1 Problems

Problem 1 (1 pt). Let a_n be a sequence defined by $a_1 = 9$ and

$$a_{n+1} = \frac{2025a_n}{2025 - 2a_n}.$$

How many terms in the sequence are integers?

Problem 2 (1 pt). There are n people, each of them speaking at most 3 languages. Between any three people, at least two speak a common language. Find the least n such that there always exists a language spoken by at least three people.

Problem 3 (2 pts). Let x be the value of

$$(1 + \tan 1^\circ)(1 + \tan 2^\circ)(1 + \tan 3^\circ)\dots(1 + \tan 45^\circ).$$

Determine the number of positive factors of $\lfloor x \log_2 x \rfloor$.[1]

Problem 4 (2 pts). Determine the largest integer n smaller than 2^{127} such that

$$\binom{n-1}{k} - (-1)^k$$

is divisible by n for all $k = 0, 1, 2, \dots, n-1$.

Problem 5 (3 pts). Let p be the largest prime that divides

$$1^{p-2} + 2^{p-2} + 3^{p-2} + \cdots + 99^{p-2} + 100^{p-2}.$$

Determine the number of digits of p.

Problem 6 (3 pts). Determine the product of all positive integers n with at least 4 factors such that n is the sum of the squares of its 4 smallest factors.

Problem 7 (3 pts). Ten thousand balls labelled 1 to 10,000 are to be put into two identical boxes so that each box contains at least one ball and the greatest common divisor of the product of the labels of all the balls in one box and the product of the labels of all the balls in the other box is 1. Determine the number of ways that this can be done.

Problem 8 (5 pts). Alice has 2025^{2025} cards in a row, where the card in position i has the label $i + 2025! \pmod{2025^{2025}}$.[2] Alice starts by colouring each card either red or blue. Afterwards, she is allowed to make several moves, where each move consists of choosing two cards of different colours and swapping them. Determine the minimum number of moves she has to make (assuming she chooses the colouring optimally) to put the cards in order (i.e. card i is at position i).

[1] $\lfloor X \rfloor$ refers to the greatest integer lesser than or equal to X. For instance, $\lfloor \pi \rfloor = 3$, $\lfloor 2 \rfloor = 2$ and $\lfloor -1.1 \rfloor = -2$.

[2] 2025! represents the *factorial* of 2025, which is defined as $2025 \times 2024 \times 2023 \times \cdots \times 3 \times 2 \times 1$.

2 Answers

P1. 27

P2. 9

P3. 48

P4. $2^{127} - 1$

P5. 22

P6. 130

P7. $2^{561} - 1$

P8. $2025^{2025} - 3^{1010}\,5^{505}$

3 Solutions

Problem 1

Source: Adapted from 26th PMO Qualifying II Q14

Taking reciprocals, we see that

$$\frac{1}{a_{n+1}} = \frac{1}{a_n} - \frac{2}{2025},$$

with $1/a_1 = 1/9 = 225/2025$. From here, it is easy to see that

$$a_2 = \frac{2025}{223},\ a_3 = \frac{2025}{221},\ a_4 = \frac{2025}{219},\ a_5 = \frac{2025}{217}, \ldots,$$

that is, the denominator keeps decreasing by 2. 2025 has a total of 27 factors lesser than or equal to 225 (accounting for negative factors), all of which are odd. Thus, the answer is 27.

Problem 2

Source: 2011 Romanian JBMO TST #2 Q5

We claim that $n = 9$ is minimal. Indeed, for $n \leq 8$, we can construct an example where each language is spoken by at most 2 people. For $n = 8$, split the people into two groups of 4. Set any pair of persons in each group to speak a different language for a total of $6 + 6 = 12$ languages, each spoken by 2 persons, each person speaking 3 languages. For $n \leq 8$, simply remove $8 - n$ people from the $n = 8$ case.

We now prove that when $n = 9$, there exists a language spoken by at least three people.

Claim 1. *When $n = 9$, there always exists a language spoken by at least three people.*

Proof. Seeking a contradiction, suppose each language is spoken by at most two people. By the pigeon-hole principle, each person A can speak with at most 3 others, say B, C and D. Likewise, E can speak with (at most) three others, say F, G and H. There is left at least another person, say Z, and in the group A, E, Z, no language is spoken in common, a contradiction. □

Problem 3

Source: 2003 Purple Comet! Math Meet Q25

It is easy to see that

$$x = \frac{(\cos 1^\circ + \sin 1^\circ)\,(\cos 2^\circ + \sin 2^\circ)\ldots(\cos 45^\circ + \sin 45^\circ)}{\cos 1^\circ \cos 2^\circ \ldots \cos 45^\circ}.$$

By R-formula, we know that $\cos x^\circ + \sin x^\circ = \sqrt{2}\sin((x+45)^\circ)$, so

$$x = \sqrt{2}^{45}\,\frac{\sin 46^\circ \sin 47^\circ \dots \sin 90^\circ}{\cos 1^\circ \cos 2^\circ \dots \cos 45^\circ}.$$

Since $\cos x^\circ = \sin((90-x)^\circ)$, the expression simplifies to

$$x = \sqrt{2}^{45}\,\frac{\cos 44^\circ \cos 43^\circ \dots \cos 0^\circ}{\cos 1^\circ \cos 2^\circ \dots \cos 45^\circ} = \sqrt{2}^{45}\,\frac{\cos 0^\circ}{\cos 45^\circ} = \sqrt{2}^{46} = 2^{23}.$$

Thus,

$$x \log_2 x = 2^{23} \cdot 23^1,$$

whence the number of factors is $(23+1)(1+1) = 48$.

