An Elementary Analytic Number Theory Problem

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1 Introduction

Let $\mu(n)$ be the Möbius function. Define

$$A(x) = \sum_{1\le n\le x} \frac{\mu(n)}{n} \quad \text{and} \quad M(x) = \sum_{1\le n\le x} \mu(n).$$

In this note, we examine the following equivalence:

$$A(x) = o(1) \iff M(x) = o(x). \tag{$*$}$$

2 The Forward Direction

The forward direction is fairly trivial.

Lemma 1. Suppose $A(x) = o(1)$. Then $\int_1^x A(t)\,\mathrm{d}t = o(x)$.

Proof. Fix $\varepsilon > 0$. Since $A(x) = o(1)$, there exists some $T \in \mathbb{N}$ such that for all $t > T$, we have the bound $|A(t)| < \varepsilon$. It readily follows that

$$\begin{aligned}\left|\frac{1}{x}\int_1^x A(t)\,\mathrm{d}t\right| &\le \frac{1}{x}\int_1^x |A(t)|\,\mathrm{d}t \\ &= \frac{1}{x}\left(\int_1^T |A(t)|\,\mathrm{d}t + \int_T^x |A(t)|\,\mathrm{d}t\right) \\ &< \frac{C_T + \varepsilon(x-T)}{x} \\ &= \frac{C_T - T}{x} + \varepsilon,\end{aligned}$$

where C_T is a constant depending solely on T. In the limit as $x \to \infty$, we have

$$\left|\frac{1}{x}\int_1^x A(t)\,\mathrm{d}t\right| < \varepsilon,$$

so $\int_1^x A(t)\,\mathrm{d}t = o(x)$ as desired. ∎

Claim A. The forwards direction of ($*$) holds:

$$A(n) = o(1) \implies M(x) = o(x).$$

Date: June 29, 2025
Latest revision: https://asdia.dev/expository/elem-ant.pdf

Proof. By Abel's summation formula, we have

$$M(x) = \sum_{1 \le n \le x} \left(\frac{\mu(n)}{n} \cdot n \right) = xA(x) - \int_1^x A(t)\,\mathrm{d}t.$$

Under our hypothesis and Lemma 1,

$$M(x) = xo(1) + o(x) = o(x),$$

which was what we wanted. □

3 The Backward Direction

The proof presented below is due to Diamond[1].

Define

$$S(x) = xA(x) = \sum_{1 \le n \le x} \mu(n) \frac{x}{n}.$$

Lemma 2. We have the identity

$$S(x) = 1 + \sum_{1 \le n \le x} \mu(n) \left\{ \frac{x}{n} \right\}.$$

Proof. It is well known that

$$1 = \sum_{1 \le n \le x} \sum_{d|n} \mu(d).$$

Switching the order of summation, we obtain

$$\begin{aligned} 1 &= \sum_{1 \le d \le x} \mu(d) \sum_{\substack{n \le x \\ d|n}} 1 \\ &= \sum_{1 \le d \le x} \mu(d) \left[\frac{x}{d} \right], \end{aligned}$$

from which it follows

$$\begin{aligned} S(x) &= \sum_{1 \le n \le x} \mu(n) \left(\left[\frac{x}{n} \right] + \left\{ \frac{x}{n} \right\} \right) \\ &= 1 + \sum_{1 \le n \le x} \mu(n) \left\{ \frac{x}{n} \right\}. \end{aligned}$$

□

Lemma 3. Fix $N \in \mathbb{N}$. Then

$$\sum_{\frac{x}{N} < n \le x} \mu(n) \left\{ \frac{x}{n} \right\} = \int_1^N M\left(\frac{x}{t}\right) \mathrm{d}t - \sum_{2 \le n \le N} M\left(\frac{x}{n}\right).$$

Proof. By Abel's summation formula, one has

$$\begin{aligned} \sum_{\frac{x}{N} < n \le x} \mu(n) \frac{x}{n} &= M(x) - NM\left(\frac{x}{N}\right) - \int_{\frac{x}{N}}^x M(t)\,\mathrm{d}\left(\frac{x}{t}\right) \\ &= M(x) - NM\left(\frac{x}{N}\right) + \int_1^N M\left(\frac{x}{t}\right) \mathrm{d}t. \end{aligned} \tag{3.1}$$

Since $[\frac{x}{n}] = k$ for $\frac{x}{k+1} < n \leq \frac{x}{k}$, we have

$$\begin{aligned}\sum_{\frac{x}{N}<n\leq x} \mu(n)\left[\frac{x}{n}\right] &= \sum_{1\leq k\leq N-1} k \sum_{\frac{x}{k+1}<n\leq\frac{x}{k}} \mu(n) \\ &= \sum_{1\leq k\leq N-1} k\left(M\left(\frac{x}{k}\right) - M\left(\frac{x}{k+1}\right)\right) \\ &= M(x) - NM\left(\frac{x}{N}\right) + \sum_{2\leq k\leq N} M\left(\frac{x}{k}\right). \qquad (3.2)\end{aligned}$$

Subtracting (3.2) from (3.1), we obtain

$$\sum_{\frac{x}{N}<n\leq x} \mu(n)\left\{\frac{x}{n}\right\} = \int_1^N M\left(\frac{x}{t}\right)\,\mathrm{d}t - \sum_{2\leq n\leq N} M\left(\frac{x}{n}\right)$$

as desired. □

Claim B. The backward direction of (∗) holds:

$$A(x) = o(1) \impliedby M(x) = o(x).$$

Proof. Fix $N \in \mathbb{N}$. We first split our result from Lemma 2 and then apply Lemma 3:

$$\begin{aligned}S(x) &= 1 + \sum_{1\leq n\leq x} \mu(n)\left\{\frac{x}{n}\right\} \\ &= 1 + \sum_{n\leq\frac{x}{N}} \mu(n)\left\{\frac{x}{n}\right\} + \sum_{\frac{x}{N}<n\leq x} \mu(n)\left\{\frac{x}{n}\right\} \\ &= 1 + \sum_{n\leq\frac{x}{N}} \mu(n)\left\{\frac{x}{n}\right\} + \int_1^N M\left(\frac{x}{t}\right)\,\mathrm{d}t - \sum_{2\leq n\leq N} M\left(\frac{x}{n}\right).\end{aligned}$$

Using our hypothesis that $M(x) = o(x)$, we obtain the bound

$$\begin{aligned}|S(x)| &\leq 1 + \sum_{n\leq\frac{x}{N}} \left|\mu(n)\left\{\frac{x}{n}\right\}\right| + \int_1^N \left|M\left(\frac{x}{t}\right)\right|\,\mathrm{d}t + \sum_{2\leq n\leq N} \left|M\left(\frac{x}{n}\right)\right| \\ &\leq 1 + \frac{x}{N} + \int_1^N o\left(\frac{x}{t}\right)\,\mathrm{d}t + \sum_{2\leq n\leq N} o\left(\frac{x}{n}\right) \\ &= \frac{x}{N} + o(x).\text{[1]}\end{aligned}$$

In the limit as $N \to \infty$, we have $S(x) = o(x)$, so $A(x) = \frac{S(x)}{x} = o(1)$. □

References

[1] Harold G. Diamond. Elementary methods in the study of the distribution of prime numbers. *Bulletin (New Series) of the American Mathematical Society*, 7(3):553 – 589, 1982.

[1] Easy exercise: show that $\int_1^N o\left(\frac{x}{t}\right)\,\mathrm{d}t = o(x)$ and $\sum_{2\leq n\leq N} o\left(\frac{x}{n}\right) = o(x)$