A GENERALIZATION OF THE CAUCHY INTEGRAL

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1. Introduction

In this paper, we showcase three methods of evaluating definite integrals of the form

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^{n+1}} \, \mathrm{d}x,$$

where $a \in \mathbb{R}^+$ and $n \in \mathbb{R}_0^+$. The first method (Section 2) exploits a recurrence relation involving the indefinite integral

$$\int \frac{1}{(x^2 + a^2)^{n+1}} \, \mathrm{d}x \, .$$

The second method (Section 3) follows a standard complex analysis argument to evaluate I_n using contour integration. Both the first and second methods hold only for $n \in \mathbb{N}_0$, yielding

$$I_n = \frac{\pi(2n-1)!!}{n! \, 2^n a^{2n+1}}. \tag{1}$$

The third method (Section 4) uses a clever substitution to transform I_n into the Beta function, yielding a closed form valid for $n \in \mathbb{R}_0^+$:

$$I_n = \frac{1}{a^{2n+1}} B\left(\frac{1}{2}, n + \frac{1}{2}\right). \tag{2}$$

On our quest to derive closed forms for I_n, we evaluate several general definite integrals of which I_n is a specific case, such as

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{(x^2 + a^2)^{n+1}} \, \mathrm{d}x = \frac{\pi}{e^{\omega a}} \frac{\theta_n(\omega a)}{n! \, 2^n a^{2n+1}}$$

and

$$\int_0^{\infty} \frac{x^{\mu-1}}{(x^\nu + a^\nu)^{n+1}} \, \mathrm{d}x = \frac{1}{\nu a^{\nu n + \nu - \mu}} B\left(\frac{\mu}{\nu}, 1 + n - \frac{\mu}{\nu}\right).$$

Next, in Section 5, we reconcile the gap between discreteness and continuity by showing that (2) is a generalization of (1). Lastly, in Section 6, we use our results to derive a relationship between I_n and the series expansion of $\frac{1}{\sqrt{1-t}}$, which gives us several stunning identities, such as

$$\pi = \sqrt{1-t} \sum_{n=0}^{\infty} B\left(\frac{1}{2}, n + \frac{1}{2}\right) t^n,$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n} \frac{B(n+1, k+1)}{B(n, k)} = 1.$$

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2. Method 1: Recurrence Relation

In this section, we evaluate I_n by considering the recurrence relation given by the indefinite integral

$$J_n = \int \frac{1}{(x^2+a^2)^{n+1}} \, \mathrm{d}x \, .$$

The recurrence relation is as such:

Lemma 1. *For all $n \in \mathbb{N}$, $a \in \mathbb{R}^+$,*

$$J_n = \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2} J_{n-1}.$$

Proof. Consider

$$J_{n-1} = \int \frac{1}{(x^2+a^2)^n} \, \mathrm{d}x \, .$$

Integrating by parts with $u = \frac{1}{(x^2+a^2)^n}$ and $\mathrm{d}v = 1$, we obtain

$$\begin{aligned} J_{n-1} &= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2}{(x^2+a^2)^{n+1}} \, \mathrm{d}x \\ &= \frac{x}{(x^2+a^2)^n} + 2n \left(\int \frac{1}{(x^2+a^2)^n} \, \mathrm{d}x - a^2 \int \frac{1}{(x^2+a^2)^{n+1}} \, \mathrm{d}x \right) \\ &= \frac{x}{(x^2+a^2)^n} + 2n \left(J_{n-1} - a^2 J_n \right) . \end{aligned}$$

Isolating J_n, we obtain

$$J_n = \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2} J_{n-1}$$

as desired. □

With our recurrence relation in place, we are now ready to evaluate I_n.

Theorem 1. *For all $n \in \mathbb{N}_0$, $a \in \mathbb{R}^+$,*

$$I_n = \frac{\pi(2n-1)!!}{n!\, 2^n a^{2n+1}}.$$

Proof. By the fundamental theorem of calculus, we have

$$\begin{aligned} I_n &= [J_n]_{-\infty}^{\infty} \\ &= \left[\frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2} J_{n-1} \right]_{-\infty}^{\infty} \\ &= \left[\frac{x}{2na^2(x^2+a^2)^n} \right]_{-\infty}^{\infty} + \frac{2n-1}{2na^2} I_{n-1} \end{aligned}$$

Observe that for all $n \in \mathbb{N}$,

$$\begin{aligned} \left| \lim_{x \to \pm\infty} \frac{x}{2na^2(x^2+a^2)^n} \right| &= \lim_{x \to \pm\infty} \left| \frac{x}{2na^2(x^2+a^2)^n} \right| \\ &\leq \frac{1}{2na^2} \lim_{x \to \pm\infty} \left| \frac{x}{x^{2n}} \right| \\ &= 0. \end{aligned}$$

