Particular Values of the Alternating Tornheim Series

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1 Introduction

In 1985, Subbarao and Sitaramachandrarao[2] introduced an alternating analogue to Tornheim's double series given by

$$S(a,b,c) = \sum_{m,n\geq 1} \frac{(-1)^{m+n}}{m^a n^b (m+n)^c}.$$

Here, the parameters a, b and c are non-negative integers. Note that the double series is symmetric in m and n, so $S(a,b,c) = S(b,a,c)$.

In this note, we evaluate all convergent $S(a,b,c)$ with parameters $a,b,c \in \{0,1\}$ using purely elementary methods. Our results are summarized below:

$$\text{(A)} \qquad S(1,1,0) = \ln^2 2, \qquad S(1,0,1) = \frac{1}{2}\ln^2 2, \qquad \text{(C)}$$

$$\text{(B)} \qquad S(0,0,1) = \ln 2 - \frac{1}{2}, \qquad S(1,1,1) = \frac{1}{4}\zeta(3). \qquad \text{(D)}$$

Quite clearly, $S(0,0,0)$ and $S(1,0,0)$ are divergent and are hence omitted.

2 Evaluating $S(1,1,0)$

$S(1,1,0)$ is perhaps the easiest of the four double series to evaluate.

Theorem A. $S(1,1,0) = \ln^2 2$.

Proof. By definition, we have

$$S(1,1,0) = \sum_{m,n\geq 1} \frac{(-1)^{m+n}}{mn} = \sum_{m\geq 1} \frac{(-1)^{m-1}}{m} \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}.$$

Recognizing each sum as the series expansion of $\ln(1+x)$ at $x=1$ finishes the proof. □

In general, $S(a,b,0)$ can easily be evaluated using the Dirichlet η function. Indeed, one can trivially show that

$$S(a,b,0) = \eta(a)\,\eta(b).$$

This converges if and only if a and b are positive integers.

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3 Evaluating $S(0,0,1)$

Theorem B. $S(0,0,1) = \ln 2 - \frac{1}{2}$.

Proof. We have

$$\begin{aligned}
S(0,0,1) &= \sum_{m,n\geq 1} \frac{(-1)^{m+n}}{m+n} \\
&= \sum_{m,n\geq 1} (-1)^{m+n} \int_0^1 x^{m+n-1}\,\mathrm{d}x \\
&= \int_0^1 x \sum_{m\geq 1}(-x)^{m-1} \sum_{n\geq 1}(-x)^{n-1}\,\mathrm{d}x \\
&= \int_0^1 \frac{x}{(1+x)^2}\,\mathrm{d}x \\
&= \ln 2 - \frac{1}{2},
\end{aligned}$$

where the integral in the second-last line can be evaluated with the transformation $1 + x \mapsto x$. ◻

We can use this result to "evaluate" Grandi's series.

Corollary 1. $1 - 1 + 1 - 1 + \cdots$ "=" $\frac{1}{2}$.

Proof. We have

$$\ln 2 - \frac{1}{2} = \sum_{m,n\geq 1} \frac{(-1)^{m+n}}{m+n}.$$

Let $k = m + n \geq 2$. Noting that $m + n = k$ has $k - 1$ solutions over $\mathbb{N}$, we have

$$\ln 2 - \frac{1}{2} = \sum_{k\geq 2} \frac{(-1)^k(k-1)}{k} = \sum_{k\geq 2}(-1)^k + \left(-1 + \sum_{k\geq 1} \frac{(-1)^{k-1}}{k}\right).$$

The first term is Grandi's series, while the second term evaluates to $-1 + \ln 2$. Thus,

$$\sum_{k\geq 2}(-1)^k = 1 - 1 + 1 - 1 + \cdots \text{ “=” } \frac{1}{2}.$$

◻

4 Evaluating $S(1,0,1)$

Theorem C. $S(1,0,1) = \frac{1}{2}\ln^2 2$.