Problem 4

Source: Adapted from 2020 SMO Open Round 1 Q18

Claim 2. *If n is prime, then $\binom{n-1}{k} - (-1)^k \equiv 0 \pmod{n}$ for all $0 \le k \le n-1$.*

Proof. Recall that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for $1 \le k \le n-1$. Note also that $\binom{n}{k}$ is divisible by n for $1 \le k \le n-1$. Thus,

$$\binom{n-1}{k-1} + \binom{n-1}{k} \equiv 0 \pmod{n}.$$

We now prove our claim by induction. The base case $k = 1$ is trivial:

$$\binom{n-1}{1} \equiv -\binom{n-1}{1-1} = -1 = (-1)^1 \pmod{n}.$$

Now suppose $\binom{n-1}{m} \equiv (-1)^m$ for some positive integer m. Then

$$\binom{n-1}{m+1} \equiv -\binom{n-1}{m} = -(-1)^m = (-1)^{m+1} \pmod{n}.$$

This closes the induction and we are done. □

$2^{127} - 1$ is a Mersenne prime, hence the largest n is $2^{127} - 1$.

Problem 5

Source: Adapted from 2024 SMO Open Round 1 Q19

Multiplying the given expression by $100!$ yields

$$\frac{100!}{1} \cdot 1^{p-1} + \frac{100!}{2} \cdot 2^{p-1} + \dots + \frac{100!}{100} \cdot 100^{p-1} \equiv 0 \pmod{p}.$$

However, Fermat's Little Theorem states that $a^{p-1} \equiv 1 \pmod{p}$ for all natural numbers a such that $p \nmid a$. Assuming that $p > 100$, we have that

$$\frac{100!}{1} + \frac{100!}{2} + \dots + \frac{100!}{100} \equiv 0 \pmod{p}.$$

The largest prime factor of the LHS is 2284070837741348234617 which has 22 digits, so the answer is 22.

Problem 6

Source: 2018 SMO Junior Round 2 Q4

Let the smallest factors of n be d_1, d_2, d_3, d_4, arranged in increasing order (i.e. $d_1 < d_2 < d_3 < d_4$), so that

$$n = d_1^2 + d_2^2 + d_3^2 + d_4^2.$$

If n is odd, then all its factors must also be odd, but this implies

$$n \equiv 1^2 + 1^2 + 1^2 + 1^2 \equiv 0 \pmod 2,$$

a contradiction, so n must be even. From this, we immediately gather $d_1 = 1$ and $d_2 = 2$, so

$$n - 5 = d_3^2 + d_4^2.$$

Since n is even, the LHS (and thus the RHS) is odd, so exactly one of d_3 and d_4 is even.

Case 1. Suppose d_3 is even. Write $d_3 = 2m$, where m is a positive integer. Since $d_3 > d_2 = 2$, we require $m > 1$. If $m > 2$, then m is a divisor between d_2 and d_3, a contradiction. So we must have $m = 2$, whence $d_3 = 4$. Thus,

$$n = 1^2 + 2^2 + 4^2 + d_4^2 = 21 + d_4^2,$$

since $d_4 \mid n$, we see that $d_4 \mid 21$, so $d_4 = 7$ (note that $d_4 \neq 3$ since $d_4 > d_3 = 4$). But this yields $n = 70$, which is not divisible by $d_3 = 4$, so we have no solution in this case.

Case 2. Suppose d_4 is even. Write $d_4 = 2m$, where m is a positive integer. If $m = 2$, then $d_4 = 4$, which leaves $d_3 = 3$, yielding $n = 30$. But 30 is not divisible by 4, so $n = 30$ is not a solution. If $m > 2$, then m is also a divisor, so $d_3 = m$, which gives

$$n = 1^2 + 2^2 + m^2 + (2m)^2 = 5 + 5m^2.$$

Since $m \mid n$, we see that $m \mid 5$, whence $m = 5$, yielding $n = 130$, which can be verified to be a solution.

Thus, the only solution is $n = 130$, and that is our answer.

Problem 7

Source: Adapted from 2018 SMO Junior Round 2 Q3

Let $\mathcal{P}_<$ be the set of all primes between 1 and 5000, and let $\mathcal{P}_>$ be the set of all primes between 5000 and 10,000, along with the number 1. Call also the two boxes Box A and Box B.

