On the Critical Density of Minesweeper Boards

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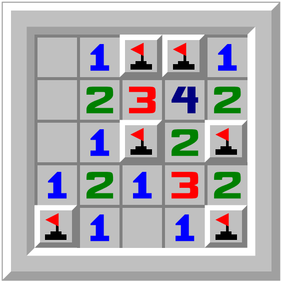
1 Introduction

1 Introduction: What is Minesweeper?

- 1 player logic puzzle video game
- Mines are randomly distributed among cells
- Safe cell opened: number of mined adjacent cells revealed
- Mined cell opened: player loses
- Objective: open all safe cells



1.1 Research Problem

We investigate the relationship between the **critical density** and the dimensions of a Minesweeper board.

Critical Density

The maximum density of mines (the ratio of the number of mines to the number of cells) that allows the player to win the game with certainty, assuming perfect play.

Paper	Critical Density	Limitations
Dempsey and Guinn	0.20 - 0.30	No guesses taken, only select square boards considered.
Sinha et. al	0.20 - 0.30	Focus of paper was not on critical density.
Qing et. al	0.2035 ± 0.002	Definition of "critical density" differs from ours.

1.4 Claim of Originality

- All lemmas, propositions and proofs presented are original.
- Definitions, unless otherwise stated, are also original.

2 Preliminaries

Definition 1.

A Minesweeper board B is given by the 6-tuple

$$B = (\mathcal{D}, \mathcal{A}, \mathcal{M}, \mathcal{S}, M, S) \tag{2.1}$$

where:

- $\mathcal{D}$ represents the dimensions of B,
- $\mathcal{A}$ is the set of cells on B,
- $\mathcal{M}$ and $\mathcal{S}$ represent the assigned state of B, and
- M and S represent the known state of B.

Definition 2.

The dimensions $\mathcal{D}$ of a board is defined as

$$\mathcal{D} = (p, q, m) \tag{2.2}$$

where $p, q \in \mathbb{N}$ are the length and width of the board respectively and $m \in \mathbb{N}_0$ is the number of mines on the board.

Notation	Description	Definition No.
$\mathcal{A}$	Set of **all** cells	3
$\mathcal{M}$	Set of all **mined** cells	4
$\mathcal{S}$	Set of all **safe** cells	5
M	Set of all **known mined** cells	6
S	Set of all **known safe** cells	7
U	Set of all **unknown** cells	8

2.1 Minesweeper Board: Adjacent Cells

Notation	Description	Definition No.
$K(a)$	Set of all cells adjacent to a	9
$K_X(a)$	Set of all cells adjacent to a that are also in X	10
$N(a)$	Number of mined cells adjacent to a	11

Definition 12.

The set $\mathcal{B} \subseteq S$ is defined as

$$\mathcal{B} = \{a \in S \colon |K_U(a)| > 0\} \tag{2.10}$$

Elements of $\mathcal{B}$ are termed "boundary cells".

Definition 13.

Given two boards B_1 and B_2, we say B_2 is a continuation of B_1, denoted $B_1 \Rightarrow B_2$, if the following statements hold.

$$\mathcal{D}_1 = \mathcal{D}_2 \tag{2.11}$$

$$M_1 \subseteq M_2 \tag{2.12}$$

$$S_1 \subseteq S_2 \tag{2.13}$$

3 Inference Algorithm

- Solving a Minesweeper board is equivalent to solving a constraint satisfaction problem (CSP).
- Information about the board can be converted into constraints via the indicator function of $\mathcal{M}$.

Definition 15.

The indicator function of $\mathcal{M}$ is the function $\mathbf{1}_{\mathcal{M}}\colon \mathcal{A} \to \{0, 1\}$ such that

$$\mathbf{1}_{\mathcal{M}}(a) = \begin{cases} 0, & a \notin \mathcal{M} \\ 1, & a \in \mathcal{M} \end{cases} \tag{3.1}$$

Proposition 1.

