Finite Orders of the Möbius Transformation

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1 Introduction

Definition 1. A *Möbius transformation* is a function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by

$$f(z) = \frac{az+b}{cz+d},$$

where $z \in \mathbb{C}$, and a, b, c and d are complex constants with $ad - bc \neq 0$.

We denote $\mathcal{M}$ to be the set of all Möbius transformations. It is readily seen that $\mathcal{M}$ forms a group when equipped with function composition.[1]

Definition 2. The *period* of a Möbius transformation f is the least positive integer m such that $f^m = \mathrm{id}_{\mathbb{C}_\infty}$. If m does not exist, then f has infinite period.

In this note, we prove the following result:[2]

Theorem A. *A Möbius transformation $f(z) = (az+b)/(cz+d)$ has period 2 if and only if $a + d = 0$.*

In the following sections, we present two proofs of the above theorem.

2 Elementary Algebra Proof

Proof of Theorem A. ($\Longrightarrow$) Suppose f has period 2. Then for all $z \in \mathbb{C}_\infty$, we have

$$f^2(z) = \frac{\left(a^2+bc\right)z + b\left(a+d\right)}{c\left(a+d\right)z + \left(bc+d^2\right)} = z \implies \left[cz^2 + (d-a)z - b\right](a+d) = 0. \qquad (*)$$

By the definition of a period, there exists some $w \in \mathbb{C}_\infty$ such that $f(w) \neq w$. Equivalently,

$$\frac{aw+b}{cw+d} \neq w \implies cw^2 + (d-a)w - b \neq 0.$$

Hence, from $(*)$, we must have $a + d = 0$.

($\Longleftarrow$) Suppose $a + d = 0$. For all $z \in \mathbb{C}_\infty$, we have

$$f^2(z) = \frac{\left(a^2+bc\right)z + b\left(a+d\right)}{c\left(a+d\right)z + \left(bc+d^2\right)} = \frac{\left(a^2+bc\right)z}{bc+(-a)^2} = z,$$

where we note that $bc + (-a)^2 = -(ad - bc) \neq 0$, so the reduction of the fraction is valid. Thus, $f^2 = \mathrm{id}_{\mathbb{C}_\infty}$. To complete the proof, we must show that $f \neq \mathrm{id}_{\mathbb{C}_\infty}$. Indeed, for

$$f(z) = \frac{az+b}{cz-a} = z \iff cz^2 - 2az - b = 0$$

to hold for all $z \in \mathbb{C}_\infty$, we require $c = 0$ and $a = 0$, which implies $ad - bc = 0$, a contradiction. □

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[1] As it turns out, $\mathcal{M}$ is precisely the automorphism group of $\mathbb{C}_\infty$.

[2] Adapted from Nanyang Junior College H3 Mathematics Preliminary Examination 2021.

3 Abstract Algebra Proof

Definition 3. We define the equivalence relation $\sim$ between two matrices $\mathbf{A}, \mathbf{B} \in \mathrm{GL}_2(\mathbb{C})$ as

$$\mathbf{A} \sim \mathbf{B} \iff (\exists \lambda \in \mathbb{C}^*)\, \mathbf{A} = \lambda \mathbf{B}.$$

The fact that $\sim$ indeed satisfies reflexivity, symmetry and transitivity is immediate.

Definition 4. The *projective linear group* $\mathrm{PGL}_2(\mathbb{C})$ is the group of 2×2 complex matrices up to scalar multiplication:

$$\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}) / \sim .$$

Proposition 5. *The projective linear group* $\mathrm{PGL}_2(\mathbb{C})$ *is isomorphic to the group of Möbius functions* $\mathcal{M}$.

The following proof is taken from [1].

