1 Preamble

The AM-GM inequality and Bernoulli's inequality are two classical results in real analysis.

(AG) AM-GM Inequality. *For* $x_1, \ldots, x_n \geq 0$*,*

$$\frac{x_1 + \cdots + x_n}{n} \geq (x_1 \ldots x_n)^{1/n}.$$

(B) Bernoulli's Inequality. *For* $n \in \mathbb{N}$ *and* $x > -1$*,*

$$1 + \frac{x}{n} \geq (1+x)^{1/n}.$$

It is easily shown that (AG) and (B) are equivalent.

By considering the concavity of $\ln x$, we can derive a generalized inequality that interpolates (B):

(GB) Generalized Bernoulli's Inequality. *For* $n \in \mathbb{N}$ *and* $x > -1$*,*

$$1 + \frac{x}{n} \geq \left(1 + \frac{2x}{n}\right)^{1/2} \geq \left(1 + \frac{3x}{n}\right)^{1/3} \geq \cdots \geq \left(1 + \frac{nx}{n}\right)^{1/n}.$$

Definition. The kth elementary symmetric polynomial in n variables $x_1, \ldots, x_n$ is defined as

$$e_k(x_1, \ldots, x_n) = \sum_{\substack{I \subseteq [n] \\ |I| = k}} \prod_{i \in I} x_i,$$

where $1 \leq k \leq n$. We also define $e_0(x_1, \ldots, x_n) = 1$ and $e_{n+1}(x_1, \ldots, x_n) = 0$. The corresponding kth elementary symmetric mean in n variables is defined as

$$E_k(x_1, \ldots, x_n) = \frac{e_k(x_1, \ldots, x_n)}{\binom{n}{k}}.$$

For example,

$$E_3(x, y, z, w) = \frac{e_3(x, y, z, w)}{\binom{4}{3}} = \frac{xyz + yzw + zwx + wxy}{4}.$$

Motivated by the observation that

$$E_1(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n} \geq (x_1 \ldots x_n)^{1/n} = E_n(x_1, \ldots, x_n),$$

we can similarly interpolate (AG) to obtain the following inequality:

(M) Maclaurin's Inequality. *Let* $n \in \mathbb{N}$ *and* $x_1, \ldots, x_n \geq 0$*. Then*

$$E_1(x_1, \ldots, x_n) \geq E_2(x_1, \ldots, x_n)^{1/2} \geq E_3(x_1, \ldots, x_n)^{1/3} \geq \cdots \geq E_n(x_1, \ldots, x_n)^{1/n}.$$

Using the identity

$$E_k(x_1, \ldots, x_n) = \left(1 - \frac{k}{n}\right) E_k(x_1, \ldots, x_{n-1}) + \frac{k}{n} E_{k-1}(x_1, \ldots, x_{n-1})\, x_n,$$

we can prove that (M) and (GB) are equivalent.

2 Motivating Maclaurin's Inequality as a Generalization of the AM-GM Inequality

Let x and y be the side lengths of a rectangle. Can we construct a square that "best approximates" this rectangle? Of course, our construction depends on which quantity we wish to preserve.

- If we preserve the perimeter of the rectangle, the resulting square has side length

$$l_1 = \frac{x+y}{2},$$

which is the arithmetic mean.

- If we preserve the area of the rectangle, the resulting square has side length

$$l_2 = \sqrt{xy},$$

which is the geometric mean.

The AM-GM inequality hence gives

$$l_1 \geq l_2. \tag{$*$}$$

We can ask the same question for higher-dimensional analogues of rectangles. For instance, suppose we have a cuboid with side lengths x, y, z.

- If we preserve the perimeter, the resulting cube has side length

$$l_1 = \frac{x+y+z}{3}.$$

- If we preserve the total area of all faces, the resulting cube has side length

$$l_2 = \sqrt{\frac{xy+yz+zx}{3}}.$$

- If we preserve the total volume, the resulting cube has side length

$$l_3 = \sqrt[3]{xyz}.$$

Continuing the pattern in $(*)$, we have

$$l_1 \geq l_2 \geq l_3.$$

In general, for a n-dimensional orthotope with side lengths $x_1, \ldots, x_n$, we have

$$l_1 \geq l_2 \geq l_3 \geq \cdots \geq l_n,$$

where

$$l_k = \sqrt[k]{\frac{e_k(x_1, \ldots, x_n)}{\binom{n}{k}}}.$$ [1]

This is precisely Maclaurin's inequality!