We thus get the first order recurrence relation

$$I_n = \frac{2n-1}{2na^2} I_{n-1}$$

which has the solution

$$I_n = I_0 \prod_{k=1}^{n} \frac{2k-1}{2ka^2}.$$

By the definition of the factorial and double factorial, we have $n! = \prod_{k=1}^{n} k$ and $(2n-1)!! = \prod_{k=1}^{n}(2k-1)$. Hence,

$$I_n = I_0 \cdot \frac{(2n-1)!!}{n!\,2^n a^{2n}}.$$

By our definition of I_n, we have

$$\begin{aligned} I_0 &= \int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,\mathrm{d}x \\ &= \left[\frac{1}{a}\arctan\frac{x}{a}\right]_{-\infty}^{\infty} \\ &= \frac{\pi}{a}. \end{aligned}$$

Thus, we finally obtain

$$I_n = \frac{\pi(2n-1)!!}{n!\,2^n a^{2n+1}}.$$

□

3. Method 2: Complex Analysis

Using techniques from complex analysis, one can show that the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,\mathrm{d}x$$

evaluates elegantly to $\frac{\pi}{e}$. Indeed, this integral is commonly used as an example to showcase the power of evaluating definite integrals using contour integration. In this section, we extend this classic integral by examining the more general form

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{(x^2+a^2)^{n+1}}\,\mathrm{d}x,$$

where n, ω and a are free variables. We then use this result to find a closed form for I_n.

Lemma 2. *For all $n \in \mathbb{N}_0$, $a \in \mathbb{R}^+$ and $\omega \in \mathbb{R}_0^+$,*

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{(x^2+a^2)^{n+1}}\,\mathrm{d}x = \frac{\pi}{e^{\omega a}} \frac{\theta_n(\omega a)}{n!\,2^n a^{2n+1}},$$

where $\theta_n(x)$ is the nth reverse Bessel polynomial.

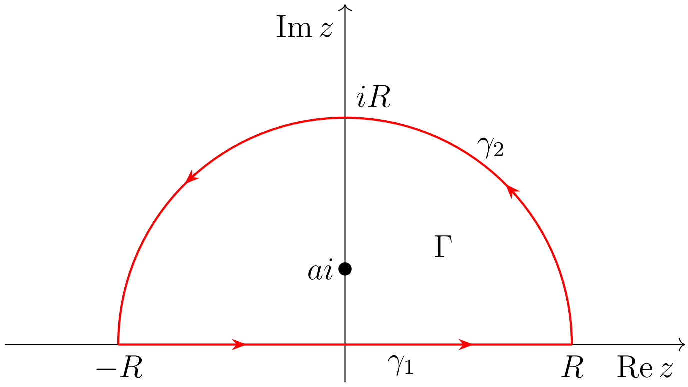

FIGURE 1. The semicircular, anti-clockwise contour Γ and the enclosed singularity ai.

Proof. Let

$$f(z) = \frac{e^{i\omega z}}{(z^2 + a^2)^{n+1}}.$$

Let $\gamma_1 = [-R, R]$ and $\gamma_2 = \{z = Re^{i\theta}, \theta \in (0, \pi)\}$, where $R \in \mathbb{R}^+$. Consider the integral $\oint_\Gamma f(z)\,\mathrm{d}z$ over the closed contour $\Gamma = \gamma_1 \cup \gamma_2$ as shown in Figure 1. We clearly have

$$\oint_\Gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z\,.$$

Now observe that $\int_{\gamma_2} f(z)\,\mathrm{d}z$ vanishes by a simple bounding argument:

$$\begin{aligned}
\lim_{R\to\infty} \left| \int_{\gamma_2} f(z)\,\mathrm{d}z \right| &= \lim_{R\to\infty} \left| \int_0^\pi \frac{e^{i\omega Re^{i\theta}}}{((Re^{i\theta})^2 + a^2)^{n+1}} iRe^{i\theta}\,\mathrm{d}\theta \right| \\
&\le \lim_{R\to\infty} \int_0^\pi \left| \frac{e^{i\omega Re^{i\theta}}}{((Re^{i\theta})^2 + a^2)^{n+1}} iRe^{i\theta} \right| \mathrm{d}\theta \\
&\le \lim_{R\to\infty} \frac{R}{R^{2n+2}} \int_0^\pi e^{-\omega R \sin\theta}\,\mathrm{d}\theta \\
&= 0.
\end{aligned}$$