Proof. We have

$$\begin{aligned}
S(1,0,1) &= \sum_{m,n\geq 1} \frac{(-1)^{m+n}}{m(n+m)} \\
&= \sum_{m,n\geq 1} \frac{(-1)^{m+n}}{m} \int_0^1 x^{m+n-1}\,\mathrm{d}x \\
&= \int_0^1 \sum_{m\geq 1} \frac{(-1)^{m-1}x^m}{m} \sum_{n\geq 1}(-x)^{n-1}\,\mathrm{d}x \\
&= \int_0^1 \frac{\ln(1+x)}{1+x}\,\mathrm{d}x.
\end{aligned}$$

Denote the above integral by I. Integrating by parts reveals that $I = \ln^2 2 - I$, so

$$S(1,0,1) = I = \frac{1}{2}\ln^2 2$$

as desired. $\square$

5 Evaluating $S(1,1,1)$

We begin by proving several integral identities.

Proposition 2. *Let α and β be non-negative integers. Then*

$$\int_0^1 x^\alpha \ln^\beta x \, \mathrm{d}x = \frac{(-1)^\beta \beta!}{(\alpha+1)^{\beta+1}}.$$

Proof. Define

$$I(\beta) = \int_0^1 x^\alpha \ln^\beta x \, \mathrm{d}x.$$

Integrating by parts yields the recurrence relation

$$I(\beta) = -\frac{\beta}{\alpha+1} I(\beta-1).$$

With the initial condition $I(0) = \frac{1}{\alpha+1}$, we immediately get the desired claim. $\square$

Proposition 3. *We have*

$$\int_0^1 \frac{\ln^2(1+x)}{x} \, \mathrm{d}x = \frac{1}{4}\zeta(3).$$

The following proof is taken from [1] (with typos corrected).

Proof. Recall the identity $2a^2+2b^2 = (a+b)^2+(a-b)^2$. Taking $a = \ln(1-x)$ and $b = \ln(1+x)$ and rearranging, we have

$$\ln^2(1+x) = \frac{1}{2}\ln^2(1-x^2) + \frac{1}{2}\ln^2\left(\frac{1-x}{1+x}\right) - \ln^2(1-x).$$

Dividing by x and integrating over $(0,1)$, we have

$$\begin{aligned}\int_0^1 \frac{\ln^2(1+x)}{x} \, \mathrm{d}x &= \frac{1}{2}\underbrace{\int_0^1 \frac{\ln^2(1-x^2)}{x} \, \mathrm{d}x}_{1-x^2 \mapsto x} + \frac{1}{2}\underbrace{\int_0^1 \frac{\ln^2\left(\frac{1-x}{1+x}\right)}{x} \, \mathrm{d}x}_{\frac{1-x}{1+x} \mapsto x} - \underbrace{\int_0^1 \frac{\ln^2(1-x)}{x} \, \mathrm{d}x}_{1-x \mapsto x} \\ &= \frac{1}{4}\int_0^1 \frac{\ln^2 x}{1-x} \, \mathrm{d}x + \int_0^1 \frac{\ln^2 x}{1-x^2} \, \mathrm{d}x - \int_0^1 \frac{\ln^2 x}{1-x} \, \mathrm{d}x.\end{aligned}$$

The latter two integrals can be combined and simplified, giving

$$\int_0^1 \frac{\ln^2 x}{1-x^2} \, \mathrm{d}x - \int_0^1 \frac{\ln^2 x}{1-x} \, \mathrm{d}x = -\int_0^1 \frac{x \ln^2 x}{1-x^2} \, \mathrm{d}x = -\frac{1}{8}\int_0^1 \frac{\ln^2 x}{1-x} \, \mathrm{d}x,$$

where we applied the transformation $x^2 \mapsto x$ to obtain the last step. Altogether, the target integral becomes

$$\int_0^1 \frac{\ln^2(1+x)}{x} \, \mathrm{d}x = \frac{1}{8}\int_0^1 \frac{\ln^2 x}{1-x} \, \mathrm{d}x.$$

Using the series expansion of $\frac{1}{1-x}$ and switching the order of summation and integration yields

$$\int_0^1 \frac{\ln^2(1+x)}{x}\,\mathrm{d}x = \frac{1}{8}\sum_{k\geq 0}\int_0^1 x^k \ln^2 x\,\mathrm{d}x.$$

By Proposition 2 and the definition of the ζ function, we finally arrive at the desired result:

$$\int_0^1 \frac{\ln^2(1+x)}{x}\,\mathrm{d}x = \frac{1}{8}\sum_{k\geq 0}\frac{2}{(k+1)^3} = \frac{1}{4}\zeta(3).$$

□

We now evaluate $S(1,1,1)$. As the result of Proposition 3 hints, we will show that $S(1,1,1)$ is equal to $\int_0^1 \frac{\ln^2(1+x)}{x}\,\mathrm{d}x$.