$|X \cap \mathcal{M}| = \sum_{a \in X} \mathbf{1}_{\mathcal{M}}(a)$ *for all* $X \subseteq \mathcal{A}$.

Definition 16.

The local constraint of a cell $a \in S$ is defined as

$$N(a) = \sum_{b \in K(a)} \mathbf{1}_{\mathcal{M}}(b) \tag{3.3}$$

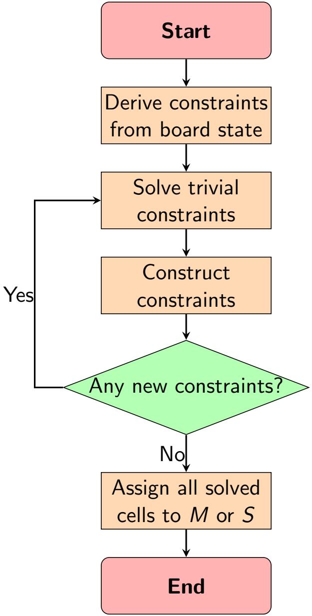
Definition 17.

The global constraint of a board with m mines is defined as

$$m = \sum_{a \in \mathcal{A}} \mathbf{1}_{\mathcal{M}}(a) \tag{3.4}$$

$$a \in \begin{cases} M, & \mathbf{1}_{\mathcal{M}}(a) = 1 \\ S, & \mathbf{1}_{\mathcal{M}}(a) = 0 \\ U, & \mathbf{1}_{\mathcal{M}}(a) \text{ unconstrained} \end{cases} \tag{3.5}$$

3.2 Inference Algorithm: Sketch



4 Guessing Algorithm

Definition 25.

The set of all boards that continue a board B is denoted $\overset{\Rightarrow}{B}$.

Definition 26.

The set $E \subseteq U$ is defined as

$$E = \{a \in U \colon |K_{\mathcal{B}}(a)| > 0\} \tag{4.1}$$

Elements of E are termed "exposed cells".

Definition 27.

The set $F \subseteq U$ is defined as

$$F = \{a \in U \colon |K_{\mathcal{B}}(a)| = 0\} \tag{4.2}$$

Elements of F are termed "floating cells".

4.1 Safety

Definition 28.

The safety of a cell $a \in U$ is denoted $P_S(a)$ and is defined as

$$P_S(a) = \mathbb{P}(a \in \mathcal{S}) \tag{4.3}$$

$$P_S(a) = \frac{|\{B' \in \overset{\Rightarrow}{B} : a \in \mathcal{S}'\}|}{|\overset{\Rightarrow}{B}|} \tag{4.4}$$

- A configuration C of a board B is a unique way to "assign" all cells in E into either $\mathcal{M}$ or $\mathcal{S}$.
- The weight W_C of a configuration C is the number of unique ways to "assign" the remaining mines into the remaining unknown cells.

$$|\overset{\Rightarrow}{B}| = \sum_C W_C \tag{4.5}$$

Definition 30.

A proper configuration C of a board B is defined by the tuple

$$C = (C_M, C_S) \tag{1}$$

such that

- C_M and C_S are a partition of E
- there exists a board $B' \in \overset{\Rightarrow}{B}$ such that $C_M \subseteq \mathcal{M}'$ and $C_S \subseteq \mathcal{S}'$.

Definition 32.

The set of all proper configurations of a board B is denoted $\mathrm{Con}_P(B)$.

Definition 33.

Given a proper configuration $C \in \mathrm{Con}_P(B)$, its weight W_C is defined as

$$W_C = |\{B' \in \overset{\Rightarrow}{B} \colon C_M \subseteq \mathcal{M}' \wedge C_S \subseteq \mathcal{S}\}| \tag{4.7}$$

Proposition 4.

