$ and $BE = ED = y$. By the power of a point theorem, we have $BA^2 = BE \cdot BD$ and $CE^2 = CD \cdot CA$, whence

$$BA = \sqrt{2}y, \quad CE = \sqrt{2}x.$$

By the alternate segment theorem, we have $\angle CED = \angle EAD$ and $\angle BAE = \angle BDA$. Hence, $\triangle BAE$ is similar to $\triangle BDA$, and $\triangle CED$ is similar to $\triangle CAE$. Thus,

$$\frac{BE}{BA} = \frac{AE}{DE} \implies AE = \frac{x}{\sqrt{2}}, \quad \frac{CD}{CE} = \frac{ED}{AE} \implies AE = \sqrt{2}y.$$

It follows that $x = 2y$. Using the cosine rule on $\triangle ADE$, we obtain $\cos\angle ADE = 3/4$, whence $\sin\angle ADE = \sqrt{7}/4$. By the extended sine rule,

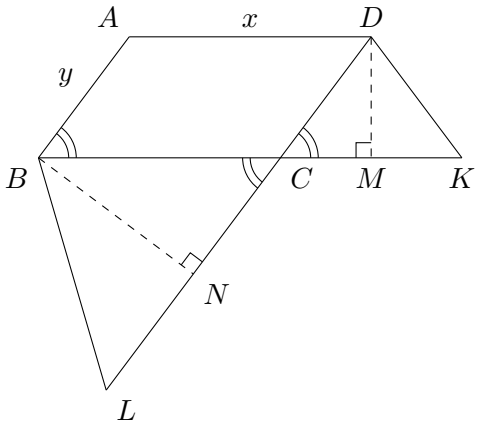
$$\frac{\sqrt{2}y}{\sqrt{7}/4} = 2\left(\frac{64}{7}\right)^{1/4} \implies y = 7^{1/4}.$$

We finally get

$$[ABC] = 2[ABD] = 4[AED] = 4\left(\frac{1}{2} \cdot 2y \cdot y \cdot \sin\angle ADE\right) = 7.$$

**Question 22** [Ans: 48]

$ABCD$ is a parallelogram such that $\angle ABC < 90^\circ$ and $\sin\angle ABC = \frac{4}{5}$. The point $K$ is on the extension of $BC$ such that $DC = DK$; the point $L$ is on the extension of $DC$ such that $BC = BL$. The bisector of $\angle CDK$ intersects the bisector of $\angle LBC$ at $Q$. Suppose the circumradius of the triangle $ABD$ is 25. Find the length of $KL$.




**Solution.** Let $x$ and $y$ be the lengths $AD$ and $BC$ respectively. Let $M$ be the intersection between the bisector of $\angle CDK$ and $BK$; let $N$ be the intersection between the bisector of $\angle LBC$ and $DL$.

Observe that

$$[\triangle ABD] = \frac{xy \cdot BD}{4 \cdot 25} = \frac{1}{2}xy\sin\angle ABC \implies BD = 40.$$

Note that $\cos\angle ABC = 3/5$ and $\cos\angle BCD = -3/5$ (since $\angle BCD = 180^\circ - \angle ABC$). Using the cosine rule on $\triangle BCD$, we obtain

$$x^2 + y^2 + \frac{6}{5}xy = 40^2.$$

Since $\angle BNC = \angle DMC = 90^\circ$, and $\angle BCN = \angle DCM = \angle ABC$, by AAA, we have that $\triangle BNC$ is similar to $\triangle DMC$, $\triangle BNC \equiv \triangle BNL$, $\triangle BMC \equiv \triangle BMK$, and that

$$BN = \frac{4}{5}x, \quad NC = \frac{3}{5}x, \quad DM = \frac{4}{5}y, \quad MC = \frac{3}{5}y.$$

It also follows that $BDMN$ is cyclic. Applying Ptolemy's theorem, we have

$$KL \cdot 40 + xy = \left(x + \frac{6}{5}y\right)\left(y + \frac{6}{5}x\right).$$

Expanding, we finally have

$$KL = \frac{1}{40} \cdot \frac{6}{5}\left(x^2 + y^2 + \frac{6}{5}xy\right) = 48.$$

**Question 23** [Ans: 7]

A group of 200 monkeys is given the task of picking up all 3000 peanuts on the ground. Determine the maximum number $k$ such that there must be $k$ monkeys picking up the same number of peanuts. (It is possible that some lazy monkeys may not pick up any peanuts at all).

**Solution.** Consider the worst-case scenario, where there are $n$ monkeys picking 0 peanuts, $n$ monkeys picking 1 peanut, etc. The total number of peanuts picked can be calculated as

$$\underbrace{n\left(0+1+2+\cdots+\left(\left\lfloor\frac{200}{n}\right\rfloor-1\right)\right)}_{n\lfloor 200/n\rfloor \text{ monkeys}}+\underbrace{\left\lfloor\frac{200}{n}\right\rfloor+\left(\left\lfloor\frac{200}{k}\right\rfloor+1\right)+\cdots}_{\text{remaining monkeys}}.$$

The smallest value of $n$ where the above expression is less than 3000 is 7. Hence, $k=7$.

**Question 24** [Ans: 7]

A chain of $n$ identical circles $C_1, C_2, \ldots, C_n$ of equal radii and centres on the $x$-axis lie inside the ellipse $E: \frac{x^2}{2023}+\frac{y^2}{333}=1$ such that $C_1$ is tangent to $E$ internally at $(-\sqrt{2023}, 0)$, $C_n$ is tangent to $E$ internally at $(\sqrt{2023}, 0)$, and $C_i$ is tangent to $C_{i+1}$ externally for $i=1,\ldots,n-1$. Determine the smallest possible value of $n$.

**Solution.** The curvature $\kappa$ of an ellipse $x^2/a^2+y^2/b^2=1$ is given by

$$\kappa(\theta)=\frac{ab}{\left(a^2\sin^2\theta+b^2\cos^2\theta\right)^{3/2}}.$$

Taking $a^2=2023$, $b^2=333$ and $\theta=0$, we have that the curvature of $E$ at $(\sqrt{2023},0)$ is $\sqrt{2023}/333$. The maximum radius of $C_n$ is thus $333/\sqrt{2023}$ (since $\kappa=1/R$). It immediately follows that

$$\min n=\left\lceil\frac{2\sqrt{2023}}{2\cdot 333/\sqrt{2023}}\right\rceil=7.$$

**Question 25** [Ans: 6]

Let $p>2023$ be a prime. Determine the number of positive integers $n$ such that $(n-p)^2+2023(2023-2n-2p)$ is a perfect square.

**Solution.** Observe that the given expression is nearly a perfect square. Indeed, it can be rewritten as $k^2=(n-p-2023)^2-4\cdot 2023p$. We hence obtain

$$2^2\cdot 7\cdot 17^2\cdot p=(n-p-2023)^2-k^2=(n-p-2023-k)(n-p-2023+k).$$

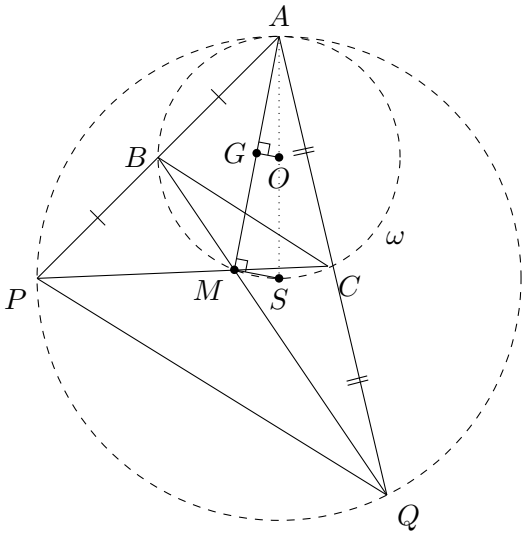
Observe that the two terms on the right have the same parity, thus both must have a factor of 2. Furthermore, because $n-p-2023+k>n-p-2023-k$, it must be that $n-p-2023+k$ has the factor of $p$ (since $p>2023$). Since there are 3 remaining factors, there are hence ${}^3P_2=6$ possible $n$.

### 9.2.2. Round 2 Solutions

**Resources:** Review by *Glen Lim*, AoPS threads

**Question 1**

In a scalene triangle $ABC$ with centroid $G$ and circumcircle $\omega$ centred at $O$, the extension of $AG$ meets $\omega$ at $M$; lines $AB$ and $CM$ intersect at $P$; and lines $AC$ and $BM$ intersect at $Q$. Suppose the circumcentre $S$ of the triangle $APQ$ lies on $\omega$ and $A$, $O$, $S$ are collinear. Prove that $\angle AGO = 90°$.




**Solution.** Consider the homothety $H$ centred at $A$ that sends $O$ to $S$ (i.e. $H$ has a scale factor of $\frac{AS}{AO} = 2$). Since homotheties preserve circumcircles and circumcentres, it follows that $H$ sends $\triangle ABC$ to $\triangle APQ$. This means that $B$ and $C$ are the midpoints of $AP$ and $AQ$ respectively, whence $M$ is the centroid of $\triangle APQ$. Because $AS$ is a diameter of $\omega$, we have $\angle AMS = 90°$. However, because homotheties preserve centroids and angles, we immediately get $\angle AGO = \angle AMS = 90°$ as desired.

**Question 2** [Ans: Yes]

A grid of cells is tiled with dominoes such that every cell is covered by exactly one domino. A subset $S$ of dominoes is chosen. Is it true that at least one of the following two statements is false?

1. There are 2022 more horizontal dominoes than vertical dominoes in $S$.
2. The cells covered by the dominoes in $S$ can be tiled completely and exactly by L-shaped tetrominoes.

**Solution** (bxiao31415)**.** Let $(i, j)$ be coloured in the following manner:

- 0 if both $i$ and $j$ are even;
- 1 if $i$ is odd and $j$ is even;
- 2 if $i$ is even and $j$ is odd; and

- 3 if both $i$ and $j$ are odd.

As an example, the following $4 \times 4$ grid shows the colouring scheme:

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 1 |
| 3 | 2 | 3 | 2 |
| 0 | 1 | 0 | 1 |
| 3 | 2 | 3 | 2 |

Seeking a contradiction, suppose both (1) and (2) are true. Observe that each horizontal domino covers squares whose sum is 1 mod 4, while each vertical domino covers squares whose sum is $-1$ mod 4. Hence, the total sum covered by $S$ is $2022 \equiv 2 \bmod 4$. However, each L-shape tetromino (which contains one horizontal and one vertical domino) covers squares who sum is 0 mod 4, a contradiction. Thus, at least one of the statements is false.

### Question 3

Let $n \geq 2$ be a positive integer. For a positive integer $a$, let $Q_a(x) = x^n + ax$. Let $p$ be a prime and let $S_a = \{b \mid 0 \leq b \leq p-1, \exists c \in \mathbb{Z}, Q_a(c) \equiv b \pmod{p}\}$. Show that $\frac{1}{p-1}\sum_{a=1}^{p-1} |S_a|$ is an integer.

*Remark.* This question is identical to 2022/Open/R2/Q5.

### Question 4

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$, such that

$$f(x+y)((f(x)-f(y))^2 + f(xy)) = f(x^3) + f(y^3)$$

for all integers $x$, $y$.

**Solution** (Ld_minh4354)**.** Let $P(x, y)$ be the assertion that $f(x+y)((f(x)-f(y))^2 + f(xy)) = f(x^3) + f(y^3)$. From $P(x, x)$, one has

$$f(2x)f(x^2) = 2f(x^3).$$

$P(0, 0)$ hence gives us $f(0)^2 = 2f(0)$, whence $f(0) = 0$ or $f(0) = 2$. This gives us two main cases:

*Case 1*. Suppose $f(0) = 0$. Then $P(x, 0)$ gives us

$$P(x, 0): \qquad f(x)^3 = f(x^3).$$

From $P(1, 1)$, we hence have

$$P(1, 1): \qquad f(2)f(1) = 0.$$

Thus, $f(1) = 0$ or $f(2) = 0$.

*Subcase 1A*. Suppose $f(1) = 0$. $P(-1, 1)$ clearly gives $f(-1) = 0$. We now prove inductively that $f(x) \equiv 0$. The base case $x = 0$ has already been assumed. Now suppose that $f(k) = 0$ for some $k \in \mathbb{Z}$. Applying this inductive hypothesis to $P(k, 1)$ and $P(-k, 1)$, we immediately get $f(k+1) = f(k-1) = 0$. This closes the induction. One solution is hence $f(x) \equiv 0$.

*Subcase 1B*. Suppose $f(2) = 2$. Then we have the following:

$$\begin{aligned} P(x,0): &\qquad f(x)^3 &&= f(x^3) \\ P(1,0): &\qquad f(1) &&\in \{-1,0,1\} \\ P(-1,0): &\qquad f(-1) &&\in \{-1,0,1\} \end{aligned}$$

If $f(1) = 0$, then by Subcase 1A, we would have $f(x) \equiv 0$, contradicting $f(2) = 2$. If $f(-1) = 0$, then by $P(-1,-1)$, we would have $f(-2)f(1) = 0$. However, if $f(-2) = 0$, then by $P(-2,2)$, one gets $0 = 8$, a contradiction. Thus, $f(1), f(-1) \neq 0$.

From $P(1,-1)$, we have $f(1) + f(-1) = 0$. Suppose $f(1) = -1$. By $P(2,1)$, we have $f(3) = \frac{8}{11} \notin \mathbb{Z}$, a contradiction. Thus, $f(1) = 1$ and $f(-1) = -1$. We now show that $f(x) \equiv x$ via induction. The base case has already been settled (namely $x = \{-1,0,1\}$). Now suppose that $f(x) = x$ on $[-k,k]$ for some $k \in \mathbb{Z}$. From $P(k,1)$ and $P(-k,1)$, applying the inductive hypothesis yields

$$\begin{aligned} P(k,1): &\qquad f(k+1)(k^2-k+1) = k^3+1 &&\implies f(k+1) = k+1 \\ P(-k,1): &\qquad f(-k-1)(k^2-k+1) = -k^3-1 &&\implies f(-k-1) = -k-1 \end{aligned}$$

This closes the induction. We hence have a second solution, namely $f(x) \equiv x$.

*Case 2*. Suppose $f(0) = 2$. From $P(x,0)$ one gets

$$P(x,0): \qquad f(x)(f(x)^2 - 4f(x) + 6) = f(x^3) + 2. \tag{1}$$

Taking $P(1,0)$ hence gives us a cubic in $f(1)$:

$$P(1,0): \qquad f(1)^3 - 4f(1)^2 + 5f(1) - 2 = 0.$$

We thus have $f(1) = 1$ or $f(1) = 2$.

*Subcase 2A*. Suppose $f(1) = 1$. Taking $P(-1,1)$, one has

$$P(-1,1): \qquad 2f(-1)^2 - 3f(-1) + 1 = 0,$$

whence $f(-1) = 1$. Note that $f(-1)$ is an integer and hence cannot be $\frac{1}{2}$. We now show via induction that $f(x) = 1$ when $x$ is odd, and $f(x) = 2$ when $x$ is even. The base cases (namely $x \in \{-1,0,1\}$) have already been settled. Let $k$ be some integer. Suppose that $f(k) = 1$ when $k$ is odd and $f(k) = 2$ when $k$ is even. From $P(k,1)$, we have

$$P(k,1): \qquad f(k+1) = \frac{f(k^3)+1}{f(k^2) - f(k) + 1},$$

from which it follows that $f(k+1) = 2$ when $k+1$ is even, and $f(k+1) = 1$ when $k+1$ is odd. Also, from $P(k,-1)$, we have

$$P(k,-1): \qquad f(k-1) = \frac{f(k^3)+1}{f(k)^2 - 2f(k) + f(-k) + 1},$$

from which it follows that $f(k-1) = 2$ when $k-1$ is even, and $f(k-1) = 1$ when $k-1$ is odd. This closes the induction. We hence obtain a third solution, namely

$$f(x) \equiv \begin{cases} 1, & x \text{ odd} \\ 2, & x \text{ even} \end{cases}.$$

*Subcase 2B*. Suppose $f(1) = 2$. From $P(1,-1)$, we obtain a quadratic in $f(-1)$:

$$P(1,-1): \qquad 2f(-1)^2 - 7f(-1) + 6 = 0,$$

which has solutions 2 and $\frac{3}{2}$. Thus, $f(-1) = 2$ (since $f(-1) \in \mathbb{Z}$). We now show that $f(x) \equiv 2$ via induction. The base cases ($x \in \{-1, 0, 1\}$) have already been settled. Now suppose $f(k) = 2$ for some $k \in \mathbb{Z}$. From $P(k, 1)$ and (1), we get

$$P(k,1): \qquad f(k+1) = \frac{f(k)(f(k)^2 - 4f(k) + 6)}{f(k)^2 - 3f(k) + 4} = 2.$$

Likewise, $P(k, -1)$ gives

$$P(k,-1): \qquad f(k-1) = \frac{f(k)(f(k)^2 - 4f(k) + 6)}{f(k)^2 - 3f(k) + 4} = 2.$$

This closes the induction. We hence get our fourth and final solution: $f(x) \equiv 2$.