[1] An n-dimensional orthotope has a k-dimensional volume of $\left(2^{n-k} e_k(x_1, \ldots, x_n)\right)$ and $\left(2^{n-k}\binom{n}{k}\right)$ k-dimensional faces. Thus, $2^{n-k}\binom{n}{k} l_k^k = 2^{n-k} e_k(x_1, \ldots, x_n)$, or $l_k = \sqrt[k]{e_k(x_1, \ldots, x_n)/\binom{n}{k}}$.

3 Outline of Question

a. Prove that (AG) $\iff$ (B).

b. By considering the concavity of $\ln x$, prove (GB). Determine when equality holds.

c. i) Show that

$$E_k(x_1, \dots, x_n) = \left(1 - \frac{k}{n}\right) E_k(x_1, \dots, x_{n-1}) + \frac{k}{n} E_{k-1}(x_1, \dots, x_{n-1})\, x_n$$

for $1 \le k \le n$.

ii) Hence, prove that (GB) $\implies$ (M). Determine when equality holds.

d. i) Explain why $|\mathcal{S}_k(n)| = \binom{n}{k}$.

ii) Deduce that

$$E_k(1, \dots, 1) = E_{k-1}(1, \dots, 1) = 1,$$

where the arguments of E_k and E_{k-1} both contain $(n-1)$ 1's.

iii) Hence, prove that (M) $\implies$ (GB).

4 Solution

Part (a). We begin with the forwards direction. For all $x > -1$, we have by (AG)

$$\left(1 + \frac{x}{n}\right)^n = \left[\frac{(x+1) + \overbrace{1 + 1 + \cdots + 1}^{(n-1) \text{ times}}}{n}\right]^n \ge (1 + x) \cdot 1 \cdot \ldots \cdot 1 = 1 + x.$$

Taking nth roots, we get

$$1 + \frac{x}{n} \ge (1 + x)^{1/n},$$

which is (B).

We now prove the backwards direction. Define

$$A_n = \frac{x_1 + \cdots + x_n}{n} \quad \text{and} \quad G_n = (x_1 \ldots x_n)^{1/n}.$$

Since $x_1, \dots, x_n$ are positive,

$$\frac{x_1 + \cdots + x_{n-1} + x_n}{x_1 + \cdots + x_{n-1}} > 1.$$

It readily follows that

$$n\left(\frac{A_n}{A_{n-1}} - 1\right) > -1.$$

Invoking (B) on the above object, we see that

$$1 + \frac{1}{n}\left[n\left(\frac{A_n}{A_{n-1}} - 1\right)\right] \ge \left[1 + n\left(\frac{A_n}{A_{n-1}} - 1\right)\right]^{1/n}.$$

Taking nth powers and simplifying, we get

$$\left(\frac{A_n}{A_{n-1}}\right)^n \ge \frac{nA_n - (n-1)A_{n-1}}{A_{n-1}} = \frac{x_n}{A_{n-1}},$$

so

$$A_n^n \geq x_n A_{n-1}^{n-1}.$$

Repeatedly applying this inequality, we obtain

$$A_n^n \geq x_n x_{n-1} \dots x_2 x_1 = G_n^n$$

and (AG) holds.

Part (b). Because $\ln x$ is concave,

$$\ln(au+bv) \geq a \ln u + b \ln v$$

for $u, v \in \mathbb{R}$ and $a, b \geq 0$ with $a+b=1$. Taking

$$a = \frac{1}{k+1}, \quad b = \frac{k}{k+1}, \quad u = 1, \quad v = 1 + \frac{(k+1)x}{n},$$

where $k = 1, \dots, n-1$, we see that

$$\frac{1}{k} \ln\left(1 + \frac{kx}{n}\right) = \frac{1}{k} \ln(au+bv) \geq \frac{1}{k}(a \ln u + b \ln v) = \frac{1}{k+1} \ln\left(1 + \frac{(k+1)x}{n}\right).$$

Exponentiating both sides,

$$\left(1 + \frac{kx}{n}\right)^{1/k} \geq \left(1 + \frac{(k+1)x}{n}\right)^{1/(k+1)}.$$

Chaining the above inequality for $k = 1, \dots, n-1$, we obtain

$$1 + \frac{x}{n} \geq \left(1 + \frac{2x}{n}\right)^{1/2} \geq \left(1 + \frac{3x}{n}\right)^{1/3} \geq \dots \geq \left(1 + \frac{nx}{n}\right)^{1/n}$$

so we are done.

Equality is achieved when $u = v$, which is equivalent to $x = 0$.