Given a proper configuration $C \in \mathrm{Con}_P(B)$, its corresponding weight $W_C = \binom{|F|}{m-|M|-|C_M|}$

Key Observation

$$a \in \mathcal{S}' \text{ for some } B' \in \overset{\Rightarrow}{B} \iff a \in C_S \text{ for some } C \in \mathrm{Con}_P(B)$$

$$\left|\{B' \in \overset{\Rightarrow}{B} \colon a \in \mathcal{S}'\}\right| = \sum_{\substack{C \in \mathrm{Con}_P(B) \\ a \in S_C}} W_C \tag{4.9}$$

$$\begin{aligned} P_S(a) &= \frac{\left|\{B' \in \overset{\Rightarrow}{B} \colon a \in \mathcal{S}'\}\right|}{|\overset{\Rightarrow}{B}|} \\ &= \sum_{\substack{C \in \mathrm{Con}_P(B) \\ a \in S_C}} W_C \Bigg/ \sum_{C \in \mathrm{Con}_P(B)} W_C \end{aligned} \tag{4.10}$$

Proposition 5, 6.

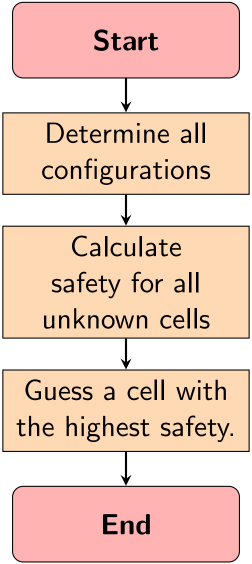
$\mathbb{E}[|F \cap \mathcal{M}|] = m - |M| - \sum_{a \in E} \mathbb{P}(a \in \mathcal{M})$

$$\mathbb{E}[|\overset{\Rightarrow}{B}|] = \binom{|F|}{\mathbb{E}[|F \cap \mathcal{M}|]} \tag{4.19}$$

$$\mathbb{E}[|\{B' \in \overset{\Rightarrow}{B} : a \in \mathcal{S}'\}|] = \binom{|F| - 1}{\mathbb{E}[|F \cap \mathcal{M}|]} \tag{4.20}$$

$$\begin{aligned} P_S(a) &= \frac{|\{B' \in \overset{\Rightarrow}{B} : a \in \mathcal{S}'\}|}{|\overset{\Rightarrow}{B}|} \\ &= \binom{|F| - 1}{\mathbb{E}[|F \cap \mathcal{M}|]} \Bigg/ \binom{|F|}{\mathbb{E}[|F \cap \mathcal{M}|]} \qquad (4.21, 4.22) \\ &= 1 - \frac{\mathbb{E}[|F \cap \mathcal{M}|]}{|F|} \end{aligned}$$

4.2 Guessing Algorithm: Sketch



5 First Click

- First click is guaranteed safe
- Only three distinct cells: corner, edge, center
- Differ only by $|K(a)|$ (3, 5, 8 respectively)

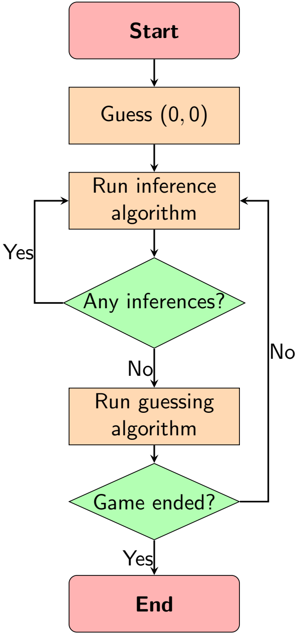
Proposition 7.

For distinct cells $a, b \in F$, if $|K(a)| > |K(b)|$, then $\mathbb{P}(N(a) = 0) \leq \mathbb{P}(N(b) = 0)$.

- Smallest $|K(a)|$ achieved at corners ($|K(a)| = 3$) $\Rightarrow$ starting in a corner is optimal.
- Always start at (0, 0)

6 Solver

6 Solver: Sketch



7 Critical Density

Definition 34.