To conclude, the following four functions are the only solutions to the given functional equation:

$$f(x) \equiv 0, \quad f(x) \equiv 2, \quad f(x) \equiv x, \quad f(x) \equiv \begin{cases} 1, & x \text{ odd} \\ 2, & x \text{ even} \end{cases}$$

**Question 5** [Ans: $x \in (0, 120]$]

Determine all real numbers $x$ between 0 and 180 such that it is possible to partition an equilateral triangle into finitely many triangles, each of which has an angle of $x^\circ$.

**Solution** (oneplusone)**.** We claim that $x \in (0, 120]$. We split our proof into 3 cases:

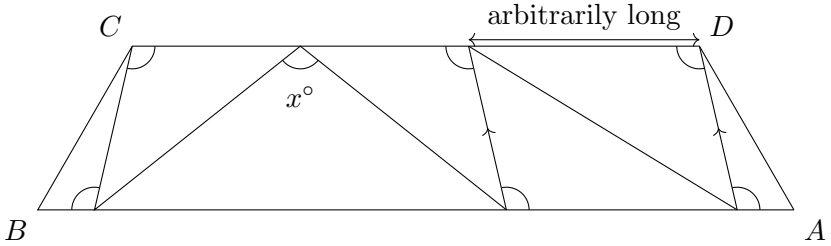
*Case 1*. Suppose $x > 120$. Let there be a total of $n$ "small triangles" in the partition of the original equilateral triangle. Let $A$ be the set of vertices of the original equilateral triangle. Let $B$ be the set of vertices that lie on an edge. Let $C$ be the remaining vertices. Now observe that angle sum of all $n$ "small triangles" must be $180n^\circ$. Since each vertex in $A$, $B$ and $C$ contributes $60^\circ$, $180^\circ$ and $360^\circ$ respectively, we have the equality

$$60 \cdot 3 + 180\,|B| + 360\,|C| = 180n \implies |B| + 2\,|C| = n - 1. \tag{1}$$

Now observe that the vertices in $A$ cannot have an angle $x^\circ$. On the hand, each vertex in $B$ and $C$ can have at least 1 and 2 such angles respectively. Thus, the total number of $x^\circ$ angles is at least $n$ (by our assumption) and at most $|B| + 2\,|C|$. With (1), we get the contradiction $n \le n - 1$. Thus, $x > 120$ is impossible.

*Case 2*. Suppose $x = 120^\circ$. This is clearly achievable; Let $O$ be the centre of the equilateral triangle $\triangle ABC$. Then $\triangle AOB$, $\triangle BOC$ and $\triangle COA$ all contain a $120^\circ$ angle.

*Case 3*. Suppose $x < 120^\circ$. For $R > 0$, let an $R$-trapezoid be a trapezoid similar to the trapezium $ABCD$, where $\angle A = \angle D = 60^\circ$ and $\angle B = \angle C = 120^\circ$, with $AB = CD = 1$ and $BC = R$. We call a shape constructible if it can be partitioned into triangles, each of which has an angle $x^\circ$. Firstly, observe that for $R$ sufficiently large, the $R$-trapezoid is constructible. This is shown in the figure below:




It thus follows that the 1-trapezoid is constructible: simply slice it horizontally into sufficiently thin $R$-trapezoids (for sufficiently large $R$). Since an equilateral triangle can be partitioned into 3 1-trapezoids, it must also be constructible.

Thus, $x \in (0, 120]$.

# 10. SMO 2024

## 10.1. Junior Section

### 10.1.1. Round 1 Solutions

**Resources:** Review by *Way Tan*

**Question 1** [Ans: D]

If $x^2 + 4x + 16 = 0$, what is the value of $x^3$?

(A) 4 (B) 8 (C) 16 (D) 64 (E) 128

**Solution.** By the quadratic formula, we have

$$x = -2 + 2\sqrt{3}i = 4\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) = 4e^{\pm\frac{2\pi}{3}i}.$$

Thus, $x^3 = 4^3 e^{\pm 2\pi i} = 64$.

**Question 2** [Ans: D]

Let $a$ be a real number that satisfies $-1 < a < 0$. Which of the following is true?

(A) $\pi^a < \frac{1}{\pi} < (\frac{1}{\pi})^a$
(B) $\pi^a < (\frac{1}{\pi})^a < \frac{1}{\pi}$
(C) $\frac{1}{\pi} < (\frac{1}{\pi})^a < \pi^a$
(D) $\frac{1}{\pi} < \pi^a < (\frac{1}{\pi})^a$
(E) $(\frac{1}{\pi})^a < \pi^a < \frac{1}{\pi}$

**Solution.** We are tasked with ordering $\pi^{-1}$, $\pi^a$ and $\pi^{-a}$. This is equivalent to ordering $-1$, $a$ and $-a$. Since $-1 < a < 0$, we clearly have $-1 < a < -a$. Thus, $\pi^{-1} < \pi^a < \pi^{-a}$.

**Question 3** [Ans: B]

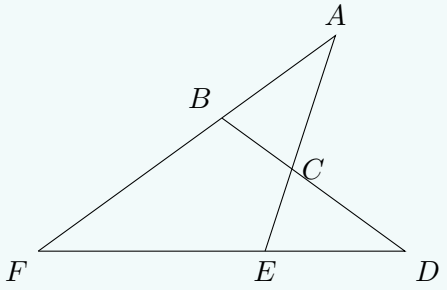
How many non-congruent triangles are there whose sides have integer lengths and the longest side has length 10 units?

(A) 25 (B) 30 (C) 35 (D) 40 (E) 45

**Solution.** Let $a$ and $b$ be the lengths of the legs of the triangle. By the triangle inequality, we must have $a + b > 10$. Without loss of generality, let $1 \leq a \leq b \leq 10$. Let $a$ be fixed. Then $b \in \{\max\{11 - a, a\}, \ldots, 10\}$. This gives $\min\{a, 11 - a\}$ possible values of $b$. Summing from $a = 1$ to $a = 10$, we have 30 possible pairs of side lengths and hence 30 possible triangles, which are guaranteed to be non-congruent by construction.

**Question 4** [Ans: C]

In the diagram below, the points $B$ and $E$ lie on $AF$ and $DF$ respectively, and $AE$ and $BD$ intersect at $C$. If $AB = AC$, $BD = BF$ and $EA = EF$, find $\angle BAC$.




(A) $30°$ (B) $33°$ (C) $36°$ (D) $38°$ (E) $40°$

**Solution.** Let $\angle BAC = \alpha$. Since $\triangle FEA$ is isosceles, we have $\angle BFD = \alpha$. Since $\triangle FBD$ is isosceles, we also have $\angle BDF$. Thus, $\angle FBD = 180° - 2\alpha$. Meanwhile, because $\triangle BAC$ is isosceles, $\angle ABC = 90° - \frac{1}{2}\alpha$. Hence,

$$\angle FBD + \angle ABD = 180° \implies (180° - 2\alpha) + \left(90° - \frac{1}{2}\alpha\right) = 180° \implies \alpha = 36°.$$

**Question 5** [Ans: C]

Let $x$, $y$ and $z$ be real numbers such that $x \neq 0$, $y - z \neq 0$ and $z + x \neq 0$. If $\dfrac{2}{x} = \dfrac{4}{y-z} = \dfrac{5}{z+x}$, find the value of $\dfrac{7x-y}{y+2z}$.

(A) $\dfrac{11}{17}$ (B) $-\dfrac{11}{17}$ (C) $\dfrac{7}{13}$ (D) $-\dfrac{7}{13}$ (E) $1$

**Solution.** Taking reciprocals, we have

$$\frac{x}{2} = \frac{y-z}{4} = \frac{z+x}{5} \implies y = \frac{7}{2}x, \quad z = \frac{3}{2}x.$$

Hence,

$$\frac{7x-y}{y+2x} = \frac{7x - \frac{7}{2}x}{\frac{7}{2}x + 2(\frac{3}{2}x)} = \frac{7}{13}.$$

**Question 6** [Ans: 76]

Let $N$ be a 2-digit whole number. When 2692 is divided by $N$, the remainder is 13, and when 2978 is divided by $N$, the remainder is 14. Find the sum of all the possible values of $N$.

**Solution.** We are given that $2692 \equiv 13 \pmod{N}$ and $2978 \equiv 14 \pmod{N}$. Thus,

$$2679 \equiv 2964 \equiv 0 \pmod{N} \implies 285 = 2964 - 2679 \equiv 0 \pmod{N}.$$

Since $285 = 3 \cdot 5 \cdot 19$, the only two-digit factors of 285 are 15, 19, 57, 95. However, $N$ clearly cannot end in 5. Thus, $N$ can only be 19 or 57. Testing these two values, we see that they indeed satisfy the given conditions. Thus, the sum of all the possible values of $N$ is $19 + 57 = 76$.

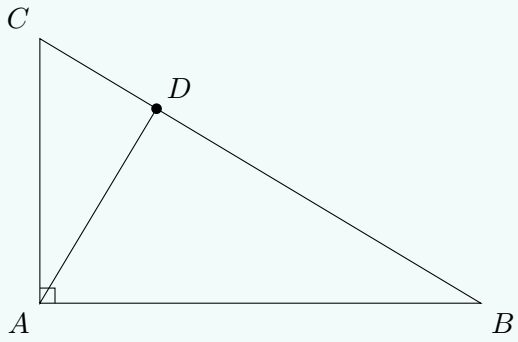
**Question 7** [Ans: 89]

If $x$ is a positive integer such that $2^x + 2^{89} = 512^{10}$, find the value of $x$.

**Solution.** Note that $512 = 2^9$. We hence have $2^x = 2^{90} - 2^{89} = 2^{89}$, whence $x = 89$.

**Question 8** [Ans: 17]

The diagram shows a right-angled triangle $ABC$. The sides $AC$ and $AB$ are in the ratio $3 : 5$. The point $D$ lies on $BC$ such that $AD$ is perpendicular to $BC$. Furthermore, $DB$ is 8 cm longer than $CD$. What is the length of $BC$ in cm?




**Solution.** Let $AC = 3x$ and $CD = y$. Then $AB = 5x$ and $DB = y + 8$. By the geometric mean theorem, we have

$$BD \cdot DC = AD^2 \implies y(y+8) = AD^2. \tag{1}$$

Applying Pythagoras' theorem to $\triangle ABC$ and $\triangle ADC$, we see that

$$AB^2 + AC^2 = BC^2 \implies 34x^2 = (2y+8)^2$$

and

$$AD^2 + DC^2 = AC^2 \implies AD^2 = 9x^2 - y = \frac{9}{34}(2y+8)^2 - y^2. \tag{2}$$

Equating (1) and (2), we see that

$$y(y+8) = \frac{9}{34}(2y+8)^2 - y^2,$$

which gives $y = \frac{9}{2}$, whence $BC = 2y + 8 = 17$.

**Question 9** [Ans: 1825]

Let $n$ be a positive integer. Suppose that $a_1, a_2, a_3, \ldots$ is a sequence of numbers defined by

$$a_1 = \sqrt{(n+3)(n-1)+4}, \quad a_k = \sqrt{(n+2k+1)a_{k-1}+4} \quad \text{for } k \geq 2.$$

If $a_{100} = 2024$, find the value of $n$.

**Solution.** We claim that $a_k = n + 2k - 1$ for $k \geq 2$. Consider the base case $k = 1$: Observe $(n+3)(n-1)+4 = [(n+1)+2]\,[(n+1)-2]+4 = (n+1)^2$. Hence, $a_1 = n + 1$. Now suppose that $a_m = n + 2m - 1$ for some positive integer $m$. Observe that

$$\begin{aligned} a_{m+1} &= \sqrt{(n+2(m+1)+1)a_m + 4} = \sqrt{(n+2m+1+2)(n+2m+1-2)+4} \\ &= \sqrt{(n+2m+1)^2 - 2^2 + 4} = \sqrt{(n+2m+1)^2} = n + 2m + 1 = n + 2(m+1) - 1. \end{aligned}$$

This closes the induction. We hence see that $a_{100} = n + 200 - 1 = 2024$, whence $n = 1825$.

**Question 10** [Ans: 10]

Let $N$ be the smallest positive integer such that the sum of its digits is 2024. What is the sum of the digits of the number $N+2$.

**Solution.** Since $N$ is minimal, it must be of the form $\overline{a9\cdots9}$, where $a$ is an integer between 1 and 9. This gives $a+9+\cdots+9=2024$. Reducing modulo 9 yields $a=8$. Thus, $N=\overline{89\cdots9}$, whence $N+2=\overline{90\cdots1}$. The digit sum of $N+2$ is hence $9+1=10$.

**Question 11** [Ans: 19]

If $a$ and $b$ are non-zero real numbers such that $\dfrac{1}{b}-\dfrac{1}{a}=4$, find the value of $\dfrac{3a+7ab-3b}{a-3ab-b}$.

**Solution.** Multiplying through by $ab$ yields

$$a-b=4ab \implies ab=\frac{a-b}{4}.$$

Thus,

$$\frac{3a+7ab-3b}{a-3ab-b}=\frac{(3a-9ab-3b)+16ab}{a-3ab-b}=3+\frac{4(a-b)}{a-3(\frac{a-b}{4})-b}=3+\frac{4}{1-\frac{3}{4}}=19.$$

**Question 12** [Ans: 5]

If the 5-digit whole number $\overline{11ab6}$ is a perfect square, find the value of $a+b$.

**Solution.** Let $\overline{11ab6}=m^2$. Then $10000<m^2<12000$, whence $100<m<109$. Also note that the ones digit of $m$ is either 4 or 6. We hence have $m=104$ or $m=106$. Testing both cases, we see that $104^2=10816$ while $106^2=11236$, whence $a+b=2+3=5$.

**Question 13** [Ans: 9920]

Let $N=1\times2+2\times3+3\times4+\cdots+30\times31$. Find the value of $N$.

**Solution.** Clearly,

$$N=\sum_{i=1}^{30}i(i+1)=\sum_{i=1}^{30}\left(i^2+i\right).$$

Using the standard formulae for power sums, we end up with

$$N=\frac{30(30+1)(2\cdot30+1)}{6}+\frac{30(30+1)}{2}=9920.$$

**Question 14** [Ans: 23]

Let $x$ and $y$ be non-zero real numbers where $x \neq y$. If $x^2+\sqrt{3}y = 4$ and $y^2+\sqrt{3}x = 4$, find the value of $\left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$.

**Solution.** Let $x \mapsto \sqrt{3}x$ and $y \mapsto \sqrt{3}y$, which obviously do not affect the value of $(\frac{y}{x})^2 + (\frac{x}{y})^2$. The given equations simplify to $3x^2 + 3y = 4$ and $3y^2 + 3x = 4$.

Summing the two equations together and completing the square, we have

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{19}{6}.$$

Observe that this describes a circle symmetric about the line $y = x$. Subtracting the two equations and completing the square, we have

$$\left(x - \frac{1}{2}\right)^2 = \left(y - \frac{1}{2}\right)^2.$$

Combining this with the symmetry of $(x, y)$ about $y = x$, it follows that $(x, y)$ lies on the line $y = -x + 1$. Hence,

$$-x + 1 = \frac{4 - 3x^2}{3} \implies x, y = \frac{3 \pm \sqrt{21}}{6},$$

where $x$ and $y$ take different branches of the square root. Thus,

$$\left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2 = \frac{x^4 + y^4}{(xy)^2} = \frac{\left(3 + \sqrt{21}\right)^4 + \left(3 - \sqrt{21}\right)^4}{\left[\left(3 + \sqrt{21}\right)\left(3 - \sqrt{21}\right)\right]^2} = 23.$$

**Question 15** [Ans: 72]

Find the sum of all 2-digit even numbers $N$ with the following property: $N$ is a multiple of the product of its two digits.

**Solution.** Let $N = 10a + b$, where $a$ and $b$ are integers such that $1 \leq a \leq 9$ and $b \in \{0, 2, 4, 6, 8\}$. The given condition translates to $10a + b = kab$, where $k$ is a positive integer. Let $b = 2b'$, where $b' \in \{0, 1, \ldots, 4\}$. Then $(kb' - 5)a = b'$.

*Case 1*. Suppose $b' = 0$. Then $a = 0$, a contradiction.

*Case 2*. Suppose $b' = 1$. Then $(k - 5)a = 1$, whence $a$ can only be 1 (when $k = 6$). Hence, $N = 12$ is a solution.

*Case 3*. Suppose $b' = 2$. Then $(2k - 5)a = 2$. Then $a$ can only be 2 (when $k = 3$). Hence, $N = 24$ is a solution.

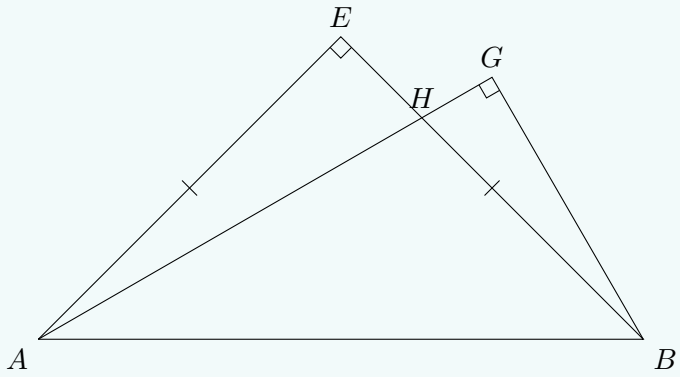
*Case 4*. Suppose $b' = 3$. Then $(3k - 5)a = 3$. Then $a$ can only be 3 (when $k = 2$). Hence, $N = 36$ is a solution.