Part (c)(i). Define

$$\mathcal{S}_k(n) = \{I : I \subseteq [n],\, |I| = k\} \quad \text{and} \quad P(I) = \prod_{i \in I} x_i.$$

Note that

$$e_k(x_1, \dots, x_n) = \sum_{I \in \mathcal{S}_k(n)} P(I).$$

For each $I \in \mathcal{S}_k(n)$, either I contains n or it doesn't. We hence obtain a recursive formula for $\mathcal{S}_k(n)$.

$$\begin{aligned}
\mathcal{S}_k(n) &= \{I : I \subseteq [n],\, |I| = k,\, n \notin I\} \cup \{I : I \subseteq [n],\, |I| = k,\, n \in I\} \\
&= \{I : I \subseteq [n-1],\, |I| = k\} \cup \{I \cup \{n\} : I \subseteq [n-1],\, |I| = k-1\} \\
&= \mathcal{S}_k(n-1) \cup [\mathcal{S}_{k-1}(n-1) + \{x_n\}].^{2}
\end{aligned}$$

We thus get the following recursion for $e_k(x_1, \dots, x_n)$ too:

$$\begin{aligned}
e_k(x_1, \dots, x_n) &= \sum_{I \in \mathcal{S}_k(n)} P(I) \\
&= \sum_{I \in \mathcal{S}_k(n-1)} P(I) + \sum_{I \in \mathcal{S}_{k-1}(n-1)} P(I \cup \{x_n\}) \\
&= \sum_{I \in \mathcal{S}_k(n-1)} P(I) + \sum_{I \in \mathcal{S}_{k-1}(n-1)} P(I)\, x_n \\
&= e_k(x_1, \dots, x_{n-1}) + e_{k-1}(x_1, \dots, x_{n-1})\, x_n.
\end{aligned}$$

[2] This notation sucks.

Note that this formula still holds in the extreme cases where $k = 1, n$ due to the way we defined $e_0(x_1, \ldots, x_{n-1})$ and $e_n(x_1, \ldots, x_{n-1})$.

Dividing through by $\binom{n}{k}$, we obtain our desired result

$$\begin{aligned}
E_k(x_1, \ldots, x_n) &= \frac{1}{\binom{n}{k}} e_k(x_1, \ldots, x_n) \\
&= \frac{1}{\binom{n}{k}} e_k(x_1, \ldots, x_{n-1}) + \frac{1}{\binom{n}{k}} e_{k-1}(x_1, \ldots, x_{n-1})\, x_n \\
&= \frac{\binom{n-1}{k}}{\binom{n}{k}} E_k(x_1, \ldots, x_{n-1}) + \frac{\binom{n-1}{k-1}}{\binom{n}{k}} E_{k-1}(x_1, \ldots, x_{n-1})\, x_n \\
&= \left(1 - \frac{k}{n}\right) E_k(x_1, \ldots, x_{n-1}) + \frac{k}{n} E_{k-1}(x_1, \ldots, x_{n-1}) x_n.
\end{aligned}$$

Part (c)(ii). We induct on n. The $n = 1$ case is trivial, so we take $n = 2$ as our base case. (GB) for $n = 2$ states that

$$1 + \frac{x}{2} \geq (1 + x)^{1/2}$$

for $x > -1$. For $x_1, x_2 \geq 0$, we have

$$\begin{aligned}
E_1(x_1, x_2) = \frac{x_1 + x_2}{2} &= x_2 \left[1 + \frac{1}{2}\left(\frac{x_1}{x_2} - 1\right)\right] \\
&\geq x_2 \left[1 + \left(\frac{x_1}{x_2} - 1\right)\right]^{1/2} = (x_1 x_2)^{1/2} = E_2(x_1, x_2)^{1/2}.
\end{aligned}$$

Our base case $n = 2$ thus holds.

Now assume that (M) holds for $n - 1$ variables, where $n \geq 3$. To simplify notation, write

$$E_k = E_k(x_1, \ldots, x_n) \qquad \text{and} \qquad \varepsilon_k = E_k(x_1, \ldots, x_{n-1})$$

for $1 \leq k \leq n - 1$. Note that $\varepsilon_0 = 1$ and $\varepsilon_n = 0$. We can rewrite the result in Part (c)(i) as

$$E_k = \left(1 - \frac{k}{n}\right) \varepsilon_k + \frac{k}{n} \varepsilon_{k-1} x_n.$$