The mine density ρ of a Minesweeper board with dimensions (p, q, m) is given by

$$\rho = \frac{m}{pq} \tag{7.1}$$

Definition 35.

The win rate of a Minesweeper board is denoted W.

- Consider a board B with dimensions (p, q, m)
- Plot W as ρ varies from 0 to 1
- For low mine densities, W will be quite high, while for high mine densities, W will be quite low
- Model the plotted curve with the logistic function

Definition 36.

The logistic function $L\colon [0, 1] \to \mathbb{R}$ is defined as

$$L(\rho) = \frac{L}{1 + e^{k(\rho - Q)}} + b \tag{7.2}$$

for some constants $L, k, Q, b \in \mathbb{R}$ such that $L, k > 0$.

- The critical density P is indicated by a sharp drop in the plot of W against ρ
- Hence, we define P to be the mine density at which the minimum of $L'(\rho)$ is achieved

Definition 37.

The critical density P of a Minesweeper board is defined as

$$P = \arg\min_{\rho} L'(\rho) \tag{7.3}$$

Proposition 8.

$L''(\rho) = 0 \iff \rho = Q.$

$$P = \begin{cases} 0, & Q < 0 \\ Q, & 0 \leq Q \leq 1 \\ 1, & Q > 1 \end{cases} \tag{7.6}$$

8 Methodology

8.1 W-correction

- As ρ tends towards 1, W suddenly shoots up to 1
- Due to sheer luck: only realistic way of winning is to correctly guess the remaining $pq - m - 1$ safe cells at random.
- Approximate win rate of

$$\prod_{k=1}^{pq-m-1} \frac{pq - m - 1 - k}{pq - k}$$

.

Strategy 1.

If the win rate of a $p \times q \,/\, m$ board has been simulated to be 0, then we take the win rate of a $p \times q \,/\, m'$ board for all $m' > m$ to also be 0.

- Works well against larger boards
- Fails against smaller boards (not enough possible values of m to allow W to fall to 0)

Strategy 2.

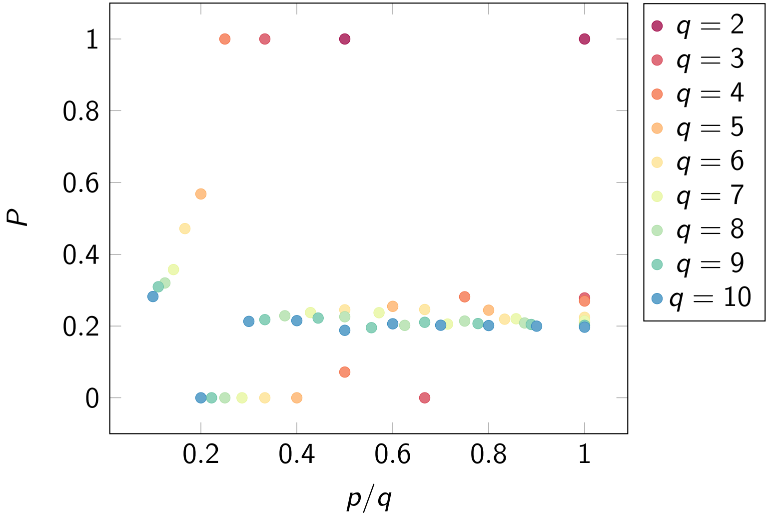
Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two board dimensions such that $m_1 < m_2$. If $\rho_1 > 0.75$ and $W_1 < W_2$, we immediately take W_2 to be 0.