*Case 5*. Suppose $b' = 4$. Then $(4k - 5)a = 4$. Since $4k - 5 \neq 1, 2, 4$, there are no solutions in this case.

Thus, the only possible values of $N$ are 12, 24 and 36. The desired answer is hence $12 + 24 + 36 = 72$.

**Question 16** [Ans: 25]

In the diagram below, $AEB$ is an isosceles right-angled triangle and $ABG$ is a 30°-60°-90° right-angled triangle with $\angle GAB = 30°$. The sides $AG$ and $BE$ intersect at $H$. If the area of triangle $AHE$ is 50 cm$^2$, find the area of triangle $BGH$ in cm$^2$.




**Solution.** Observe that $ABGE$ is cyclic, where $AB$ is the diameter of $(ABGE)$. Let $\alpha = \angle EAB$. Then $\angle EBA = \alpha$. Thus, $\angle EAH = \alpha - 30°$ and $\angle GBH = 60° - \alpha$. Since $\angle EAH = \angle GBH$, we obtain $\alpha = 45°$. We thus have $AE = \sqrt{2}r$, where $r = \frac{1}{2}AB$ is the radius of $(ABGE)$. Also, $GB = AB \sin 30° = r$. Since $\triangle AEH$ and $\triangle BGH$ are similar, we see that

$$\frac{HB}{GB} = \frac{HA}{EA} \implies \frac{HB}{EH} = \frac{1}{\sqrt{2}}.$$

The ratio of similarity is thus $\frac{1}{\sqrt{2}}$, whence $[BGH] = \left(\frac{1}{\sqrt{2}}\right)^2 [AEH] = 25$ cm$^2$.

**Question 17** [Ans: 2451]

Find the smallest positive integer $n$ such that $\sqrt{n} - \sqrt{n-1} < \dfrac{1}{99}$.

**Solution.** Squaring the inequality, we see that

$$2\sqrt{n(n-1)} > 2n - 1 - 99^{-2}.$$

Squaring once more, we obtain

$$1 + (2 - 4n)99^{-2} + 99^{-4} < 0.$$

Rearranging, we obtain

$$n > \frac{99^2 + 99^{-2} + 2}{4} \approx 2450.75,$$

whence $\min n = 2451$.

**Question 18** [Ans: 32]

Let $a$, $b$ and $c$ be real numbers such that $a+b+c = 8$ and $ab+bc+ca = 0$. Find the maximum value of $3(a+b)$.

**Solution.** From the second equation, we have $c = -\frac{ab}{a+b}$. Substituting this into the first equation and clearing denominators yields

$$(a+b)^2 - 8(a+b) - ab = 0.$$

By the AM-GM inequality, we have $ab \leq \frac{1}{4}(a+b)^2$. Hence,

$$(a+b)^2 - 8(a+b) - \frac{1}{4}(a+b) \leq 0 \implies 3(a+b)^2 - 32(a+b) \leq 0.$$

Thus, $\max(a+b) = \frac{32}{3}$, whence $\max 3(a+b) = 32$.

**Question 19** [Ans: 16]

In the table below, every row and column is an infinite arithmetic progression.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | $\cdots$ |
| 3 | 6 | 9 | 12 | 15 | $\cdots$ |
| 5 | 9 | 13 | 17 | 21 | $\cdots$ |
| 7 | 12 | 17 | 22 | 27 | $\cdots$ |
| 9 | 15 | 21 | 27 | 33 | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

How many times does the number 2025 appear in the table?

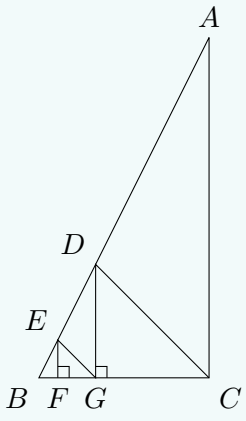
**Solution.** Let $a_{ij}$ be the number in the $i$th row and $j$th column of the table, where $i, j > 0$. Observe that $a_{i+1,j} - a_{ij} = j + 1$. Since $a_{1j} = 2j - 1$, it follows that

$$a_{ij} = 2j - 1 + (i-1)(j+1) = (i+1)(j+1) - 3.$$

Consider $a_{ij} = 2025$. Then $(i+1)(j+1) = 2028 = 2^2 \cdot 3 \cdot 13^2$. The number of possible pairs of $(i, j)$ is thus the number of positive pairs of factors of 2028 (excluding 1). This gives a total of $3 \cdot 2 \cdot 3 - 2 = 16$ possibilities. Note that we subtract 2 as we need to exclude $(1, 2028)$ and $(2028, 1)$.

**Question 20** [Ans: 3]

In the diagram below, $ABC$ is a right-angled triangle. Points $D$ and $E$ lie on $AB$ while points $F$ and $G$ lie on $BC$ such that $\triangle EFG$ and $\triangle DGC$ are right-angled isosceles triangles. It is given that $DC = 3EG$ and the area of $\triangle DGC = 1$ cm$^2$. What is the area of $\triangle ADC$ in cm$^2$?




**Solution.** Since $\triangle EFG$ and $\triangle DGC$ are similar, we get $\frac{EF}{DG} = \frac{EG}{DC} = \frac{1}{3}$. Since $\triangle DEG$ and $\triangle FEG$ have the same height ($FG$), we get

$$[DEG] = 3\,[FEG] = 3 \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{3}.$$

Observe that $\triangle DEG$ and $\triangle ADC$ are also similar. Since $\frac{DC}{EG} = 3$, we obtain

$$[ADC] = 3^2\,[DEG] = 3.$$

**Question 21** [Ans: 504]

How many different 4-tuples $(a,\ b,\ c,\ d)$ are there, where $a$, $b$, $c$ and $d$ are positive integers, such that

$$a > b > c > d, \quad a+b+c+d = 2024 \quad \text{and} \quad a^2 - b^2 + c^2 - d^2 = 2024?$$

**Solution.** Observe that

$$a^2 - b^2 + c^2 - d^2 = a+b+c+d \implies (a-b-1)(a+b) + (c-d-1)(c+d) = 0.$$

Since $a+b$ and $c+d$ are positive, it must be that one of $a-b-1$ and $c-d-1$ is negative, or both are zero.

*Case 1*. Suppose $a-b-1 < 0$. Then $a-b < 1$. However, since $a > b$, we know that $a-b > 0$, a contradiction. A similar argument holds for the case where $c-d-1 < 0$.

*Case 2*. Suppose $a-b-1 = 0$ and $c-d-1 = 0$. Then $a = b+1$ and $c = d+1$. Substituting this into the two equations, we see that

$$b+d = 1011.$$

Observe that $b \geq d+2$, whence

$$b + (d+2) = 1013 \implies b \geq 507$$

Additionally, $\max b$ occurs when $d = 1$ and $c = 2$, thus

$$(\max b + 1) + \max b + 2 + 1 = 2024 \implies \max b = 1010.$$

There are hence $1010 - 507 + 1 = 504$ possible values for $b$, giving a total of 504 different 4-tuples.

**Question 22** [Ans: 360]

Points $A$ and $B$ lie on the graph of $y = x^2 + 5x - 8$ such that $A$, $B$ and the origin $O$ are collinear and $|OB| = 2\,|OA|$. It is given that $A$ lies in the first quadrant. Find $|AB|^2$.

**Solution.** We have that $A$ and $B$ are the intersections between $y = x^2+5x-8$ and $y = kx$, where $k > 0$. Solving the two equations simultaneously, we end up with $A(x_A, kx_A)$ and $B(x_B, kx_B)$, where

$$x_A = \frac{k-5+\sqrt{k^2-10k+57}}{2}, \qquad x_B = \frac{k-5-\sqrt{k^2-10k+57}}{2}.$$

Since $|OB| = 2\,|OA|$, we get

$$x_B^2 + k^2 x_B^2 = 4x_A^2 + 4k^2 x_A^2 \implies \left(k^2-1\right)\left(4x_A^2 - x_B^2\right) = 0.$$

Clearly $k \neq 1$. Hence, we have $4x_A^2 - x_B^2 = (2x_A - x_B)(2x_A + x_B) = 0$. Since $|x_A| < |x_B|$, we have $2x_A + x_B = 0$. This gives

$$2\left(k-5+\sqrt{k^2-10k+57}\right) + \left(k-5-\sqrt{k^2-10k+57}\right) = 0,$$

which implies that

$$3(k-5)+\sqrt{k^2-10k+57}=0 \implies \sqrt{k^2-10k+57}=3(5-k).$$

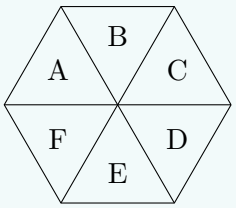
Squaring, we get

$$k^2-10k+57=225-90k+9k^2 \implies k^2-10k+21=0,$$

whence $k=3$ or $k=7$. Testing these two values, we see that only $k=3$ satisfies the condition $|OB|=2\,|OA|$, where $|OA|=\sqrt{40}$ and $|OB|=\sqrt{160}$, thus $|AB|^2=(3\sqrt{40})^2=360$.

**Question 23** [Ans: 15630]

The diagram below shows a regular hexagon that is divided into six congruent triangular regions $A$, $B$, $C$, $D$, $E$ and $F$. Two triangular regions are adjacent if they share a common side. For example, $A$ and $B$ are adjacent but $A$ and $C$ are not adjacent. In how many ways can we colour these regions $A$, $B$, $C$, $D$, $E$ and $F$ using six different colours such that the adjacent regions do not receive the same colour? (Note that not all six colours need to be used in colouring the six regions and non-adjacent regions can receive the same colour).




**Solution.** Let $A_k$ be the event that at least $k$ pairs of adjacent regions have the same colour. By the inclusion-exclusion principle, the number of colourings such that no two pairs have the same colour is given by

$$|A_0|-|A_1|+|A_2|-|A_3|+|A_4|-|A_5|+|A_6|\,.$$

Now observe that for $0\le k\le 5$, we have

$$|A_k|=\binom{6}{k}6^{6-k}:$$

$\binom{6}{k}$ ways to choose the $k$ pairs, and $6^{6-k}$ ways to colour the $6-k$ groups of regions (where we group a pair of regions together). Meanwhile, for $k=6$, we obviously have

$$|A_6|=6.$$

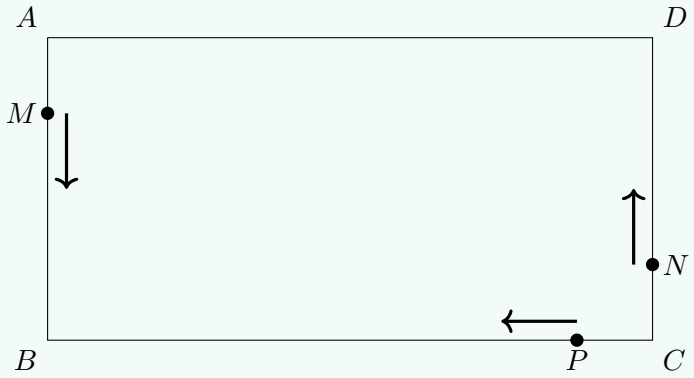
Thus, the desired answer is

$$\sum_{k=0}^{5}\binom{6}{k}(-1)^k6^{n-k}+6=(6-1)^6+(6-1)=15630.$$

*Remark.* In general, the number of ways to colour a cycle graph with $n$ nodes ($C_n$) with $\lambda$ colours is given by $(\lambda-1)^n+(-1)^n(\lambda-1)$. This is called the chromatic polynomial of $C_n$. In our case, we have $n=\lambda=6$.

**Question 24** [Ans: 45]

The diagram below shows three toy cars moving in a rectangular circuit $ABCD$ where $AB = 10$m and $BC = 20$m. Toy cars $M$ and $N$ start from the vertices $A$ and $C$ respectively and move in an anti-clockwise direction with constant speeds 10 m/min and 4 m/min respectively. Toy car $P$ starts from $C$ and moves in a clockwise direction with a constant speed of 8 m/min.




The three toy cars start their motion at the same time. Assume that when any two toy cars meet, there is no collision and the cars will continue with their motion. Find the total time elapsed in minutes when all the three toy cars meet simultaneously for the fifth time.

**Solution.** Let displacement be measured along the circuit $ABCD$ with reference to $C$, taking anti-clockwise as positive. For example, the point $D$ has displacement 10 m, while the point $B$ has displacement $-20$ m. Let $t$ min be the time elapsed. We can parameterize the displacement of $M$, $N$ and $P$ using $t$:

$$s_M = 10t + 30, \qquad s_N = 4t, \qquad s_P = -8t.$$

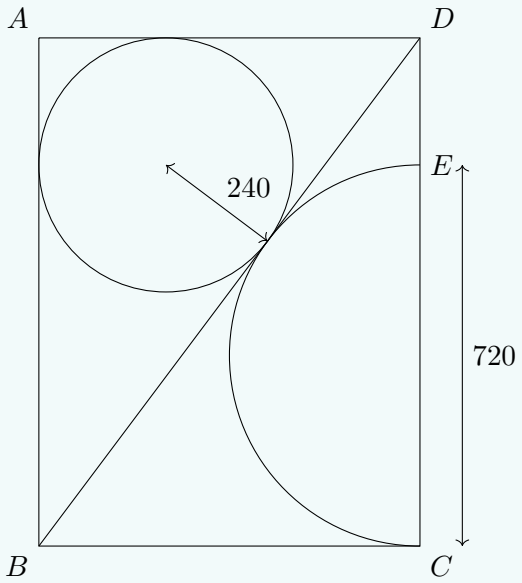
When the three toy cars meet simultaneously, their displacement must be equivalent modulo 60, which is the perimeter of the circuit. We hence wish to solve the following system of congruences:

$$10t + 30 \equiv 4t \equiv -8t \pmod{60}.$$

Firstly, we have $4t \equiv -8t \pmod{60}$. This gives $12t \equiv 0 \pmod{60}$, whence $t \equiv 0 \pmod{5}$. Let $t = 5k$ for some positive integer $k$. Substituting this into the congruence $10t + 30 \equiv 4t \pmod{60}$ yields $30kt + 30 \equiv 0 \pmod{60} \implies kt \equiv 1 \pmod{2}$. Thus, $k$ is odd, whence $t \in \{5, 15, 25, \cdots\}$. The fifth-smallest value of $t$ is thus 45.

**Question 25** [Ans: 3360]

In the diagram below, $ABCD$ is a rectangle. A circle of radius 240 mm is inscribed in $\triangle ABD$ and $BD$ is a common tangent to both the circle and a semicircle whose diameter $CE$ lies on $CD$. It is given that $CE = 720$ mm. Find the perimeter of rectangle $ABCD$ in mm.




**Solution.** Let $AB = x$ and $AD = y$. Recall that $A = rs$, where $A$ is the area of a triangle, $r$ is the inradius of the triangle, and $s$ is the semiperimeter of the triangle. Applying this formula on $\triangle ABD$, we have

$$\frac{xy}{2} = 240\left(\frac{x+y+\sqrt{x^2+y^2}}{2}\right) = 120\left(x+y+\sqrt{x^2+y^2}\right). \tag{1}$$

Let $F$ be the reflection of $B$ across $CD$. Applying this formula on $\triangle BDF$, we have

$$2\left(\frac{xy}{2}\right) = 360\left(\frac{\sqrt{x^2+y^2}+\sqrt{x^2+y^2}+2y}{2}\right) \implies \frac{xy}{2} = 180\left(\sqrt{x^2+y^2}+y\right). \tag{2}$$

Equating (1) and (2), we see that

$$120\left(x+y+\sqrt{x^2+y^2}\right) = 180\left(\sqrt{x^2+y^2}+y\right) \implies y = \frac{3x}{4}.$$

Substituting this back into (2), we have

$$\frac{1}{2}\left[x\left(\frac{3x}{4}\right)\right] = 180\left[\sqrt{x^2+\left(\frac{3x}{4}\right)^2}+\frac{3x}{4}\right] \implies x = 960.$$

Thus, $y = 720$, whence the perimeter of $ABCD = 2(x+y) = 3360$ mm.

### 10.1.2. Round 2 Solutions

**Resources:** Review by *Way Tan*, AoPS threads

**Question 1** [Ans: $\sqrt{2}\sqrt{\sqrt{2}-1}$]

Let $ABC$ be an isosceles right-angled triangle of area 1. Find the length of the shortest segment that divides the triangle into two parts of equal area.

**Solution.** Let $A(0,\sqrt{2})$, $B(0,0)$ and $C(\sqrt{2},0)$. Let $DE$ be the line segment that divides $\triangle ABC$ into two regions of area $\frac{1}{2}$. Without loss of generality, suppose $D \in BC$. Let $D(t,0)$, where $t \in [0,\sqrt{2}]$.