Proof. Define $\varphi : \mathrm{GL}_2(\mathbb{C}) \to \mathcal{M}$ with mapping

$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{az+b}{cz+d}.$$

Let $\mathbf{A}, \mathbf{B} \in \mathrm{GL}_2(\mathbb{C})$ with

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Then

$$\varphi(\mathbf{A})\,\varphi(\mathbf{B}) = \frac{(aa' + bc')\,z + (ab' + bd')}{(ca' + dc')\,z + (cb' + dd')} = \varphi(\mathbf{AB}),$$

so φ is a group homomorphism. Further, φ is surjective, since every Möbius transformation $f(z) = (az+b)/(cz+d)$ can be identified with the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which lies in $\mathrm{GL}_2(\mathbb{C})$ by the condition that $ad - bc \neq 0$.

Next, suppose some $\mathbf{A} \in \mathrm{GL}_2(\mathbb{C})$ is in the kernel of φ. Then

$$\frac{az+b}{cz+d} \equiv z,$$

which immediately implies $b = c = 0$ and $a = d$, so $\mathbf{A}$ is a scalar multiple of the identity matrix:

$$\ker \varphi = \{\lambda \mathbf{I} : \lambda \in \mathbb{C}^*\} = [\mathbf{I}]\,.$$

Thus, by the First Isomorphism Theorem,

$$\mathcal{M} \cong \mathrm{GL}_2(\mathbb{C}) / \ker \varphi = \mathrm{PGL}_2(\mathbb{C}).$$

□

With the above result, we can rephrase our question on the periods of Möbius transformations to a question on the order of matrices in $\mathrm{PGL}_2(\mathbb{C})$.

Proposition 6. *Suppose* $f \in \mathcal{M}$ *is represented by* $\mathbf{A} \in \mathrm{GL}_2(\mathbb{C})$. *Then the period of* f *is equal to the order of* $[\mathbf{A}]$ *in* $\mathrm{PGL}_2(\mathbb{C})$.

Proof. Recall that the period of f is the least positive integer m such that $f^m = \mathrm{id}_{\mathbb{C}_\infty}$. In the context of $\mathrm{PGL}_2(\mathbb{C})$, the identity is $[\mathbf{I}]$, while f^m corresponds to $[\mathbf{A}^m] = [\mathbf{A}]^m$. Translating the definition of order, we have that m is the least positive integer such that $[\mathbf{A}]^m = [\mathbf{I}]$. But this is precisely the definition of the order of an element in a group, so $\mathrm{ord}([\mathbf{A}]) = m$. □

Theorem A is thus equivalent to the following claim:

Theorem B. *Let* $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. *Then the order of* $[\mathbf{A}]$ *in* $\mathrm{PGL}_2(\mathbb{C})$ *is 2 if and only if* $\operatorname{tr} \mathbf{A} = 0$.

Proof. ($\Longrightarrow$) Suppose $[\mathbf{A}]$ has order 2. Then there exists some $\lambda \in \mathbb{C}^*$ such that $\mathbf{A}^2 = \lambda \mathbf{I}$. By the Cayley-Hamilton theorem, we have

$$\lambda \mathbf{I} = \mathbf{A}^2 = \operatorname{tr}(\mathbf{A})\,\mathbf{A} - \det(\mathbf{A})\,\mathbf{I} \iff \operatorname{tr}(\mathbf{A})\,\mathbf{A} = (\lambda + \det(\mathbf{A}))\,\mathbf{I}.$$

But $[\mathbf{A}] \neq [\mathbf{I}]$, which forces $\operatorname{tr}(\mathbf{A}) = 0$ as desired.

($\Longleftarrow$) Suppose $\operatorname{tr} \mathbf{A} = 0$. We must have $[\mathbf{A}] \neq [\mathbf{I}]$, since $\mathbf{A} = \lambda \mathbf{I}$ for some $\lambda \in \mathbb{C}^*$ would force $\mathbf{A} = \mathbf{0} \notin \mathrm{GL}_2(\mathbb{C})$, a contradiction. Next, by the Cayley-Hamilton theorem, we have

$$\mathbf{A}^2 = -\det(\mathbf{A})\,\mathbf{I}.$$

Since $\det(\mathbf{A})$ is non-zero, we have $[\mathbf{A}]^2 = [\mathbf{I}]$, thus $\mathrm{ord}(\mathbf{A}) = 2$. □

4 A Generalization to Higher Orders

Theorem C. *The equivalence class* $[\mathbf{A}] \in \mathrm{PGL}_2(\mathbb{C})$ *has finite order* $m > 1$ *if and only if*

$$\frac{\operatorname{tr}(\mathbf{A})^2}{\det(\mathbf{A})} = 4\cos^2 \frac{\pi k}{m},$$

where k *is an integer coprime to* m.