By our induction hypothesis,

$$\varepsilon_{k-1}^{1/(k-1)} \geq \varepsilon_k^{1/k}$$

for $2 \leq k \leq n - 1$. We can rewrite this in two ways:

$$\varepsilon_{k-1} \geq \varepsilon_k^{(k-1)/k} \qquad \text{and} \qquad \varepsilon_{k+1} \leq \varepsilon_k^{(k+1)/k}$$

for $1 \leq k \leq n - 1$. We thus obtain

$$E_k \geq \left(1 - \frac{k}{n}\right) \varepsilon_k + \frac{k}{n} \varepsilon_k^{(k-1)/k} x_n = \varepsilon_k \left[1 + \frac{k}{n}\left(\varepsilon_k^{-1/k} x_n - 1\right)\right] \tag{1}$$

and

$$E_{k+1} \leq \left(1 - \frac{k+1}{n}\right) \varepsilon_k^{(k+1)/k} + \frac{k+1}{n} \varepsilon_k x_n = \varepsilon_k^{(k+1)/k} \left[1 + \frac{k+1}{n}\left(\varepsilon_k^{-1/k} x_n - 1\right)\right]. \tag{2}$$

Let $c_k = \varepsilon_k^{-1/k} x_n - 1$. Note that $\varepsilon_k^{-1/k} x_n > 0$, so $c_k > -1$. By (1), (2) and (GB), we obtain

$$E_k^{1/k} \geq \varepsilon_k^{1/k} \left(1 + \frac{k c_k}{n}\right)^{1/k} \geq \varepsilon_k^{1/k} \left(1 + \frac{(k+1) c_{k+1}}{n}\right)^{1/(k+1)} \geq E_{k+1}^{1/(k+1)}.$$

Since this is true for $1 \le k \le n-1$, we have

$$E_1 \ge E_2^{1/2} \ge E_3^{1/3} \ge \cdots \ge E_n^{1/n},$$

so (M) holds for n variables. This closes the induction.

Equality holds in (M) when $c_k = 0$ for all $1 \le k \le n-1$, so

$$x_n = \varepsilon_k^{1/k} \le \varepsilon_1 = \frac{x_1 + \cdots + x_{n-1}}{n-1}.$$

Because each $E_k(x_1, \ldots, x_n)$ is symmetric in x_i, we may assume without loss of generality that x_n is maximal, so

$$\frac{x_1 + \cdots + x_{n-1}}{n-1} \le x_n.$$

Thus, equality occurs only when $x_1 = \cdots = x_n$.

Part (d)(i). Recall that

$$\mathcal{S}_k(n) = \{I : I \subseteq [n],\, |I| = k\}.$$

$\mathcal{S}_k(n)$ is hence the set of all k-subsets of $[n]$. Since there are $\binom{n}{k}$ ways to choose k elements from $[n]$ to form I, it follows that $|\mathcal{S}_k(n)| = \binom{n}{k}$.

Part (d)(ii). We have

$$E_k(1, \ldots, 1) = \frac{1}{\binom{n-1}{k}} \sum_{I \in \mathcal{S}_k(n-1)} 1 = \frac{|\mathcal{S}_k(n-1)|}{\binom{n-1}{k}} = \frac{\binom{n-1}{k}}{\binom{n-1}{k}} = 1.$$

Similarly,

$$E_{k-1}(1, \ldots, 1) = \frac{1}{\binom{n-1}{k-1}} \sum_{I \in \mathcal{S}_{k-1}(n-1)} 1 = \frac{|\mathcal{S}_{k-1}(n-1)|}{\binom{n-1}{k-1}} = \frac{\binom{n-1}{k-1}}{\binom{n-1}{k-1}} = 1.$$

Part (d)(iii). Fix $x > -1$ and let $x_1 = \cdots = x_{n-1} = 1$ and $x_n = 1 + x$. By Parts (c)(i) and (d)(ii), for $1 \le k \le n$,

$$E_k(1, \ldots, 1, 1+x) = \left(1 - \frac{k}{n}\right) E_k(1, \ldots, 1) + \frac{k}{n} E_{k-1}(1, \ldots, 1)\,(1+x) = 1 + \frac{kx}{n}.$$

(M) thus states

$$1 + \frac{x}{n} \ge \left(1 + \frac{2x}{n}\right)^{1/2} \ge \left(1 + \frac{3x}{n}\right)^{1/3} \ge \cdots \ge \left(1 + \frac{nx}{n}\right)^{1/n},$$

which is exactly (GB).

References

[1] I. Ben-Ari and K. Conrad. Maclaurin's Inequality and a Generalized Bernoulli Inequality. *Mathematics Magazine*, 87(1):14–24, 2014.

[2] L. Maligranda. The AM-GM Inequality is Equivalent to the Bernoulli Inequality. *The Mathematical Intelligencer*, 34(1):1–2, 2012.