*Case 1*. Suppose $E \in AB$. Then $E(0,h)$, with $h \in [0,\sqrt{2}]$. Observe that one of the regions that $DE$ splits $ABC$ into is the right-angled triangle $\triangle EBD$. We thus have

$$[EBD] = \frac{1}{2} \implies \frac{th}{2} = \frac{1}{2} \implies h = \frac{1}{t}.$$

Let $s = |DE|$. By the distance formula, we have $s = \sqrt{t^2 + t^{-2}}$. We now consider the stationary points of $s$:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0 \implies \frac{2t - 2t^{-3}}{2\sqrt{t^2+t^{-2}}} = 0 \implies t = 1.$$

It is not too hard to show that $s$ attains a minimum when $t = 1$. We thus have

$$\min |DE| = \sqrt{1^2 + 1^{-2}} = \sqrt{2}. \tag{1}$$

*Case 2*. Suppose $E \in AC$. Note that the equation of the line $AC$ is $y = \sqrt{2} - x$. Thus, $E(h, \sqrt{2}-h)$, where $h \in [0,\sqrt{2}]$. Observe that one of the regions that $DE$ splits $ABC$ into is the triangle $\triangle EDC$. We thus have

$$[EDC] = \frac{1}{2} \implies \frac{1}{2}\left(\sqrt{2}-t\right)\left(\sqrt{2}-h\right) = \frac{1}{2} \implies h = \sqrt{2} - \frac{1}{\sqrt{2}-t}.$$

Let $s = |DE|$. By the distance formula, we have

$$s = \sqrt{\left(\sqrt{2} - t - \frac{1}{\sqrt{2}-t}\right)^2 + \left(\frac{1}{\sqrt{2}-t}\right)^2} = \sqrt{(u-u^{-1})^2 + u^{-2}} = \sqrt{u^2 + 2u^{-2} - 2},$$

where $u = \sqrt{2} - t$. We now consider the stationary points of $s$:

$$\frac{\mathrm{d}s}{\mathrm{d}u} = 0 \implies \frac{2u - 4u^{-3}}{2\sqrt{u^2+2u^{-2}-2}} = 0 \implies u = 2^{1/4},$$

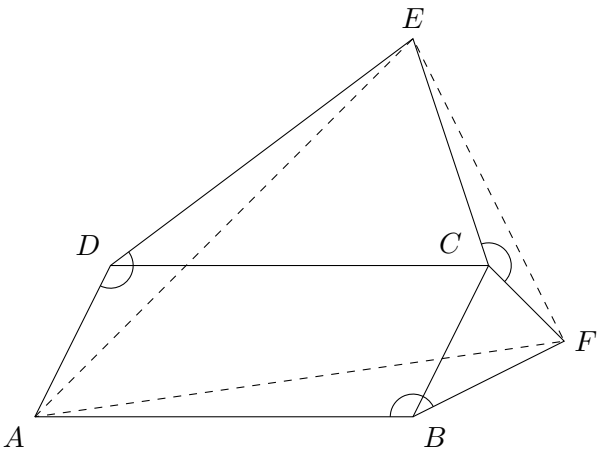
whence $t = \sqrt{2} - 2^{1/4} \in (0,\sqrt{2})$. It is not too hard to show that $s$ attains a minimum when $u = 2^{1/4}$. We thus have

$$\min |DE| = \sqrt{\left(2^{1/4}\right)^2 + 2\left(2^{1/4}\right)^{-2} - 2} = \sqrt{2}\sqrt{\sqrt{2}-1}. \tag{2}$$

Comparing (1) and (2), we see that $\sqrt{2}\sqrt{\sqrt{2}-1} < \sqrt{2}$. Thus, the shortest length of $DE$ is $\sqrt{2}\sqrt{\sqrt{2}-1}$.

### Question 2

Let $ABCD$ be a parallelogram and points $E$, $F$ be on its exterior. If triangles $BCF$ and $DEC$ are similar, i.e. $\triangle BCF \sim \triangle DEC$, prove that triangle $AEF$ is similar to these two triangles.




**Solution.** Let $\angle CBF = \alpha$ and $\angle CFB = \beta$. Since $\triangle BCF$ and $\triangle DEC$ are similar, we have $\angle EDC = \alpha$ and $\angle ECD = \beta$. Let $\angle ADC = \angle ABC = \gamma$. Note that $\angle ADE = \angle ABF = \alpha + \gamma$. Furthermore, observe that

$$\angle ECF = 360^\circ - \angle ECD - \angle DCB - \angle BCF = 360^\circ - (\beta) - (180^\circ - \gamma) - (180^\circ - \alpha - \beta) = \alpha + \gamma.$$

We thus have

$$\angle ADE = \angle ABF = \angle ECF = \alpha + \gamma.$$

Let $k$ be the ratio of similarity between $\triangle BCF$ and $\triangle DEC$. Then $EC = k \cdot CF$, $ED = k \cdot BC$ and $DC = k \cdot BF$. By the cosine rule, we can write the side lengths of $\triangle AEF$ in terms of $CF$, $BC$ and $BF$.

$$\begin{aligned} EF^2 &= CF^2 + k^2 CF^2 - 2k \cdot CF^2 \cos(\alpha + \gamma) \\ AE^2 &= BC^2 + k^2 BC^2 - 2k \cdot BC^2 \cos(\alpha + \gamma) \\ AF^2 &= BF^2 + k^2 BF^2 - 2k \cdot BF^2 \cos(\alpha + \gamma) \end{aligned}$$

This immediately gives

$$\frac{EF}{CF} = \frac{AE}{BC} = \frac{AF}{BF} = \sqrt{1 + k^2 - 2k \cos(\alpha + \gamma)},$$

which is a constant. Thus, $\triangle AEF$ is similar to $\triangle BCF$. □

### Question 3

Seven triangles of area 7 lie in a square of area 27. Prove that among the 7 triangles there are 2 that intersect in a region of area not less than 1.

**Solution 1.** Observe that the total area covered by the triangles ($7 \cdot 7 = 49$ units$^2$) is greater than the area of the square. Hence, by the pigeonhole principle, there is a minimum overlap area of $49 - 27 = 22$ units$^2$ (accounting for multiple overlaps). However, since there are only $\binom{7}{2} = 21$ possible pairs of triangles, it again follows from the pigeonhole principle that there is one pair that overlap in an area of at least 1 unit$^2$. □

**Question 4** [Ans: 3]

Suppose for some positive integer $n$, the numbers $2^n$ and $5^n$ have equal first digit. What are the possible values of this first digit?

**Solution 1.** Let the first digit of $2^n$ be $d$. Then $2^n = k \cdot 10^a$, where $k \in [d, d+1)$ and $a$ is some non-negative integer. Since $5^n = \frac{10^n}{2^n} = \frac{10}{k}10^{n-a-1}$, we see that the first digit of $5^n$ is $\left\lfloor \frac{10}{k} \right\rfloor$. Going through all possible $d = 1, \ldots, 9$, we see that $d = \left\lfloor \frac{10}{k} \right\rfloor$ only when $d = 3$ and $k \in [3, 10/3)$. Thus, the only possible value for this first digit is 3.

**Solution 2** (Zhero)**.** Let $d$ be the first digit of $2^n$ and $5^n$. Then

$$d \cdot 10^{m_1} \le 2^n < (d+1) \cdot 10^{m_1}, \qquad d \cdot 10^{m_2} \le 5^n < (d+1) \cdot 10^{m_2},$$

where $m_1$ and $m_2$ are positive integers. Multiplying the two inequalities yields

$$d^2 \cdot 10^{m_1+m_2} \le 10^n < (d+1)^2 \cdot 10^{m_1+m_2} \implies d^2 \le 10^{n-m_1-m_2} < (d+1)^2.$$

For $d = 1, \ldots, 9$, the only $d$ that has a perfect power of 10 between $d^2$ and $(d+1)^2$ is 3. Thus, the first digit can only be 3.

**Question 5** [Ans: $(-17, -130), (-17, 128), (7, -130), (7, 128)$]

Find all integer solutions of the equation

$$y^2 + 2y = x^4 + 20x^3 + 104x^2 + 40x + 2003.$$

*Remark.* This question is identical to 2024/Senior/R2/Q2.

**Solution.** Completing the square, we have

$$(y+1)^2 - (x^2 + 10x + 2)^2 = 2000,$$

which factors as

$$(y + x^2 + 10x + 3)(y - x^2 - 10x - 1) = 2000.$$

Let $A = y + x^2 + 10x + 3$ and $B = y - x^2 - 10x - 1$. We clearly have

$$\frac{A+B}{2} = y + 1 \implies y = \frac{A+B}{2} - 1$$

and

$$\frac{A-B}{2} = x^2 + 10x + 2 = (x+5)^2 - 23 \implies (x+5)^2 = \frac{A-B}{2} + 23. \tag{1}$$

Going through all possible factor pairs of 2000, we see that only

$$(a, b) \in \{(-8, -250), (250, 8)\}$$

gives a perfect square as in (1). Thus,

$$(x, y) \in \{(-17, -130), (-17, 128), (7, -130), (7, 128)\}.$$

## 10.2. Senior Section

### 10.2.1. Round 1 Solutions

**Resources:** Review by *Way Tan*

**Question 1** [Ans: D]

Find the largest positive integer $A$ such that $2^x + \dfrac{2025}{2^x} - A > 0$ for all real numbers $x$

(A) 59 (B) 69 (C) 79 (D) 89 (E) 99

**Solution.** Multiplying through by $2^x$ (which does not affect the inequality since $2^x > 0$), we get a quadratic in $2^x$:

$$(2^x)^2 - A \cdot 2^x + 2025 > 0.$$

The discriminant of this quadratic must be less than 0, hence

$$A^2 - 4 \cdot 2025 < 0.$$

Solving, we get $A < 50$, whence $\max A = 49$.

**Question 2** [Ans: B]

If $x = \dfrac{1}{\log_{\frac{2024}{2023}} 7} + \dfrac{1}{\log_{\frac{2023}{2022}} 7} + \dfrac{1}{\log_{\frac{2022}{2021}} 7}$, find $7^x$.

(A) $\dfrac{2021}{2024}$ (B) $\dfrac{2024}{2021}$ (C) $\dfrac{2022}{2024}$ (D) $\dfrac{2024}{2022}$ (E) 2024

**Solution.** Recall the change of base formula for logarithms: $\log_b a = \frac{\ln a}{\ln b}$. We hence get

$$x = \frac{\ln \frac{2024}{2023}}{\ln 7} + \frac{\ln \frac{2023}{2022}}{\ln 7} + \frac{\ln \frac{2022}{2021}}{\ln 7} = \frac{\ln \frac{2024}{2021}}{\ln 7} = \log_7 \frac{2024}{2021},$$

where we used the property $\log a + \log b = \log ab$ in the intermediate step. Thus, $7^x = \frac{2024}{2021}$.

**Question 3** [Ans: D]

Simplify $\dfrac{2024}{\sqrt{4+\sqrt{12}}} + \dfrac{2024}{\sqrt{4-\sqrt{12}}}$.

(A) 1012 (B) $1012\sqrt{3}$ (C) 2024 (D) $2024\sqrt{3}$
(E) $1012 + 1012\sqrt{3}$

**Solution.** Observe that $4 \pm \sqrt{12} = \sqrt{3}^2 \pm 2\sqrt{3} + 1 = \left(\sqrt{3} \pm 1\right)^2$. Hence, we see that

$$\frac{2024}{\sqrt{4+\sqrt{12}}} + \frac{2024}{\sqrt{4-\sqrt{12}}} = 2024\left(\frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{3}-1}\right) = 2024\sqrt{3}.$$

**Question 4** [Ans: A]

Suppose $x^{1/3}+12=y^{1/3}$ for some real numbers $x$ and $y$. Find the minimum possible value of $y-x$.

(A) 432 (B) 532 (C) 632 (D) 732

(E) None of the above

**Solution 1.** By the difference of cubes identity, one has

$$y-x=\left(y^{1/3}-x^{1/3}\right)\left(y^{2/3}+y^{1/3}x^{1/3}+x^{2/3}\right)=12\left(y^{2/3}+y^{1/3}x^{1/3}+x^{2/3}\right).$$

Substituting $y^{1/3}=x^{1/3}+12$, we see that

$$y^{2/3}+y^{1/3}x^{1/3}+x^{2/3}=3x^{2/3}+36x^{1/3}+144=3\left(x^{1/3}+6\right)^2+36.$$

Hence, $y-x\geq 12\cdot 36=432$, with equality when $x^{1/3}=-6$.

**Solution 2** (Abusing symmetry)**.** Let $x_*=-x$. We wish to find the minimum value of $x_*+y$ subject to $x_*^{1/3}+y^{1/3}=12$. By symmetry, this occurs when $x_*^{1/3}=y^{1/3}=6$, thus $\min x_*+y=6^3+6^3=432$.

**Question 5** [Ans: D]

Find the largest possible value of $\dfrac{\sqrt{2}\cos(2x)}{\sin(x)+\cos(x)}$.

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) $\frac{5}{2}$

**Solution.** Note that $\cos(2x)=\cos^2 x-\sin^2 x=(\cos x+\sin x)(\cos x-\sin x)$. Hence,

$$\frac{\sqrt{2}\cos(2x)}{\sin(x)+\cos(x)}=\sqrt{2}(\cos x-\sin x)=\sqrt{2}\cdot\sqrt{2}\cos\left(x-\frac{\pi}{4}\right),$$

where we used the R-formula in the last step. Thus, the largest possible value is clearly 2.

**Question 6** [Ans: 64]

If $\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}=4$, find the value of $15x$.

**Solution.** Squaring and simplifying, we get $\sqrt{x^2-x}=8-x$. Squaring again, we get $x^2-x=64-16x+x^2$, whence $15x=64$.

**Question 7** [Ans: 242]

Find the smallest positive integer $K$ such that

$$x^2 - 200x + y^2 = 0 \text{ and } x + y \leq K.$$

**Solution.** Completing the square on the first equation, we clearly get the equation of a circle with centre $(100, 0)$ and radius 100:

$$(x - 100)^2 + y^2 = 100^2.$$

From the second equation, we have $y \leq K - x$. Since we are interested in the extreme values of $K$, we consider $y = K - x$. This is the equation of a line with gradient $-1$ and $y$-intercept $K$. Additionally, this line is tangent to the aforementioned circle, implying that the "argument" of the point of tangency to the centre of the circle is $45°$. Since $K > 0$, the point of tangency must be $(100 + 100\cos(45°), 100\cos(45°)) = (100 + 50\sqrt{2}, 50\sqrt{2})$. Using the point slope formula, we have

$$y - 50\sqrt{2} = -x + 100 + 50\sqrt{2},$$

whence the $y$-intercept is $100 + 100\sqrt{2} \approx 241.4$. Since $K$ is an integer, $\min K = 242$.

**Question 8** [Ans: 506]

Given that $\dfrac{\cos(x)}{\sin(3x)} - \dfrac{\sin(x)}{\cos(3x)} - 2 \cdot \dfrac{\sin(4x)}{\cos(6x)} = 2024$, find the value of $\dfrac{\cos(10x)}{\sin(12x)}$.

**Solution.** Observe that

$$\frac{\cos x}{\sin 3x} - \frac{\sin x}{\cos 3x} = \frac{\cos x \cos 3x - \sin x \sin 3x}{\sin 3x \cos 3x} = \frac{2\cos 4x}{\sin 6x}.$$

Plugging this into the given equation yields

$$\frac{\cos 4x}{\sin 6x} - \frac{\sin 4x}{\cos 6x} = 1012.$$

Playing the same game, we see that

$$\frac{\cos 4x}{\sin 6x} - \frac{\sin 4x}{\cos 6x} = \frac{2\cos 10x}{\sin 12x}.$$

Thus,

$$\frac{\cos 10x}{\sin 12x} = \frac{1}{2} \cdot 1012 = 506.$$

**Question 9** [Ans: 3]

Find the smallest positive integer $k$ such that the coefficient of $x^k$ in the expansion of $\left(5x^3 + \frac{1}{\sqrt{x}}\right)^{2024}$ is **not** zero.

**Solution.** By the binomial theorem, we have

$$\left(5x^3 + \frac{1}{\sqrt{x}}\right)^{2024} = \sum_{n=1}^{2024} \binom{2024}{n} \left(5x^3\right)^n \left(x^{-\frac{1}{2}}\right)^{2024-n} = \sum_{n=1}^{2024} \binom{2024}{n} 5^n x^{\frac{7}{2}n - 1012}.$$

Let $k = \frac{7}{2}n - 1012$. Since $k > 0$, we have $n \geq 290$. Since $\binom{2024}{290}5^{290}$ is clearly non-zero, we have $\min k = \frac{7}{2} \cdot 290 - 1012 = 3$.

**Question 10** [Ans: 2048]

Let

$$P = \left(2024^2+1\right)\left(2024^{2^2}+1\right)\left(2024^{2^3}+1\right)\cdots\left(2024^{2^{10}}+1\right)\times 2025 + \frac{1}{2023}.$$

Find the smallest positive integer $N$ such that $N > \log_{2024} P$.