We can thus express our goal integral as

$$\int_\infty^\infty \frac{\cos\omega x}{(x^2 + a^2)^{n+1}}\,\mathrm{d}x = \mathrm{Re} \lim_{R\to\infty} \int_{\gamma_1} f(z)\,\mathrm{d}z = \mathrm{Re} \lim_{R\to\infty} \oint_\Gamma f(z)\,\mathrm{d}z\,.$$

We now wish to show that $\oint_\Gamma f(z)\,\mathrm{d}z = \frac{\pi}{e^{\omega a}} \frac{\theta_n(\omega a)}{n!\,2^n a^{2n+1}}$. It is clear that the only singularity enclosed by γ is at ai. Moreover, it is a pole of order $n+1$. Invoking Cauchy's residue theorem yields

$$\begin{aligned}
\oint_\Gamma f(z)\,\mathrm{d}z &= 2\pi i \,\mathrm{Res}(f, ai) \\
&= 2\pi i \cdot \frac{1}{n!} \lim_{z\to ai} \frac{\mathrm{d}^n}{\mathrm{d}z^n} (z - ai)^{n+1} f(z) \\
&= 2\pi i \cdot \frac{1}{n!} \lim_{z\to ai} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \frac{e^{i\omega z}}{(z + ai)^{n+1}}.
\end{aligned}$$

By the general Leibniz rule, we have

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{e^{i\omega z}}{(z+ai)^n} &= \sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^k}{\mathrm{d}z^k}(z+ai)^{-n-1}\frac{\mathrm{d}^{n-k}}{\mathrm{d}z^{n-k}}e^{i\omega z}\\ &= \sum_{k=0}^{n}\binom{n}{k}(-n-1)(-n-2)\dots(-n-k)(z+ai)^{-n-1-k}(i\omega)^{n-k}e^{i\omega z}\\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(n+k)!}{n!}i^{n-k}\omega^{n-k}e^{i\omega z}(z+ai)^{-n-1-k}\\ &= \sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}i^{n+k}\omega^{n-k}e^{i\omega z}(z+ai)^{-n-1-k}.\end{aligned}$$

Taking the limit as $z \to ai$, we see that

$$\begin{aligned}\lim_{z\to ai}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{e^{iz}}{(z+ai)^{n+1}} &= \sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}i^{n+k}\omega^{n-k}e^{-\omega a}(2ai)^{-n-1-k}\\ &= e^{-\omega a}i^{-1}2^{-n-1}a^{-2n-1}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\frac{(\omega a)^{n-k}}{2^k}.\end{aligned}$$

Recall that the nth reverse Bessel polynomial $\theta_n(x)$ is defined as

$$\theta_n(x) = \sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\frac{x^{n-k}}{2^k}.$$

Our series hence simplifies to $\theta_n(\omega a)$. Putting everything together, we obtain

$$\begin{aligned}\oint_\Gamma f(z)\,\mathrm{d}z &= 2\pi i\cdot\frac{1}{n!}e^{-\omega a}i^{-1}2^{-n-1}a^{-2n-1}\theta_n(\omega a)\\ &= \frac{\pi}{e^{\omega a}}\frac{\theta_n(\omega a)}{n!\,2^n a^{2n+1}}\end{aligned}$$

as desired. □

Corollary 1. *Evaluating the integral at $n = 0$ and $\omega = 1$, we get the interesting result*

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{ae^a}.$$

Taking $n = 0$ and $a = 1$ also yields another intriguing result:

$$\int_{-\infty}^{\infty}\frac{\cos\omega x}{x^2+1}\,\mathrm{d}x = \frac{\pi}{e^\omega}.$$

Corollary 2. *Taking the complex part of $\oint_\Gamma f(z)\,\mathrm{d}z$, we see that*

$$\int_{-\infty}^{\infty}\frac{\sin\omega x}{(x^2+a^2)^{n+1}}\,\mathrm{d}x = 0,$$

though this result is trivial from the substitution $x \mapsto -x$.

Remark. It is sufficient to know either

$$\int_{-\infty}^{\infty}\frac{\cos\omega x}{(x^2+1)^{n+1}}\,\mathrm{d}x = \frac{\pi}{e^\omega}\frac{\theta_n(\omega)}{n!\,2^n} \tag{3}$$

or

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+a^2)^{n+1}}\,\mathrm{d}x = \frac{\pi}{e^a}\frac{\theta_n(a)}{n!\,2^n a^{2n+1}}, \tag{4}$$

as each integral can be converted into the other by means of a substitution. For instance, the substitution $x \mapsto \frac{x}{\omega}$ converts the integral in (3) to that of (4). For our case, we consider both ω and a at the same time to simplify the workings.