Theorem D. $S(1,1,1) = \frac{1}{4}\zeta(3)$.

Proof. We have

$$\begin{aligned} S(1,1,1) &= \sum_{m,n\geq 1}\frac{(-1)^{m+n}}{mn(m+n)} \\ &= \sum_{m,n\geq 1}\frac{(-1)^{m+n}}{mn}\int_0^1 x^{m+n-1}\,\mathrm{d}x \\ &= \int_0^1 \frac{1}{x}\sum_{n\geq 1}\frac{(-1)^{n-1}x^n}{n}\sum_{m\geq 1}\frac{(-1)^{m-1}x^m}{m}\,\mathrm{d}x \\ &= \int_0^1 \frac{\ln^2(1+x)}{x}\,\mathrm{d}x = \frac{1}{4}\zeta(3) \end{aligned}$$

as desired. □

In [5], Chen showed that for non-negative a, b and c,

$$S(a+1,b+1,c+1) = \sum_{d_1+d_2=a+b}\zeta(d_1+1,\overline{d_2+c+2})\left[\binom{d_2}{a}+\binom{d_2}{b}\right],$$

where the double-ζ function is defined as

$$\zeta(\alpha,\overline{\beta}) = \sum_{1\leq m<n}\frac{(-1)^n}{m^\alpha n^\beta}.$$

In the particular case of $S(1,1,1)$, we see that

$$\sum_{1\leq m<n}\frac{(-1)^n}{mn^2} = \frac{1}{8}\zeta(3).$$

Elementary manipulation gives the identity

$$\sum_{n=2}^{\infty}\frac{(-1)^n H_{n-1}}{n^2} = \frac{1}{8}\zeta(3).$$

Using this, we deduce the following inequality.

Proposition 4. *We have*

$$\eta'(2) = \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n^2} < \frac{\zeta(2) - \zeta(3)}{4}.$$

Proof. Using the trapezium rule on $1/x$ over the interval $1 \le x \le n$ yields the inequality

$$H_{n-1} > \ln n + \frac{1}{2} - \frac{1}{2n}.$$

Thus,

$$\frac{1}{8}\zeta(3) = \sum_{n\ge 2} \frac{(-1)^n H_{n-1}}{n^2} > \sum_{n\ge 2} \frac{(-1)^n \ln n}{n^2} - \frac{1}{2}\sum_{n\ge 2} \frac{(-1)^{n-1}}{n^2} + \frac{1}{2}\sum_{n\ge 2} \frac{(-1)^{n-1}}{n^3}.$$

We recognize the last two sums as $\eta(2) - 1$ and $\eta(3) - 1$ respectively, so

$$\sum_{n\ge 2} \frac{(-1)^n \ln n}{n^2} < \frac{1}{8}\zeta(3) + \frac{1}{2}\eta(2) - \frac{1}{2}\eta(3).$$

Using the relationship $\eta(s) = (1 - 2^{1-s})\zeta(s)$, we obtain our desired result. □

References

[1] Ali Olaikhan. Ways to prove $\int_0^1 \frac{\ln^2(1+x)}{x} dx = \frac{\zeta(3)}{4}$? Mathematics Stack Exchange. `https://math.stackexchange.com/q/4391213`. Accessed 2025-09-19.

[2] M. V. Subbarao and R. Sitaramachandrarao. On Alternating Analogues of Tornheim's Double Series. *Pacific Journal of Mathematics of the American Mathematical Society*, 119(1):245–255, 1985.

[3] L. Tornheim. Harmonic double series. *American Journal of Mathematics*, 72(2):303–314, 2025.

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