**Solution.** Observe that

$$P = \frac{1}{2023} + \prod_{k=1}^{10}\left(2024^{2^k}+1\right).$$

Expanding out the product, it is easy to see that

$$\prod_{k=1}^{10}\left(2024^{2^k}+1\right) = \sum_{k=0}^{2^{11}-1} 2024^k,$$

which evaluates to $\frac{2024^{2^{11}}-1}{2024-1}$. Hence,

$$P = \frac{2024^{2^{11}}-1}{2023} + \frac{1}{2023} = \frac{2024^{2^{11}}}{2023} \implies \log_{2024} P = 2^{11} - \log_{2024} 2023.$$

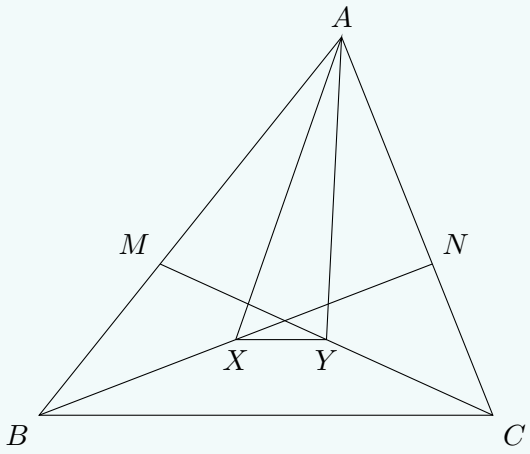
Clearly $\log_{2024} 2023 < 1$. Hence, $\min N = 2^{11} = 2048$.

**Question 11** [Ans: 145]

Let $\triangle ABC$ be a triangle with area 1000. Let $M$ and $N$ be points on $AB$ and $AC$ respectively such that

$$AM : MB = 3 : 2 \text{ and } AN : NC = 7 : 3.$$

Let $X$ and $Y$ be the midpoints of $BN$ and $CM$ respectively. Find the area of $\triangle AXY$.




**Solution.** Observe that $\overrightarrow{AM} = \frac{3}{5}\overrightarrow{AB}$ and $\overrightarrow{AN} = \frac{7}{10}\overrightarrow{AC}$. By the midpoint theorem, it follows that

$$\overrightarrow{AX} = \frac{1}{2}\left(\frac{3}{5}\overrightarrow{AB} + \overrightarrow{AC}\right) \text{ and } \overrightarrow{AY} = \frac{1}{2}\left(\frac{7}{10}\overrightarrow{AC} + \overrightarrow{AB}\right).$$

Thus,

$$[AXY] = \frac{1}{2}\left|\overrightarrow{AX}\times\overrightarrow{AY}\right| = \frac{1}{8}\left|\left(\frac{3}{5}\overrightarrow{AB} + \overrightarrow{AC}\right)\times\left(\frac{7}{10}\overrightarrow{AC} + \overrightarrow{AB}\right)\right| = \frac{29}{200}\cdot\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|.$$

Since $[ABC] = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$, we see that $[AXY] = \frac{29}{200} \cdot 1000 = 145$.

**Question 12** [Ans: 8099]

Find the largest positive integer $n \leq 10000$ such that $1 + 2024n^2$ is a perfect square.

**Solution.** Let $m$ be an integer such that $1 + 2024n^2 = m^2$. Then $m^2 - 2024n^2 = 1$. We thus have a Pell equation where $D = 2024$. The fundamental solution is clearly $m = 45$ and $n = 1$. The solutions $m_i$ and $n_i$ to the Pell equation are hence given by $m_i + \sqrt{2024}n_i = (45 + \sqrt{2024})^i$. When $i = 2$, we have

$$(45 + \sqrt{2024})^2 = 4049 + 90\sqrt{2024} \implies n_2 = 90.$$

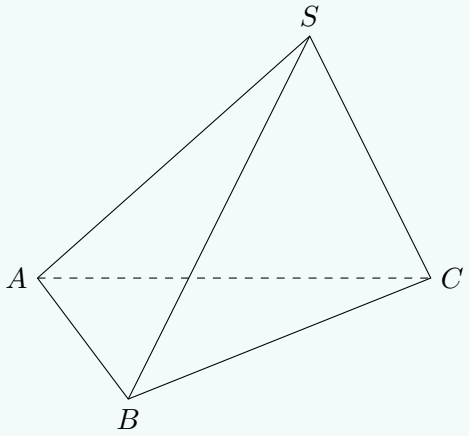
When $i = 3$, we have

$$(45 + \sqrt{2024})^3 = 364365 + 8099\sqrt{2024} \implies n_3 = 8099.$$

Taking $i \geq 4$ will very clearly give us an $n$ greater than 10000. Thus, $\max n = 8099$.

**Question 13** [Ans: 128]

In a tetrahedron $SABC$, the faces $SBC$ and $ABC$ are perpendicular to each other. The angles $\angle ASB$, $\angle BSC$, $\angle ASC$ are all $60°$, and $SB = SC = 4$. Find the square of the volume of the tetrahedron.




**Solution.** Without loss of generality, let $SBC$ lie on the $x - z$ plane, and let $ABC$ lie on the $x - y$ plane. Since $\angle BSC = 60°$ and $SB = SC = 4$, it follows that $\triangle SBC$ is equilateral. Let $B(-2, 0, 0)$ and $C(2, 0, 0)$. Then $S(0, 0, 2\sqrt{3})$. By symmetry, $A$ must lie on the $y$ axis. Hence, $A(0, a, 0)$ for some $a$ to be determined. Since $\angle ASB = 60°$, we have

$$\overrightarrow{AS} \cdot \overrightarrow{BS} = AS \cdot BS \cdot \cos 60° \implies AS = 6.$$

Let $O$ be the origin, which is also the midpoint of $BC$. Since $OA \perp BC$, by Pythagoras' theorem, we have

$$OS^2 + OA^2 = AS^2 \implies a^2 = AS^2 - OS^2 = 24.$$

Hence, $[ABC] = \frac{1}{2}BC \cdot OA = 4\sqrt{6}$. Thus, the volume of the tetrahedron is

$$[SABC] = \frac{1}{3} \cdot [ABC] \cdot OS = 8\sqrt{2} \implies [SABC]^2 = 128.$$

**Question 14** [Ans: 965]

Let $a$, $b$, $c$ be the three real roots of the cubic equation

$$2x^3 - 4x^2 - 21x - 8 = 0.$$

Given that

$$S = \frac{1}{ab + c - 1} + \frac{1}{bc + a - 1} + \frac{1}{ca + b - 1}$$

is a rational number that can be expressed as a fraction in the lowest form $\frac{m}{n}$, find the value of $m^2 + n^2$.

**Solution.** By Vieta's formula, we have $abc = 4$ and $a + b + c = 2$. Hence,

$$S = \frac{a}{a^2 - a + 4} + \frac{b}{b^2 - b + 4} + \frac{c}{c^2 - c + 4}.$$

Now observe that

$$2x^3 - 4x^2 - 21x - 8 = (x^2 - x + 4)(2x - 2) - 31x.$$

Since $a$, $b$, $c$ are roots of the cubic on the LHS, we get

$$(a^2 - a + 4)(2a - 2) - 31a = 0 \implies \frac{a}{a^2 - a + 4} = \frac{2a - 2}{31},$$

with identical expressions for $b$ and $c$. Substituting these expressions back into $S$, we have

$$S = \frac{2a - 2}{31} + \frac{2b - 2}{31} + \frac{2c - 2}{31} = \frac{2(a + b + c) - 6}{31} = \frac{-2}{31}.$$

Thus, $m^2 + n^2 = (-2)^2 + 31^2 = 965$.

**Question 15** [Ans: 25]

Consider the equation

$$\frac{\sqrt{3} - 1}{\sin x} + \frac{\sqrt{3} + 1}{\cos x} = 4\sqrt{2}.$$

For the range $0 < x < \pi/2$, the sum of the solutions of the equation can be expressed in the form $\frac{m\pi}{n}$, where $\frac{m}{n}$ is a fraction in the lowest form. Find $m + n$.

**Solution.** Observe that

$$\frac{\sqrt{3} - 1}{2} = \cos\frac{\pi}{6} - \cos\frac{\pi}{3} = 2\sin\frac{\pi}{4}\sin\frac{\pi}{12} = \sqrt{2}\sin\frac{\pi}{12}.$$

Similarly,

$$\frac{\sqrt{3} + 1}{2} = \cos\frac{\pi}{6} + \cos\frac{\pi}{3} = 2\sin\frac{\pi}{4}\cos\frac{\pi}{12} = \sqrt{2}\cos\frac{\pi}{12}.$$

Our equation hence simplifies to

$$\frac{\sin(\pi/12)}{\sin x} + \frac{\cos(\pi/12)}{\cos x} = 2 \implies \sin\frac{\pi}{12}\cos x + \cos x \sin\frac{\pi}{12} = 2\sin x\cos x,$$

which very clearly gives us

$$\sin\left(x + \frac{\pi}{12}\right) = \sin(2x).$$

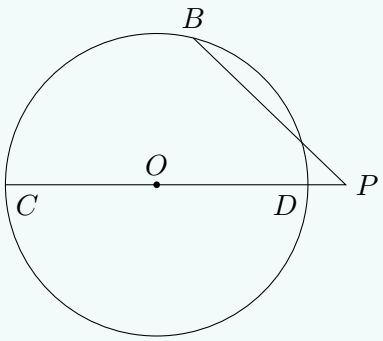
*Case 1*. Suppose $2x = x + \frac{\pi}{12} + 2n\pi$, where $n$ is an integer. Then $x = \frac{24n+1}{12}\pi$, whence the only solution in the given range is $x = \frac{\pi}{12}$.

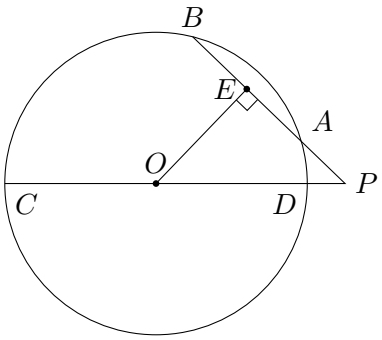
*Case 2*. Suppose $2x = (2n + 1)\pi - \left(x + \frac{\pi}{12}\right)$, where $n$ is an integer. Then $x = \frac{24n+11}{36}\pi$, whence the only solution in the given range is $x = \frac{11\pi}{36}$.

Thus, the sum of solutions is $\frac{\pi}{12} + \frac{11\pi}{36} = \frac{7\pi}{18}$, whence $m + n = 25$.

**Question 16** [Ans: 40]

An engineer constructs a circle with centre $O$ and diameter $CD$ on level ground, and builds a vertical tower of height 20 at the centre. $B$ is another point on the circumference and $P$ is on $CD$ produced such that $PB$ is a secant line of the circle. Given that $PB = 33$, $PC = 77$ and $CD = 74$, find the minimum possible distance of any point on $PB$ to the top of the tower.

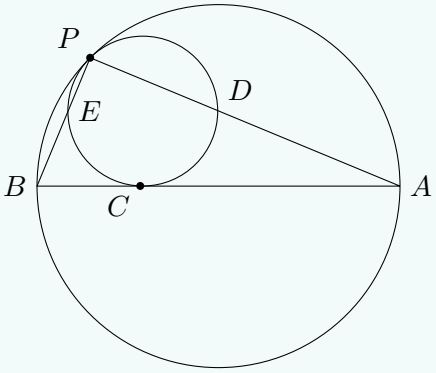


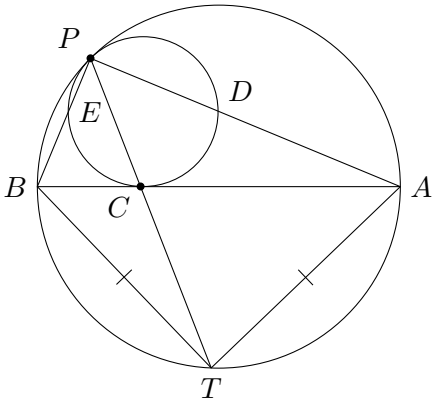




**Solution.** Let $A$ be the intersection between $PB$ and the circle. Note that $PD = PC - CD = 3$. By power of a point, we have $PD \cdot PC = PA \cdot PB$, which immediately gives $PA = 7$. Let $E$ be the midpoint of $AB$. Clearly the minimum distance is achieved at $E$, since $OE \perp PB$. Since $AB = PB - PA = 26$, we have $PE = PA + \frac{1}{2}AB = 20$. By Pythagoras's theorem, $OE^2 = OP^2 - PE^2 = 1200$. Again, by Pythagoras' theorem, it follows that the distance between $E$ and the top of the tower is $\sqrt{OE^2 + 20^2} = 40$.

**Question 17** [Ans: 9]

$P$ is a common point of tangency of two circles. $BA$ is a chord of the larger circle which is tangent to the smaller circle at a point $C$. $PB$ and $PA$ intersect the smaller circle at points $E$ and $D$ respectively. If $BA = 15$, $PE = 2$, and $PD = 3$, find the length $CA$.




**Solution.** Consider the homothety $H$ at $P$ that maps the small circle to the big circle. Since $H(E) = B$ and $H(D) = A$, it follows that $\triangle PED$ and $\triangle PBA$ are similar. Let $H(C) = T$. Then $T$ is equidistant to $A$ and $B$, whence $PT$ bisects $\angle P$. Thus, by the angle bisector theorem, it follows that

$$\frac{CA}{CB} = \frac{PA}{PB} = \frac{PD}{PE} = \frac{2}{3}.$$

Since $CA + CB = 15$, we immediately have $CA = 9$.

*Remark.* This is an application of the Shooting Lemma.

**Question 18** [Ans: 160]

On each face of a cube, an integer greater than 2 is written. Each vertex of the cube is the intersection of three unique faces, and each edge is the intersection of two unique faces. Assign to each vertex the product of the numbers written on the faces intersecting the vertex, and assign to each edge the product of the numbers written on the faces intersecting the edge. The sum of the numbers assigned to the eight vertices is equal to 2024. Find the maximum possible value of an edge.

**Solution.** Without loss of generality, let $f$, $b$, $l$, $r$, $u$, $d$ represent the numbers on the front, back, left, right, up, down faces of the cube respectively. We see that $(f+b)(l+r)(u+d) = 2024 = 2^3 \cdot 11 \cdot 23$. Since each face has an integer greater than 2, it follows that

$$f + b = 8, \qquad l + r = 11, \qquad u + d = 23.$$

Since we want to maximize only one edge, we take the edge joining $l$ and $u$, which is assigned

$$(11 - \min r)(23 - \min d) = (11 - 3)(23 - 3) = 8 \cdot 20 = 160.$$

**Question 19** [Ans: 72]

Find the sum of the squares of each of the roots of the equation

$$x^2 - 4\lfloor x \rfloor - 12 = 0,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

**Solution.** Let $x = \lfloor x \rfloor + \{x\}$, where $\{x\}$ denotes the fractional part of $x$. This turns the given equation into

$$\lfloor x \rfloor^2 - \lfloor x \rfloor (4 - 2\{x\}) + \left(\{x\}^2 - 12\right) = 0.$$

By the quadratic formula, we obtain

$$\lfloor x \rfloor = 2 - \{x\} \pm 2\sqrt{4 - \{x\}}.$$

Since $\{x\} \in [0, 1)$ and $\lfloor x \rfloor \in \mathbb{Z}$, we see that $\lfloor x \rfloor = -2, 5, 6$.

*Case 1.* $\lfloor x \rfloor = -2$ The given equation reduces to $x^2 = 4$. Thus, $x = -2$ is a solution.

*Case 2.* $\lfloor x \rfloor = 5$ The given equation reduces to $x^2 = 32$. Thus, $x = \sqrt{32}$ is a solution.

*Case 3.* $\lfloor x \rfloor = 6$ The given equation reduces to $x^2 = 36$. Thus, $x = 6$ is a solution.

Thus, the sum of the squares of the roots is $(-2)^2 + \sqrt{32}^2 + 6^2 = 72$.

**Question 20** [Ans: 609]

Calculate the remainder when $1901^{2024}$ is divided by 1216.

**Solution.** Notice that $1216 = 2^6 \cdot 19$. We can hence use the Chinese remainder theorem to tackle this problem. We obviously have $1901^{2024} \equiv 1 \pmod{19}$. We now calculate $1901^{2024} \pmod{64}$. Note that $1901 \equiv 45 \pmod{64}$. Additionally, since $\varphi(64) = 32$ and $2024 \equiv 8 \pmod{32}$, by Euler's theorem we immediately get

$$1901^{2024} \equiv 45^8 \equiv 19^8 \equiv 561^4 \equiv 23^4 \equiv 529^2 \equiv 17^2 \equiv 289 \equiv 33 \pmod{64}.$$

Let $n$ be the desired remainder. We hence have the following system of congruences:

$$n \equiv 1 \pmod{19}, \qquad n \equiv 33 \pmod{64}.$$

From the second congruence, we know $n = 33 + 64m$ for some integer $m$. Substituting this into the first congruence gives

$$33 + 64m \equiv 1 \pmod{19} \implies 7m = 6 \pmod{19}.$$

Note that the multiplicative inverse of 7 modulo 19 is 11 (since $11 \cdot 7 \equiv 1 \pmod{19}$). Hence, $m \equiv 6 \cdot 11 \equiv 9 \pmod{19}$. We thus get $m = 9 + 19k$ for some integer $k$. Substituting this back into the definition of $n$, we get

$$n = 33 + 64(9 + 19k) = 609 + 64 \cdot 19k.$$

Hence, the desired remainder is 609.