- Stops the win rate from shooting up before it can happen

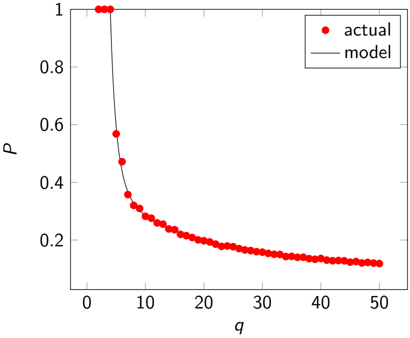
1. Generate 1000 Minesweeper boards with dimensions (p, q, m) for all $0 < m < pq$.
2. Solve boards using our Minesweeper solver
3. Record win rate for each m while enforcing W-correction.
4. Plot W against ρ.
5. Use Powell's dog-leg algorithm to get the parameters of the best-fit logistic function.
6. Calculate P from Q using Equation 7.6.

$$P = \begin{cases} 0, & Q < 0 \\ Q, & 0 \leq Q \leq 1 \\ 1, & Q > 1 \end{cases} \tag{7.6}$$

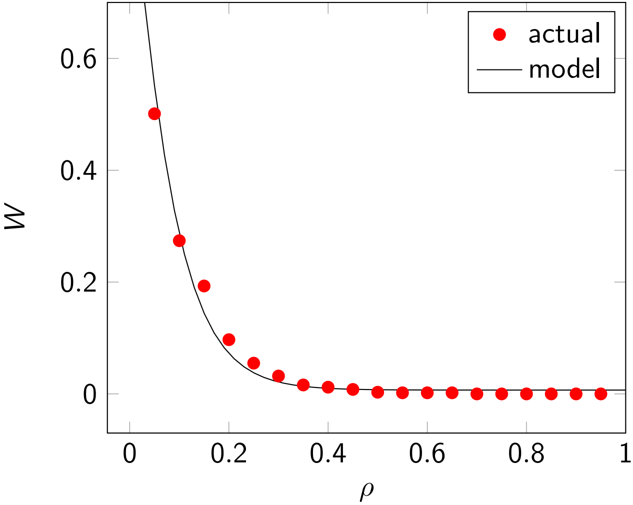
9 Results and Analysis


1
0.8
0.6
0.4
0.2
0
P
0.2
0.4
0.6
0.8
1
p/q
q = 2
q = 3
q = 4
q = 5
q = 6
q = 7
q = 8
q = 9
q = 10

- P is inversely related to q
- Motivates us to calculate the critical densities of all $1 \times q$ Minesweeper board with $q \leq 50$

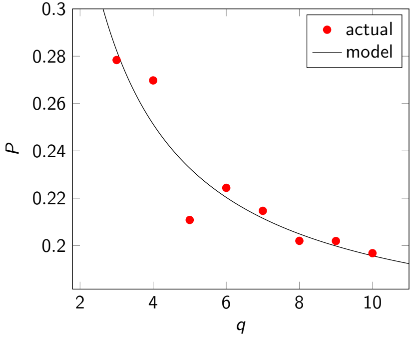


9.2 $p = 2$ Trend



9.3 $p > 2$ Trend

- Critical density P remains stable at 0.20 regardless of aspect ratio
- P decreases as q increases
- Suggests that there exists some asymptotic critical density $P^\star = \lim_{q\to\infty} P$ that is the same for all aspect ratios
- To approximate $P^\star$, we plot P for all $q \times q$ boards and fit the resulting plot to a reciprocal function.



$P^\star \approx 0.159$

10 Conclusion

We have...

- developed an original Minesweeper solver
- created an original algorithm to calculate critical density
- calculated the critical density of all board sizes smaller than 10×10
- analysed our results and concluded that the critical density converges to 0.159 for all board sizes where $p > 2$ and 0 for $p \leq 2$, respectively.

- Metric for difficulty of Minesweeper boards
- Allows players to improve and train efficiently while having fun

10.2 Limitations

- Our solver is not as accurate as more conventional solvers
- Stronger solver $\Rightarrow$ more accurate calculation of $P^\star$
- Explains why our calculated value for $P^\star$ is significantly lower than the 0.20 observed by other studies

10.3 Future Directions

- Minesweeper variants
  - The lower the asymptotic critical density, the more complex the variant
- Different definitions for critical density
  - Time taken to solve a board
  - Average number of guesses

Thank you!