Observe that if n is in Box A, then all its factors (and multiples) must also be in Box A. Without loss of generality, suppose 2 is in Box A. Let $p \in \mathcal{P}_<$. Then $2p < 10,000$ and has a factor of 2, so it must be in Box A. But p is also a factor of $2p$, so p must be in Box A as well. From this, it follows that any integer that has a prime factor in $\mathcal{P}_<$ must be in Box A. Thus, the only integers not guaranteed to be in Box A are precisely the elements of $\mathcal{P}_>$. Since $|\mathcal{P}_>| = 561$, it follows that there are $2^{561} - 1$ ways to distribute primes in $\mathcal{P}_>$ among the two boxes. Note that we subtract one as we cannot distribute all primes to Box A, since Box B would be empty which is not allowed.

Problem 8

Source: Adapted from 2022 SMO Open Round 2 Q4

We prove a more general result, where Alice starts of with n cards with a label offset of k. We claim that Alice requires a minimum of $n - \gcd(n, k)$ moves.

Consider a graph over vertices labelled 1 through n. If the card at position i has label j, draw a directed edge from i to j. Since each vertex has indegree 1 and outdegree 1, the graph is composed of disjoint cycles. We now consider the effect of swapping two cards (say, at positions i and j) on our graph.

Claim 3. *If the cards at i and j were initially in the same cycle, then the cycle will split into two cycles upon swapping the two cards.*

Proof. Without loss of generality, let the cycle that i and j are in be

$$(i', i, i'', \underbrace{\ldots}_{I}, j', j, j'', \underbrace{\ldots}_{J}).$$

Consider the effect of swapping i and j on the cycle. Since the label on the card at position i' is still i, we see that i' still maps to i. Similarly, $j' \to j$. However, the card at position j (originally at position i) now has the label i'', hence we now have $j \to i''$. Similarly, $i \to j''$. It is hence easy to see that the cycle now splits as

$$(i', i, j'', \underbrace{\ldots}_{J})(j', j, i'', \underbrace{\ldots}_{I}).$$

□

Claim 4. *If the cards at i and j were initially in different cycles, then the two cycles will merge upon swapping the two cards.*

Proof. Using a similar argument as Claim 3, the cycles

$$(i', i, i'', \underbrace{\ldots}_{I})$$

and

$$(j', j, j'', \underbrace{\ldots}_{J})$$

will merge into a single cycle

$$(i', i, j'', \underbrace{\ldots}_{J}, j', j, i'', \underbrace{\ldots}_{I})$$

upon swapping positions i and j. □

From Claims 3 and 4, it follows that every move, the number of cycles increases by at most one. We now show that the initial number of cycles is $\gcd(n, k)$.

Claim 5. *The number of cycles is initially* $\gcd(n, k)$.

Proof. Let $D = \{d \in \mathbb{Z}_n \mid 1 \leq d \leq \gcd(n, k)\}$. Let $C(m)$ be the cycle starting from some integer m. Then $C(m)$ is clearly of the form

$$(m, m + k, m + 2k, m + 3k, \ldots, m + (l - 1)k),$$

where $l = n/\gcd(n, k)$ is the smallest positive integer such that $lk \equiv 0 \pmod{n}$.

Consider $C(m)$ for any choice of m. Since $k \equiv 0 \pmod{\gcd(n,k)}$, it follows that each member of $C(m)$ is congruent to $m \pmod{\gcd(n,k)}$. In addition, from the minimality of l, all elements of $C(m)$ are distinct. Thus, there is exactly one $d \in D$ that is also in $C(m)$ (namely, $d \equiv m \pmod{\gcd(n,k)}$). Conversely, each $d \in D$ has a unique cycle m that it is a member of. Hence, the number of cycles is $\gcd(n,k)$ as desired. □

Since there are n cycles when all cards are in their correct position, Alice must make at least $n - \gcd(n,k)$ moves. We now construct a strategy that guarantees Alice can indeed win in $n - \gcd(n,k)$ moves.

Claim 6. *Alice can win in $n - \gcd(n,k)$ moves.*

Proof. Let all cards with labels $1, 2, \ldots, \gcd(n,k)$ be red, and all other cards be blue. By Claim 5, each cycle initially contains exactly one red card. For each cycle, keep swapping the red card with the card that is pointing towards it. Doing so removes one blue card every move. Since the cycle has length $n/\gcd(n,k)$, each cycle requires $n/\gcd(n,k) - 1$ moves to completely sort it. Since there are $\gcd(n,k)$ cycles to sort, Alice can sort all cycles within $n - \gcd(n,k)$ moves, as desired. □

In our problem, $n = 2025^{2025}$ and $k = 2025!$. A simple application of Legendre's formula gives $2025! = 3^{1010} 5^{505} \ldots$, so the answer is $2025^{2025} - 3^{1010}\, 5^{505}$.