**Question 21** [Ans: 521]

Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial with non-negative integer coefficients satisfying $0 \le a_i \le 17$ for all $i$. If $P(18) = 367616$, find the value of $P(3)$.

**Solution.** Observe that the coefficients correspond to the digits of 367616 in base 18. Indeed, we see that

$$367616 = 2 \cdot 18^0 + 11 \cdot 18^1 + 9 \cdot 18^3 + 3 \cdot 18^4.$$

Hence,

$$P(x) = 2 + 11x + 9x^3 + 3x^4,$$

whence $P(3) = 521$.

**Question 22** [Ans: 129]

Evaluate the sum

$$\frac{2}{1 + \tan\left(\frac{\pi}{260}\right)} + \frac{2}{1 + \tan\left(\frac{2\pi}{260}\right)} + \frac{2}{1 + \tan\left(\frac{3\pi}{260}\right)} + \cdots + \frac{2}{1 + \tan\left(\frac{129\pi}{260}\right)}.$$

**Solution.** Observe that we can pair each term with a corresponding term at the opposite end (except for the middle term):

$$\sum_{k=1}^{129} \frac{2}{1 + \tan\left(\frac{k\pi}{260}\right)} = 2\sum_{k=1}^{64} \left[\frac{1}{1 + \tan\left(\frac{k\pi}{260}\right)} + \frac{1}{1 + \tan\left(\frac{\pi}{2} - \frac{k\pi}{260}\right)}\right] + \frac{2}{1 + \tan\left(\frac{65\pi}{260}\right)}.$$

Since $\tan\left(\frac{\pi}{2} - x\right) = \cot x$, we see that

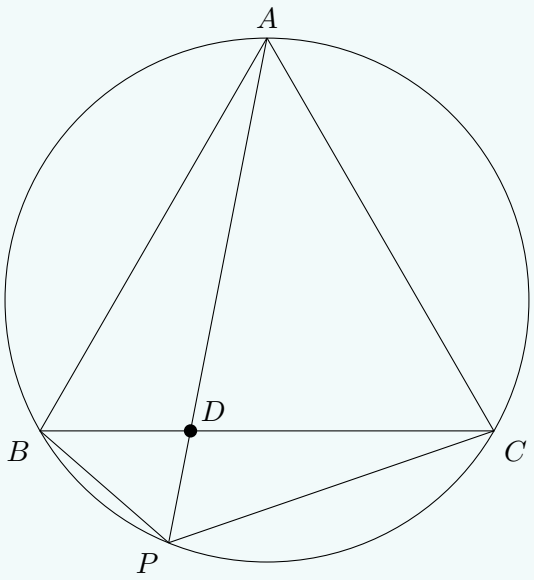
$$\sum_{k=1}^{64} \left[\frac{1}{1 + \tan\left(\frac{k\pi}{260}\right)} + \frac{1}{1 + \cot\left(\frac{k\pi}{260}\right)}\right] = \sum_{k=1}^{64} \frac{1 + \cot\left(\frac{k\pi}{260}\right) + \tan\left(\frac{k\pi}{260}\right) + 1}{1 + \cot\left(\frac{k\pi}{260}\right) + \tan\left(\frac{k\pi}{260}\right) + 1} = 64.$$

Thus,

$$\sum_{k=1}^{129} \frac{2}{1 + \tan\left(\frac{k\pi}{260}\right)} = 2 \cdot 64 + \frac{2}{1 + \tan\left(\frac{65\pi}{260}\right)} = 129.$$

**Question 23** [Ans: 21]

An equilateral triangle $ABC$ is inscribed in a circle and $P$ is a point on the minor arc $BC$. Point $D$ is the intersection of $AP$ and $BC$.




Suppose that $BP = 5$, $CP = 20$. Find the length of $AD$.

**Solution.** Let $s$ be the side length of the equilateral triangle. By Ptolemy's theorem, we know that

$$BP \cdot AC + BA \cdot CP = BC \cdot AP \implies AD + DP = 25.$$

Notice that $\triangle BDP$ is similar to $\triangle ADC$. Hence,

$$\frac{DP}{BP} = \frac{CD}{CA} \implies CD = \frac{s \cdot DP}{5}, \qquad \frac{BD}{BP} = \frac{AD}{AC} \implies BD = \frac{5AD}{s}.$$

Likewise, $\triangle ADB$ is similar to $\triangle CDP$, whence

$$\frac{CD}{CP} = \frac{AD}{AB} \implies CD = \frac{20AD}{s}.$$

We thus see that

$$s = BD + DC = \frac{25AD}{s} \implies s^2 = 25AD,$$

and

$$CD = \frac{s \cdot DP}{5} = \frac{20AD}{s} \implies DP = \frac{100AD}{s^2} = \frac{100AD}{25AD} = 4.$$

Thus, $AD = 25 - DP = 21$.

**Question 24** [Ans: 94]

Find the number of positive integers $x < 9000$ such that $x^3 + 95$ is divisible by 96.

**Solution.** We are given that

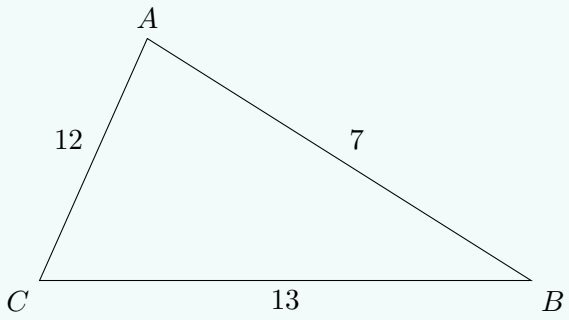
$$x^3 + 95 \equiv 0 \pmod{96} \implies x^3 - 1 \equiv 0 \pmod{96}.$$

Since $96 = 2^5 \cdot 3$, we have $x^3 - 1 \equiv 0$ modulo 3 and 32. Note that $x^3 - 1 \equiv 0 \pmod 3$ has a unique solution $x \equiv 1 \pmod 3$. Furthermore, by repeatedly applying Hensel's lemma, it

is not too hard to see that $x \equiv 1 \pmod{32}$ is the unique solution to $x^3 - 1 \equiv 0 \pmod{32}$. It thus follows by the Chinese remainder theorem that $x \equiv 1 \pmod{96}$ is the only solution to $x^3 - 1 \equiv 0 \pmod{96}$. That is, we have $x = 1 + 96k$ for integers $k \geq 0$. Since $x < 9000$, we have $k < \frac{8999}{96} \approx 93.7$. Thus, $k \in \{0, 1, \ldots, 93\}$, whence there are 94 possibilities for $x$.

**Question 25** [Ans: 26]

A scalene triangle $\triangle ABC$ has sides $AB = 7$, $AC = 12$ and $BC = 13$. Write $\tan\frac{A-B}{2}\tan\frac{C}{2}$ as a fraction $\frac{m}{n}$ in its lowest form and find $m + n$.




**Solution.** Observe that $\frac{C}{2} = 90^\circ - \frac{A+B}{2}$. Hence,

$$\tan\frac{C}{2} = \tan\left(90^\circ - \frac{A+B}{2}\right) = \cot\frac{A+B}{2}.$$

The desired product is hence

$$\tan\frac{A-B}{2}\tan\frac{C}{2} = \frac{\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{\cos\frac{A-B}{2}\sin\frac{A+B}{2}} = \frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{2\cos\frac{A-B}{2}\sin\frac{A+B}{2}}.$$

By the product-to-sum identities, this immediately simplifies to

$$\frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{2\cos\frac{A-B}{2}\sin\frac{A+B}{2}} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

By the sine rule, we have $\sin A = \frac{a}{b}\sin B$. Thus,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\frac{a}{b} - 1}{\frac{a}{b} + 1} = \frac{a-b}{a+b} = \frac{13-12}{13+12} = \frac{1}{25}.$$

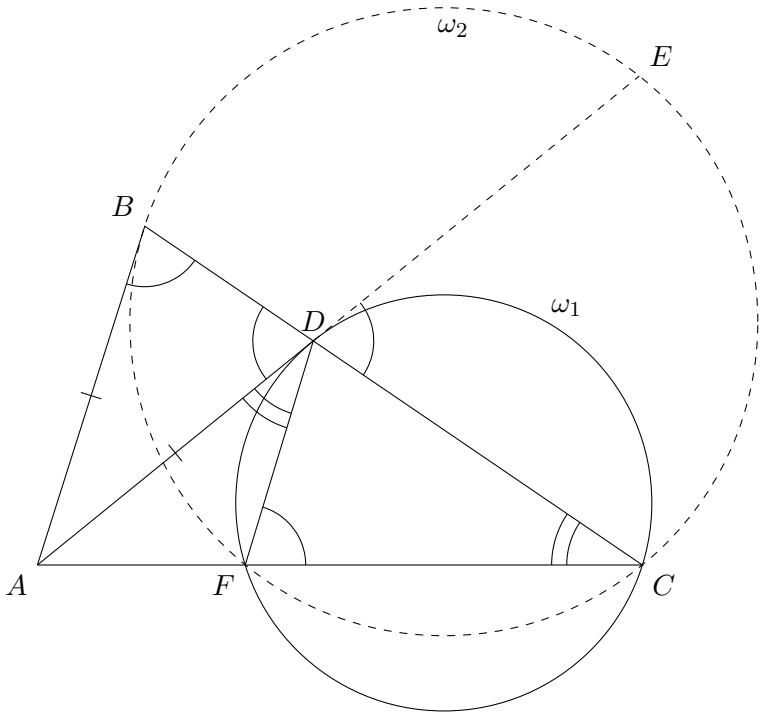
Hence, $m + n = 26$.

### 10.2.2. Round 2 Solutions

**Resources:** Review by *Way Tan*, AoPS threads

**Question 1**

In an acute triangle $ABC$, $AC > AB$, $D$ is the point on $BC$ such that $AD = AB$. Let $\omega_1$ be the circle through $C$ tangent to $AD$ at $D$, and $\omega_2$ the circle through $C$ tangent to $AB$ at $B$. Let $F$ ($\neq C$) be the second intersection of $\omega_1$ and $\omega_2$. Prove that $F$ lies on $AC$.




**Solution 1.** Let $\alpha = \angle BAD$. Since $\triangle BAD$ is isosceles, we also have $\angle BDA = \alpha$. Let $E$ be the intersection between $\omega_2$ and $AD$ extended. Then $\angle EDC = \alpha$. By the alternate segment theorem, we get $\angle DFC = \alpha$. Let $\beta = \angle DCF$ by the alternate segment theorem, we also have $\angle ADC = \beta$.

**Claim.** *$BDFA$ is cyclic.*

*Proof.* Observe that $\angle CDF = 180^\circ - \alpha - \beta$. Hence, $\angle CAD = 180^\circ - \angle ADC - \angle DCA = \alpha - \beta$. Since $\angle BAD = 180^\circ - 2\alpha$, we have

$$\angle BAF = (\alpha - \beta) + 180^\circ - 2\alpha = 180^\circ - (\alpha + \beta) = 180^\circ - \angle BDF.$$

□

Since $BDFA$ is cyclic, we have must have $\angle DFA = 180^\circ - \angle DBA = 180^\circ - \alpha$. Thus, $\angle AFD + \angle DFC = 180^\circ$, whence $AFC$ is a straight line.

**Solution 2** (Way Tan)**.** Note that $F$ and $C$ obviously lie on the radical axis of $\omega_1$ and $\omega_2$. Observe that the power of $A$ with respect to $\omega_1$ and $\omega_2$ is $AD^2$ and $AB^2$ respectively. However, because $AD = AB$, the two powers are equal, whence $A$ also lies on the radical axis. Thus, $F$ lies on $AC$. □

**Question 2** [Ans: $(-17,-130),(-17,128),(7,-130),(7,128)$]

Find all integer solutions of the equation

$$y^2+2y=x^4+20x^3+104x^2+40x+2003.$$

*Remark.* This question is identical to 2024/Junior/R2/Q5.

**Question 3** [Ans: 11]

Find the smallest positive integer $n$ for which there exist integers $x_1<x_2<\cdots<x_n$ such that every integer from 1000 to 2000 can be written as a sum of some of the integers from $x_1,x_2,\ldots,x_n$ without repetition.

**Solution** (joeym2011)**.** Observe that there are exactly $2^n$ ways $x_1,\ldots,x_n$ can be summed. Since we want at least 1001 different sums (one for each number between 1000 and 2000), we require $n\geq\lceil\log_2 1001\rceil=10$.

Now observe that we need an $x_i\equiv 1 \pmod 2$, if not we cannot attain any odd integers. Next, we also need an $x_i\equiv 2 \pmod 4$, if not we cannot attain any integer congruent to 3 (mod 4). It follows similarly that we require an $x_i\equiv 2^{k-1} \pmod{2^k}$ for $k=1,\ldots,11$. Since $k\geq 11$, it follows that $n\geq 11$.

It is trivial to see that $n=11$ is possible: simply take $x_i=2^{i-1}$ and look at the binary expansion of each integer between 1000 and 2000 to determine what $x_i$'s to sum. Hence, $\min n=11$.

**Question 4**

Suppose $p$ is a prime number and $x$, $y$, $z$ are integers satisfying $0<x<y<z<p$. If $x^3$, $y^3$, $z^3$ have equal remainders when divided by $p$, prove that $x^2+y^2+z^2$ is divisible by $x+y+z$.

**Solution** (HoshimiyaMukuro)**.** We have $y^3-x^3\equiv z^3-x^3\equiv 0 \pmod p$. By the difference of squares identity, this gives

$$(y-x)(x^2+xy+y^2)\equiv(z-x)(x^2+xz+z^2)\equiv 0 \pmod p.$$

However, because $y-x$ and $z-x$ are less than $p$, we have

$$x^2+xy+y^2\equiv x^2+xy+z^2\equiv 0 \pmod p.$$

Thus,

$$(z-y)(x+y+z)=(x^2+xz+x^2)-(x^2+xy+y^2)\equiv 0 \pmod p.$$

Once again, because $z-y$ is less than $p$, we get $x+y+z\equiv 0 \pmod p$. However, since $0<x+y+z<3p$, it follows that $x+y+z$ is either $p$ or $2p$. Now, observe that

$$3(xy+yz+zx)=2(x+y+z)^2-(x^2+xy+y^2)-(y^2+yz+z^2)-(z^2+zx+x^2)\equiv 0 \pmod p,$$

whence $xy+yz+zx\equiv 0 \pmod p$ since $p>3$. Thus, $xy+yz+zx=kp$, where $k\in\mathbb{Z}^+$. Note that

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx).$$

Dividing both sides by $x+y+z$, we have

$$\frac{x^2+y^2+z^2}{x+y+z}=(x+y+z)-\frac{2(xy+yz+zx)}{x+y+z}.$$

Clearly, $\frac{2(xy+yz+zx)}{x+y+z}$ must be an integer: if $x+y+z=p$, we get $2k$; if $x+y+z=2p$, we get $k$. Thus, $\frac{x^2+y^2+z^2}{x+y+z}$ is an integer, whence $x^2+y^2+z^2$ is divisible by $x+y+z$. □

**Question 5**

Let $a_1, a_2, \ldots$ be a sequence of positive numbers satisfying, for any positive integers $k$, $l$, $m$, $n$ such that $k+n=m+l$,

$$\frac{a_k+a_n}{1+a_ka_n}=\frac{a_m+a_l}{1+a_ma_l}.$$

Show that there exist positive numbers $b$, $c$ so that $b \le a_n \le c$ for any positive integer $n$.

**Solution** (Way Tan)**.** Taking reciprocals, we have

$$\frac{1+a_ka_n}{a_k+a_n}=\frac{1+a_ma_l}{a_m+a_l} \iff \frac{1+a_k+a_n+a_ka_n}{a_k+a_n}=\frac{1+a_m+a_l+a_ma_l}{a_m+a_l}.$$

The numerators factor nicely as

$$\frac{(1+a_k)(1+a_n)}{a_k+a_n}=\frac{(1+a_m)(1+a_l)}{a_m+a_l}.$$

Taking reciprocals once again, we arrive at the set-up of 2020/Open/R1/Q19:

$$\frac{a_k+a_n}{(1+a_k)(1+a_n)}=\frac{a_m+a_l}{(1+a_m)(1+a_l)}.$$

Letting $b_i=\frac{1}{a_i+1}-\frac{1}{2}$ as given in the hint to that question, we have after simplification,

$$b_kb_n=b_mb_n.$$

However, by the condition that $k+n=m+l$, we clearly have that $\{b_i\}$ is in geometric progression. Moreover, because $a_i \in (0,\infty)$, we have $b_i \in (-\frac{1}{2},\frac{1}{2})$. Since $b_i$ is bounded, it follows that $|r| \le 1$, where $r$ is the common ratio of $\{b_i\}$. It follows that $|b_i| \le |b_1| < \frac{1}{2}$ for all positive integers $i$. Hence, $-|b_1| \le b_i \le |b_1|$, implying that

$$\frac{1}{\frac{1}{2}+|b_1|}-1 \le a_i \le \frac{1}{\frac{1}{2}-|b_1|}-1,$$

whence $b=\frac{1}{\frac{1}{2}+|b_1|}-1$ and $c=\frac{1}{\frac{1}{2}-|b_1|}-1$ are clearly positive, and we are done. □

## 10.3. Open Section

### 10.3.1. Round 1 Solutions

**Resources:** Review by *Way Tan*

**Question 1** [Ans: 1540]

Let $S_k = 1 + 2 + 3 + \cdots + k$ for any positive integer $k$. Find $S_1 + S_2 + S_3 + \cdots + S_{20}$.