Note that $\operatorname{tr}(\mathbf{A})^2 / \det(\mathbf{A})$ is invariant under scalar multiplication, hence it is indeed meaningful to associate the equivalence class $[\mathbf{A}]$ with it.

Proof. ($\Longrightarrow$) Suppose $[\mathbf{A}]$ has finite order $m \geq 1$. Then m is the least positive integer such that there exists some $\lambda \in \mathbb{C}^*$ such that $\mathbf{A}^m = \lambda \mathbf{I}$. Let λ_1, λ_2 be the eigenvalues of $\mathbf{A}$. Then $\lambda_1^m = \lambda_2^m = \lambda$, so the ratio $\zeta = \lambda_1/\lambda_2$ is a primitive mth root of unity. Write $\zeta = \exp(2\pi \mathrm{i} k/m)$, where k is an integer coprime to m. Then

$$\frac{\operatorname{tr}(\mathbf{A})^2}{\det(\mathbf{A})} = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2} = \frac{(\zeta + 1)^2 \lambda_2^2}{\zeta \lambda_2^2} = \zeta + 2 + \zeta^{-1} = 2\left(1 + \cos\frac{2\pi k}{m}\right) = 4\cos^2\frac{\pi k}{m}.$$

($\Longleftarrow$) Suppose

$$\frac{\operatorname{tr}(\mathbf{A})^2}{\det(\mathbf{A})} = 4\cos^2 \frac{\pi k}{m}$$

for some integers m and k with $m \geq 1$ coprime to k. Write $r = \sqrt{\det(\mathbf{A})} \neq 0$ and $\theta = \pi k/m$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$. Then λ_1 and λ_2 satisfy the quadratic

$$x^2 - \operatorname{tr}(\mathbf{A})\,x + \det(\mathbf{A}) = 0,$$

which can be rewritten as

$$x^2 - (2r\cos\theta)\,x + r^2 = 0,$$

whence it is obvious that $\lambda_1 = r\mathrm{e}^{\mathrm{i}\theta}$ and $\lambda_2 = r\mathrm{e}^{-\mathrm{i}\theta}$. Since k and m are coprime, the eigenvalues are distinct, so $\mathbf{A}$ is diagonalizable as

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} r\mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & r\mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} \mathbf{P}^{-1}.$$

Taking mth powers, noting that $\theta m = k\pi$, we see that

$$\mathbf{A}^m = \mathbf{P} \begin{pmatrix} (-1)^k r^m & 0 \\ 0 & (-1)^k r^m \end{pmatrix} \mathbf{P}^{-1} = (-1)^k r^m \mathbf{I},$$

whence $[\mathbf{A}]^m = [\mathbf{I}]$. Because k and m are coprime, m is minimal, thus $[\mathbf{A}]$ has order m. □

Remark. If $m = 1$, the forwards direction holds, but the backwards direction does not. This is because we have $k = 1$, so $\theta = \pi$ and the two eigenvalues are identical. This gives rise to the possibility that $\mathbf{A}$ is not diagonalizable, which would imply that $\operatorname{ord}([\mathbf{A}]) = \infty$ and not 1 as desired.

Example 7. Note that $k = 1$ is the only (positive) integer coprime to $m = 2$. Thus, Theorem C asserts that $[\mathbf{A}]$ has order 2 if and only if

$$\frac{\operatorname{tr}(\mathbf{A})^2}{\det(\mathbf{A})} = 4\cos^2\frac{\pi}{2} = 0,$$

whence $\operatorname{tr}(\mathbf{A}) = 0$, recovering Theorem B.

References

[1] John Olsen. The Geometry of Möbius Transformations, 2010. Accessed: 11 July 2025.