**Solution.** Note that $S_k = \frac{1}{2}\left(k^2 + k\right)$. Hence, the required sum is

$$\sum_{k=1}^{20} \frac{1}{2}\left(k^2 + k\right) = \frac{1}{2}\left[\frac{20 \cdot 21 \cdot 41}{6} + \frac{20 \cdot 21}{2}\right] = 1540.$$

**Question 2** [Ans: 69]

Let $S = \sum_{r=1}^{64} r\binom{64}{r}$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $0! = 1$. Find $\log_2 S$.

**Solution.** Note that $(1+x)^{64} = \sum_{r=1}^{64} \binom{64}{r} x^r$. Differentiating with respect to $x$ gives $64(1+x)^{63} = \sum_{r=1}^{64} r\binom{64}{r} x^{r-1}$. Evaluating at $x = 1$, we have $64 \cdot 2^{63} = \sum_{r=1}^{64} r\binom{64}{r} = S$. Hence, $\log_2 S = 69$.

**Question 3** [Ans: 4]

Let $x$ be the largest number in the interval $[0, 2\pi]$ such that $(\sin x)^{2024} - (\cos x)^{2024} = 1$. Find $\lfloor x \rfloor$.
*(Note: If you think that such a number $x$ does not exist, enter your answer "99999".)*

**Solution.** Observe that $(\sin x)^{2024}, (\cos x)^{2024} \in [0, 1]$. Hence, the equation holds if and only if $(\sin x)^{2024} = 1$ and $(\cos x)^{2024} = 0$. Thus, $\sin x = \pm 1$ and $\cos x = 0$, whence $x = \frac{3\pi}{2}$ and $\lfloor x \rfloor = 4$.

**Question 4** [Ans: 99999]

Find the number of real numbers $x$ that satisfies the equation $|x - 2| + |x - 3| = |2x - 5|$.
*(Note: If you think that there are no such numbers, enter "0"; if you think that there are infinitely many such numbers, enter "99999".)*

**Solution.** Consider $x < 2$. The given equation simplifies to $(2 - x) + (3 - x) = 5 - 2x$, whence $x$ is free. There are hence infinitely many solutions.

**Question 5** [Ans: 3]

Among all the real numbers that satisfies the inequality $e^x \geq 1 + 2e^{-x}$, find the minimum value of $\lceil e^x + e^{-x} \rceil$.

**Solution.** Multiplying through by $e^x$ yields a quadratic in $e^x$: $(e^x)^2 - e^x - 2 \geq 0$. Thus, $e^x \geq 2$ (keeping in mind that $e^x > 0$). Since $e^x + e^{-x}$ is increasing for $x > 0$, the minimum value occurs when $e^x = 2$, whence the desired answer is $\lceil 2 + 1/2 \rceil = 3$.

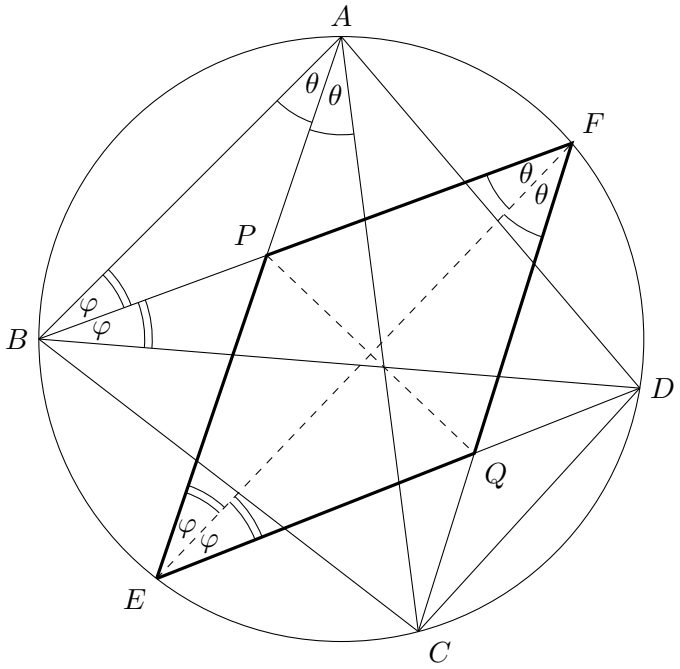
**Question 6** [Ans: 2025]

Find the smallest positive integer $C$ greater than 2024 such that the sets $A = \{2x^2 + 2x + C : x \in \mathbb{Z}\}$ and $B = \{x^2 + 2024x + 2 : x \in \mathbb{Z}\}$ are disjoint.

**Solution.** Observe that $x^2 + 2024x + 2 \equiv 2, 3 \pmod 4$, while $2x^2 + 2x + C \equiv C \pmod 4$. Thus, so long as $C \not\equiv 2, 3 \pmod 4$, we will have $A$ and $B$ disjoint. The smallest such $C$ is hence 2025, which has a residue of 1 modulo 4.

**Question 7** [Ans: 110]

Let $ABCD$ be a convex quadrilateral inscribed in a circle $\omega$. The bisector of $\angle BAC$ meets $\omega$ at $E$ ($\neq A$), the bisector of $\angle ABD$ meets $\omega$ at $F$ ($\neq B$), $AE$ intersects $BF$ at $P$ and $CF$ intersects $DE$ at $Q$. Suppose $EF = 20$, $PQ = 11$. Find the area of the quadrilateral $PEQF$.




**Solution.** Let $\angle BAC = 2\theta$ and $\angle ABD = 2\varphi$. Using angles in same segment on $ABEF$, we have $\angle BFE = \angle BAE = \theta$, and $\angle AEF = \angle ABF = \varphi$. Using angles in same segment on $AECF$, we have $\angle CFE = \angle CAE = \theta$. Using angles in same segment on $BEDF$, we have $\angle FBD = \angle FED = \varphi$. By ASA, $\triangle PEF \equiv \triangle QEF$, whence $PEQF$ is a kite. Hence, $[PEQF] = \frac{1}{2} \cdot EF \cdot PQ = 110$.

**Question 8** [Ans: 5]

Let $f(x) = \sqrt{x^2 + 1} + \sqrt{(4 - x)^2 + 4}$. Find the minimum value of $f(x)$.

**Solution.** Let $A(0, \pm 1)$, $B(x, 0)$ and $C(4, \pm 2)$. Observe that $AB = \sqrt{x^2 + 1}$ and $BC = \sqrt{(4 - x)^2 + 4}$. It follows that $f(x) = AB + BC$ attains its minimum when $ABC$ is a straight line (crucially, $B$ must be in between $A$ and $C$). To achieve this, we can set $A(0, -1)$ and $C(4, 2)$. Thus, $\min f(x) = AC = 5$.

**Question 9** [Ans: 17]

It is known that $a \geq 0$ satisfies $\sqrt{4+\sqrt{4+\sqrt{4+\sqrt{4+a}}}} = a$. Find the value of $(2a-1)^2$.

**Solution.** Observe that $\sqrt{4+a} = a$, whence $a^2 - a - 4 = 0$. Thus,

$$(2a-1)^2 = 4\left(a^2 - a - 4\right) + 17 = 17.$$

**Question 10** [Ans: 160]

A rectangle with sides parallel to the horizontal and vertical axes is inscribed in the region bounded by the graph of $y = 60 - x^2$ and the $x$-axis. If the area of the largest such rectangle has area $k\sqrt{5}$, find the value of $k$.

**Solution.** Let $A$ be the area of the rectangle. Let the rectangle have width $2x$. Then $A = 2x\left(60 - x^2\right) = 120x - 2x^3$. Hence, $\mathrm{d}A/\mathrm{d}x = 120 - 6x^2$. The sole stationary point of $A$ (which can easily be verified as a maximum) hence occurs when $x = 2\sqrt{5}$. The area of the rectangle is thus $160\sqrt{5}$, whence $k = 160$.

**Question 11** [Ans: 10]

Let $x$ be a real number satisfying the equation $x^{x^5} = 100$. Find the value of $\lfloor x^5 \rfloor$.

**Solution.** Raising the given equation to the 5th power yields $\left(x^5\right)^{x^5} = 10^{10}$, whence $x^5 = 10$.

**Question 12** [Ans: 10]

Let $a$, $b$, $c$, $d$, $e$ be distinct integers with $a+b+c+d+e = 9$. If $m$ is an integer such that

$$(m-a)(m-b)(m-c)(m-d)(m-e) = 2009,$$

determine the value of $m$.

**Solution.** Note that $2009 = 7^2 \cdot 41$. Since $a$, $b$, $c$, $d$, $e$ are distinct, the five terms must be $7$, $-7$, $41$, $1$ and $-1$. Summing, we get $5m - (a+b+c+d+e) = 41$, whence $m = 10$.

**Question 13** [Ans: 3]

Let $\{x\}$ be the fractional part of the number $x$, i.e., $\{x\} = x - \lfloor x \rfloor$. If $S = \int_0^9 \{x\}^2 \, \mathrm{d}x$, find $\lfloor S \rfloor$.

**Solution.** Observe that $\{x\}$ has period 1, and is equivalent to $x$ on the interval $[0, 1)$. Thus,

$$S = \int_0^9 \{x\}^2 \, \mathrm{d}x = 9 \int_0^1 x^2 \, \mathrm{d}x = 3.$$

**Question 14** [Ans: 81]

The solution of the inequality $|(x+1)(x-6)| > |(x+4)(x-2)|$ can be expressed as $x < a$ or $b < x < c$. If $S = |a| + |b| + |c|$, find $\lfloor 14S \rfloor$.

**Solution.** As we are interested in the extreme ends of the solution range, we consider the case of equality, i.e. $|(x+1)(x-6)| = |(x+4)(x-2)|$.

*Case 1*. Consider $(x+1)(x-6) = (x+4)(x-2)$. Expanding and simplifying, we get $x = 2/7$.

*Case 2*. Consider $(x+1)(x-6) = -(x+4)(x-2)$. Expanding and simplifying, we get $(2x-7)(x+2) = 0$, whence $x = -2$ and $x = 7/2$.

Together, it stands to reason that $a = -2$, $b = 2/7$ and $c = 7/2$, whence $14S = 81$.

**Question 15** [Ans: 2]

Given that $x, y > 0$ and $x\sqrt{2-y^2} + y\sqrt{2-x^2} = 2$, find the value of $x^2 + y^2$.

**Solution 1.** Let $X = x^2$ and $Y = y^2$. Squaring the given equation, we have

$$X(2-Y) + 2xy\sqrt{(2-X)(2-Y)} + Y(2-X) = 4.$$

This gives

$$xy\sqrt{(2-X)(2-Y)} = 2 - (X+Y) + XY.$$

Squaring once more, we obtain

$$XY(2-X)(2-Y) = [2-(X+Y)+XY]^2.$$

Upon simplification, one gets

$$(X+Y)^2 - 4(X+Y) + 4 = 0,$$

whence $x^2 + y^2 = X + Y = 2$.

**Solution 2** (Abusing uniqueness)**.** Suppose $x = y$. We thus get $x\sqrt{2-x^2} = 1$, whence $x = y = 1$. Since $x^2 + y^2$ is a constant (by the way the question is asked), we have $x^2 + y^2 = 2$.

**Question 16** [Ans: 17]

A convex polygon has $n$ sides such that no three diagonals are concurrent. It is known that all its diagonals divide the polygon into 2500 regions. Determine $n$.

**Solution.** Let $P$ be a convex polygon with $n$ sides such that no three diagonals are concurrent. By Euler's formula, we have

$$V - E + F = 1,$$

where $V$, $E$ and $F$ are the number of vertices, edges and faces of $P$ respectively. Note that we disregard the region "outside" $P$. Observe that any four of the $n$ vertices of $P$ gives a unique vertex in the interior of $P$. Hence, $V = n + \binom{n}{4}$. Next, observe that each vertex of $P$ has degree $n-1$, while each vertex in the interior of $P$ has degree 4. By the degree sum formula, $2E = n(n-1) + 4\binom{n}{4}$. We hence obtain the following expression for $F$:

$$F = \binom{n}{4} + \frac{n(n-3)}{2} + 1.$$

Setting $F = 2500$, we see that $n = 17$.

**Question 17** [Ans: 4]

Find the number of integers $n$ between $-2029$ and 2029 inclusive such that $(n+2)^2+n^2$ is divisible by 2029.

**Solution.** Re-expressing the given condition in the language of modular arithmetic, we get $(n+2)^2+n^2 \equiv 0 \pmod{2029}$. Simplifying, we obtain

$$(n+1)^2 \equiv -1 \pmod{2029}. \tag{1}$$

Now observe that

$$\left(\frac{-1}{2029}\right) = (-1)^{\frac{2029-1}{2}} = 1,$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol of $a$ and $p$. We have hence established that $-1$ is a quadratic residue modulo 2029 (i.e. there exists some integer $m$ such that $m^2 \equiv -1 \pmod{2029}$). There are hence two integer solutions to (1). However, since the solutions are 2029-periodic, we get a total of 4 solutions.

**Question 18** [Ans: 2516]

Let $f$ be a function such that for any real number $x$, we have $f(x)+2f(2-x) = x+x^2$. Find the value of $f(1)+f(2)+f(3)+\cdots+f(34)$.

**Solution.** Let $S = \sum_{n=1}^{34} f(n)$ and $T = \sum_{n=-32}^{1} f(n)$. From the given equation, we have

$$S + 2T = \sum_{n=1}^{34} \left(n + n^2\right). \tag{1}$$

Now consider the transformation $x \mapsto 2-x$. We get $f(2-x)+2f(x) = (2-x)+(2-x)^2$. Hence,

$$T + 2S = \sum_{n=1}^{34} \left[(2-n)+(2-n)^2\right]. \tag{2}$$

Simultaneously solving (1) and (2) yields

$$S = \frac{1}{3}\sum_{n=1}^{34} \left(2\left[(2-n)+(2-n)^2\right] - (n+n^2)\right) = 2516.$$

**Question 19** [Ans: 761]

Find the largest possible positive prime integer $p$ such that $p$ divides

$$S(p) = 1^{p-2} + 2^{p-2} + 3^{p-2} + 4^{p-2} + 5^{p-2} + 6^{p-2} + 7^{p-2} + 8^{p-2}.$$

**Solution.** Multiplying $S(p)$ through by 8! yields

$$\frac{8!}{1}\cdot 1^{p-1} + \frac{8!}{2}\cdot 2^{p-1} + \cdots + \frac{8!}{8}\cdot 8^{p-1} \equiv 0 \pmod{p}.$$

However, Fermat's little theorem states that $a^{p-1} \equiv 1 \pmod{p}$ for all natural numbers $a$ such that $p \nmid a$. Assuming that $p > 8$, we have that

$$\frac{8!}{1} + \frac{8!}{2} + \cdots + \frac{8!}{8} \equiv 0 \pmod{p}.$$

The LHS works out to be $109584 = 2^4 \cdot 3^2 \cdot 761$. Hence, the largest possible $p$ is 761.

**Question 20** [Ans: 1009]

Let $f$ be a function such that $f(x)+f(\frac{1}{1-x})=1+\frac{1}{x}$ for all $x \notin \{0,1\}$. Find the value of $\lfloor 180 \cdot f(10) \rfloor$.

**Solution.** Substituting $x = 10$, we get

$$f(10) + f(-\tfrac{1}{9}) = \frac{11}{10}.$$

Substituting $x = -\frac{1}{9}$, we get

$$f(-\tfrac{1}{9}) + f(\tfrac{9}{10}) = -8.$$

Substituting $x = \frac{9}{10}$, we get

$$f(\tfrac{9}{10}) + f(10) = \frac{19}{9}.$$

Solving the three equations simultaneously, we get $180 \cdot f(10) = 1009$.

**Question 21** [Ans: 2]

Let $C$ be a circle with equation $(x-a)^2+(y-b)^2=r^2$, where at least one of the $a$ and $b$ are irrational numbers. Find the maximum possible numbers of points $(p,q)$ on $C$ where both $p$ and $q$ are rational numbers.

**Solution.** Observe that it is possible for a circle with "irrational centre" to have two rational points. For instance, the circle with centre $(0,\sqrt{2})$ and radius 3 contains the rational points $(-1,0)$ and $(1,0)$.

We now show that three or more rational points is possible only if the coordinates of the centre of the circle are rational. Suppose there exists a circle with three rational points ($P$, $Q$ and $R$). Then the gradients of chords $PQ$ and $QR$ are rational. Thus, the gradients of the perpendicular bisector of $PQ$ and $QR$ are also rational. It follows that the equation of the perpendicular bisector of $PQ$ and $QR$ have rational coefficients. However, the perpendicular bisectors of any two chords must meet in the centre, implying that the coordinates of the centre are rational. This concludes the proof.

**Question 22** [Ans: 24]

On the plane there are 2024 points coloured either red or blue such that each red point is the centre of a circle passing through 3 blue points. Determine the least number of blue points.

**Solution.** Let there be $b$ blue points. Observe that the absolute maximum number of red points is equal to the number of triplets of blue points (each triplet uniquely defines a red point). That is, there are at most $\binom{b}{3}$ red points. We thus require $\binom{b}{3}+b \geq 2024$. The smallest $b$ that satisfies this is $b = 24$.

**Question 23** [Ans: 45]

It is given that the positive real numbers $x_1, \ldots, x_{2026}$ satisfy $\dfrac{x_1^2}{x_1^2+1}+\cdots+\dfrac{x_{2026}^2}{x_{2026}^2+1} = 2025$. Find the maximum value of $\dfrac{x_1}{x_1^2+1}+\cdots+\dfrac{x_{2026}}{x_{2026}^2+1}$.

**Solution 1.** From the given equation, we see that

$$\sum_{n=1}^{2026}\left(1-\frac{1}{x_n^2+1}\right) = 2025 \implies \sum_{n=1}^{2026}\frac{1}{x_n^2+1} = 1.$$

By the Cauchy-Schwarz inequality, one thus has

$$\left(\sum_{n=1}^{2026}\frac{x_n}{x_n^2+1}\right)^2 \le \left[\sum_{n=1}^{2026}\left(\frac{x_n}{\sqrt{x_n^2+1}}\right)^2\right]\left[\sum_{n=1}^{2026}\left(\frac{1}{\sqrt{x_n^2+1}}\right)^2\right] = 2025.$$

The maximum is thus 45.

**Solution 2** (Abusing symmetry)**.** Observe that the given expressions are all symmetric polynomials. Letting $x_1 = x_2 = x_3 = \cdots = x_{2026}$, one gets that $2026 \cdot \frac{x_i^2}{x_i^2+1} = 2025$, whence $x_i^2+1 = 2026$ and $x_i = 45$. Thus, the expression in question evaluates to $2026 \cdot \frac{45}{2026} = 45$.

**Question 24** [Ans: 441]

Let $n$ denote the number of ways of arranging all the letters of the word MATHEMATICS in one row such that

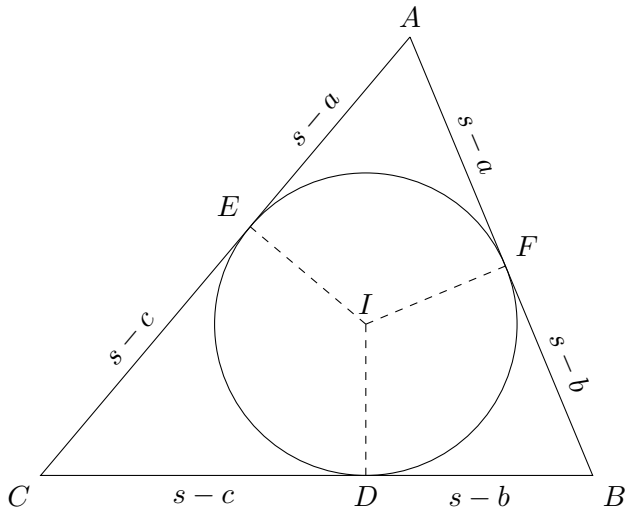
- both M's precede both T's; and
- neither the two M's nor the two T's are next to each other.

Determine the value of $\frac{n}{6!}$.

**Solution.** To ensure that both M's are both T's are not adjacent to each other, we consider the second M and the letter in front of it as one group, and the first T and the letter after it as one group. This gives a total of 9 groups. Since there are 4 groups with M's and T's, we have $\binom{9}{4}$ ways to arrange the M's and T's. Meanwhile, there are $\frac{7!}{2!}$ ways to arrange the remaining letters (note that we divide by 2! to account for the double A). Hence, $n = \binom{9}{4} \cdot \frac{7!}{2!}$, whence $\frac{n}{6!} = 441$.

**Question 25** [Ans: 146]

The incircle of the triangle $ABC$ centred at $I$ touches the sides $BC$, $CA$, $AB$ at $D$, $E$, $F$ respectively. Let $D'$ be the intersection of the extension of $ID$ with the circle through $B$, $I$, $C$; $E'$ be the intersection of the extension of $IE$ with the circle through $A$, $I$, $C$; and $F'$ the intersection of the extension of $IF$ with the circle through $A$, $I$, $B$. Suppose $AB = 52$, $BC = 56$, $CA = 60$. Find $DD' + EE' + FF'$.




**Solution.** It is well known that the points tangents to the incircle divide the triangle into lengths of $s-a$, $s-b$, $s-c$, as shown in the figure above. Here, $s$ is the semiperimeter $\frac{1}{2}(a+b+c)$, $a = BC$, $b = CA$ and $c = AB$.

From Heron's formula, one has $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$. Thus, the inradius $r$ is given by $r = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}$.

We now formulate equations involving $DD'$, $EE'$ and $FF'$. Invoking the power of a point theorem in the circumcircle of $\triangle BIC$, one has $(s-c)(s-b) = r \cdot DD'$. Likewise, we obtain the formulae $(s-c)(s-a) = r \cdot EE'$ and $(s-a)(s-b) = r \cdot FF'$. Thus,

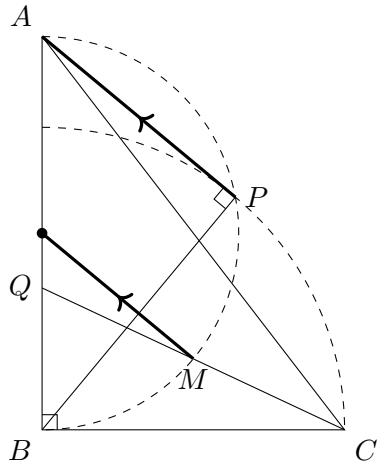
$$DD' + EE' + FF' = \frac{(s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a)}{r} = 146.$$

### 10.3.2. Round 2 Solutions

**Resources:** Review by *Way Tan*, Review by *Glen Lim*, AoPS threads

**Question 1**

In triangle $ABC$, $\angle B = 90°$, $AB > BC$, and $P$ is the point such that $BP = BC$ and $\angle APB = 90°$, where $P$ and $C$ lie on the same side of $AB$. Let $Q$ be the point on $AB$ such that $AP = AQ$, and let $M$ be the midpoint of $QC$. Prove that the line through $M$ parallel to $AP$ passes through the midpoint of $AB$.




**Solution.** We will use coordinate bashing to solve this problem. Firstly, let $A(0, a)$, $B(0, 0)$, and $C(1, 0)$. Note that we fix $C(1, 0)$ as we can always scale the triangle to make $BC = 1$.

Since $BP = BC = 1$, we know that $P$ lies on the unit circle $x^2 + y^2 = 1$. Furthermore, since $\angle APB = 90°$, $P$ also lies on the circle with diameter $AB$, which has equation $x^2+(y-\frac{1}{2}a)^2 = (\frac{1}{2}a)^2$. Solving the two equations simultaneously, along with the constraint $x > 0$ (since $P$ and $C$ lie on the same side of $AB$), we get $P(\frac{1}{a}\sqrt{a^2-1}, \frac{1}{a})$.

Now note that $AQ = AP = \sqrt{(\frac{1}{a}\sqrt{a^2-1})^2 + (\frac{1}{a} - a)^2} = \sqrt{a^2-1}$. Hence, $Q(0, a - \sqrt{a^2-1})$. Thus, $M(\frac{1}{2}, \frac{1}{2}(a - \sqrt{a^2-1}))$.

Note that the gradient of $AP$ is $\frac{a-1/a}{0-\sqrt{a^2-1}/a} = -\sqrt{a^2-1}$. Let $l$ be the line passing through $M$ parallel to $AP$. By the point-slope formula, $l$ has the equation

$$l : y - \frac{a - \sqrt{a^2-1}}{2} = -\sqrt{a^2-1}\left(x - \frac{1}{2}\right).$$

When $x = 0$, we have $y = \frac{a}{2}$. Hence, $l$ passes through $(0, \frac{1}{2}a)$, which is the midpoint of $AB$ as desired. □

*Remark.* The condition $AB > BC$ is equivalent to $a > 1$, which is crucial in ensuring that $\sqrt{a^2-1}$ remains real.

**Question 2** [Ans: $\frac{1}{2}n(n+1)$]

Let $n$ be a fixed positive integer. Find the minimum value of

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n}$$

where $x_1, x_2, \ldots, x_n$ are distinct positive integers.

**Solution** (KHOMNYO2)**.** Without loss of generality, suppose $x_1 < x_2 < \cdots < x_n$. We begin by showing $\sum_{i=1}^n x_i^3 \geq (\sum_{i=1}^n x_i)^2$ via induction. The case where $n = 1$ is trivial. Suppose that $\sum_{i=1}^k x_i^3 \geq (\sum_{i=1}^k x_i)^2$ for some $k$. Consider $(\sum_{i=1}^{k+1} x_i)^2$:

$$\begin{aligned}\left(\sum_{i=1}^{k+1} x_i\right)^2 &= \left(\sum_{i=1}^{k} x_i\right)^2 + 2x_{k+1}\sum_{i=1}^{k} x_i + x_{k+1}^2 \\ &\leq \sum_{i=1}^{k} x_i^3 + 2x_{k+1}\sum_{i=1}^{k} x_i + x_{k+1}^2 \\ &= \sum_{i=1}^{k+1} x_i^3 + x_{k+1}\left(2\sum_{i=1}^{k} x_i + x_{k+1} - x_{k+1}^2\right).\end{aligned}$$

It hence suffices to show that $x_{k+1}^2 \geq 2\sum_{i=1}^k x_i + x_{k+1}$. Rearranging, we get the equivalent statement $\frac{1}{2}x_{k+1}(x_{k+1} - 1) \geq \sum_{i=1}^k x_i$. Indeed, we see that

$$\frac{x_{k+1}(x_{k+1}-1)}{2} \geq \frac{x_k(x_k+1)}{2} = 1 + 2 + \cdots + x_k \geq x_1 + x_2 + \cdots + x_k$$

where the last inequality holds because $\{x_1, x_2, \ldots, x_k\} \subseteq \{1, 2, \ldots, x_k\}$. This closes the induction.

Applying our newly-established inequality to the problem at hand, we see that

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n} \geq \frac{(x_1 + \cdots + x_n)^2}{x_1 + \cdots + x_n} \geq x_1 + \cdots + x_n \geq 1 + \cdots + n = \frac{n(n+1)}{2}.$$

**Question 3**

Prove that for every positive integer $n$ there exists an $n$-digit number divisible by $5^n$ all of whose digits are odd.

**Solution.** We prove via induction that for all $n \in \mathbb{N}$, there must exist some $n$-digit number $a_n$ divisible by $5^n$ all of whose digits are odd. When $n = 1$, it is trivial to see that $a_1 = 5$. Now suppose $a_k$ exists for some $k \in \mathbb{N}$. Let $S = \{1, 3, 5, 7, 9\}$. We now show that there exists an $s \in S$ such that $a_{k+1} = s \cdot 10^k + a_k$.

Observe that $s \cdot 10^k + a_k = 5^k\left(s \cdot 2^k + m\right)$. Recall that $2^k$ can only end with a 2, 4, 6, or 8 in base 10. Hence, the only residues $2^k$ can take on modulo 5 are 1, 2, 3, and 4. Meanwhile, $s$ can take on any residue modulo 5. It is thus obvious that $s \cdot 2^k$ can take on any residue modulo 5. Hence, by picking $s \equiv -m \pmod 5$, we will have $5^{k+1} \mid 5^k\left(s \cdot 2^k + m\right)$ as desired. This closes the induction. □

**Question 4** [Ans: $n+1$]

Alice and Bob play a game. Bob starts by picking a set $S$ consisting of $M$ vectors of length $n$ with entries either 0 or 1. Alice picks a sequence of numbers $y_1 \le y_2 \le \cdots \le y_n$ from the interval $[0,1]$, and a choice of real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Bob wins if he can pick a vector $(z_1, z_2, \ldots, z_n) \in S$ such that

$$\sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} x_i z_i,$$

otherwise Alice wins. Determine the minimum value of $M$ so that Bob can guarantee a win.

**Solution** (DVDthe1st)**.** Note that we can rewrite Bob's winning condition as

$$\langle \mathbf{x}, \mathbf{y} \rangle \le \max_{\mathbf{z} \in S} \langle \mathbf{x}, \mathbf{z} \rangle.$$

Geometrically, Bob wins if he can find some $\mathbf{z} \in S$ whose projection on $\mathbf{x}$ is longer than the projection of $\mathbf{y}$ on $\mathbf{x}$. Hence, $\mathbf{y}$ must be in the convex hull of $S$. However, observe that the space of all possible $\mathbf{y}$ forms a simplex in $n$-dimensional space. In particular, this simplex has $n+1$ vertices of the form

$$\mathbf{v}_i = (\underbrace{0, \ldots, 0}_{i}, \underbrace{1, \ldots, 1}_{n-i}).$$

Thus, picking $S = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ will guarantee a win for Bob, whence $M = n+1$. We now show that this is indeed the minimum.

Suppose there exists some $\mathbf{v}_i \notin S$. Alice can capitalize on this by picking

$$\mathbf{x} = (\underbrace{-1, \ldots, -1}_{i}, \underbrace{1, \ldots, 1}_{n-i}) \text{ and } \mathbf{y} = \mathbf{v}_i.$$

This is because $\mathbf{x} \cdot \mathbf{y}$ attains its maximum only when $\mathbf{y} = \mathbf{v}_i$. Hence, Bob can only win by choosing $\mathbf{z} = \mathbf{v}_i$. But because $\mathbf{v}_i \notin S$, he cannot do so and will thus lose.

**Question 5** [Ans: $p+1$]

Let $p$ be a prime number. Determine the largest possible $n$ such that the following holds: it is possible to fill an $n \times n$ table with integers $a_{ik}$ in the $i$th row and $k$th column, for $1 \le i, k \le n$, such that for any quadruple $i$, $j$, $k$, $l$ with $1 \le i < j \le n$ and $1 \le k < l \le n$, the number $a_{ik}a_{jl} - a_{il}a_{jk}$ is not divisible by $p$.

**Solution** (Glen Lim)**.** Observe that the condition $p \nmid a_{ik}a_{jl} - a_{il}a_{jk}$ is equivalent to

$$\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} \not\equiv 0 \pmod{p}$$

That is, $(a_{ik}, a_{jk})$ and $(a_{il}, a_{jl})$ are linearly independent. $n$ is thus the maximum number of vectors in $\mathbb{F}_p^2$ that are pairwise linearly independent.

We now show that $n \le p+1$. Take an arbitrary non-zero vector $\mathbf{v} \in \mathbb{F}_p^2$. Then there are clearly $p-1$ non-zero vectors parallel to $\mathbf{v}$, namely $\mathbf{v}, 2\mathbf{v}, \ldots, (p-1)\mathbf{v}$. Since there are $p^2 - 1$ non-zero vectors in $\mathbb{F}_p^2$, the maximum number of pairwise linearly independent

vectors is $\frac{p^2-1}{p-1} = p+1$. Hence, $n \leq p+1$. Indeed, we can construct the following set with $p+1$ vectors that are pairwise linearly independent:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} p-1 \\ 1 \end{pmatrix} \right\}.$$

To finish off the problem, we construct a valid $(p+1) \times (p+1)$ grid. We do so as follows:

- $a_{11} = 0$ (the top left corner is 0)
- $a_{1i} = a_{i1} = 1$ for $2 \leq i \leq p+1$ (all other cells in the first row and first column are 1)
- $a_{ik} = i - k$ for $2 \leq i, j \leq p+1$ (cells in the remaining $p \times p$ grid are the difference between the row number and the column number)

As an example, the following grid shows the $p = 5$ case.

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 1 | 4 | 0 | 1 | 2 | 3 |
| 1 | 3 | 4 | 0 | 1 | 2 |
| 1 | 2 | 3 | 4 | 0 | 1 |
| 1 | 1 | 2 | 3 | 4 | 0 |

We now show that $p \nmid a_{ik}a_{jl} - a_{il}a_{jk}$.

*Case 1*. Consider the case where $i = 1$. Then $a_{1k}a_{jl} - a_{1l}a_{jk} = a_{jl} - a_{jk}$. However, each cell in the $j$th row has a different number, whence it is clear that $a_{jl} - a_{jk} \not\equiv 0 \pmod{p}$. A similar argument works for $k = 1$.

*Case 2*. Consider the case where $i, k \neq 1$. Then $a_{ik}a_{jl} - a_{il}a_{jk} = (i-k)(j-l) - (i-l)(j-k) = (i-j)(k-l)$, which clearly cannot be 0 modulo $p$.

Hence, $\max n = p+1$.