We now use our result to evaluate

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^{n+1}}\,\mathrm{d}x\,.$$

Theorem 2. *For all $n \in \mathbb{N}_0$, $a \in \mathbb{R}^+$,*

$$I_n = \frac{\pi(2n-1)!!}{n!\,2^n a^{2n+1}}.$$

Proof. Observe that our goal integral I_n corresponds to the case where $\omega = 0$. Applying Lemma 2 gives

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^{n+1}}\,\mathrm{d}x = \frac{\pi\theta_n(0)}{n!\,2^n a^{2n+1}}.$$

We now show that $\theta_n(0) = (2n-1)!!$. We do so by utilizing the recursive nature of the reverse Bessel polynomials:

$$\theta_n(x) = (2n-1)\theta_{n-1}(x) + x^2\theta_{n-2}(x).$$

At $x = 0$, we have

$$\theta_n(0) = (2n-1)\theta_{n-1}(0).$$

Since $\theta_0(0) = 1$, we have

$$\begin{aligned}\theta_n(0) &= (2n-1)(2n-3)\ldots 1\\ &= (2n-1)!!\end{aligned}$$

as desired. □

4. Method 3: The Beta Function

In this section, we show that the integral in consideration I_n can be expressed in terms of the Beta function. We first consider a more general definite integral:

Lemma 3. *For all $n \in \mathbb{R}_0^+$, $a \in \mathbb{R}^+$, $0 < \frac{\mu}{\nu} < n+1$,*

$$\int_0^{\infty} \frac{x^{\mu-1}}{(x^\nu+a^\nu)^{n+1}}\,\mathrm{d}x = \frac{1}{\nu a^{\nu n+\nu-\mu}} B\Big(\frac{\mu}{\nu}, 1+n-\frac{\mu}{\nu}\Big).$$

Proof. We begin by factoring out a^ν from the denominator:

$$\int_0^{\infty} \frac{x^{\mu-1}}{(x^\nu+a^\nu)^{n+1}}\,\mathrm{d}x = \frac{1}{a^{\nu n+\nu}} \int_0^{\infty} \frac{x^{\mu-1}}{\left(1+\left(\frac{x}{a}\right)^\nu\right)^{n+1}}\,\mathrm{d}x\,.$$

Under the substitution $t = \left(\frac{x}{a}\right)^\nu$, our integral transforms to

$$\frac{1}{a^{\nu n+\nu}} \int_0^{\infty} \frac{(at^{\frac{1}{\nu}})^{\mu-1}}{(1+t)^{n+1}} \cdot \frac{a}{\nu} t^{\frac{1-\nu}{\nu}}\,\mathrm{d}t\,.$$

Simplifying, we obtain

$$\frac{1}{\nu a^{\nu n+\nu-\mu}} \int_0^{\infty} \frac{t^{\frac{\mu}{\nu}-1}}{(1+t)^{n+1}}\,\mathrm{d}t\,.$$

Now recall that the Beta function $B(z_1, z_2)$ has the identity

$$B(z_1,z_2) = \int_0^{\infty} \frac{t^{z_1-1}}{(1+t)^{z_1+z_2}}\,\mathrm{d}t\,.$$

Looking at our integral, we have $z_1 = \frac{\mu}{\nu}$ and $z_2 = 1 + n - \frac{\mu}{\nu}$. Hence, our integral can be written as

$$\frac{1}{\nu a^{\nu n+\nu-\mu}} B\Big(\frac{\mu}{\nu}, 1 + n - \frac{\mu}{\nu}\Big).$$

□

Remark. This closed form has appeared in integral tables in the literature (Gradshteyn and Ryzhik 2007). However, no such derivation could be found.

We now use this fact to evaluate I_n.

Theorem 3. *For all $n \in \mathbb{R}_0^+$, $a \in \mathbb{R}^+$,*

$$I_n = \frac{1}{a^{2n+1}} B\left(\frac{1}{2}, n + \frac{1}{2}\right).$$

Proof. Observe that the integrand of I_n is an even function. Hence,

$$I_n = 2 \int_0^\infty \frac{1}{(x^2 + a^2)^{n+1}} \, \mathrm{d}x \, .$$

From Lemma 3, taking $\mu = 1$ and $\nu = 2$, we have

$$\begin{aligned} I_n &= 2 \cdot \frac{1}{2a^{2n+2-1}} B\left(\frac{1}{2}, 1 + n - \frac{1}{2}\right) \\ &= \frac{1}{a^{2n+1}} B\left(\frac{1}{2}, n + \frac{1}{2}\right) \end{aligned}$$

as desired. □

5. Reconciling the Discrete and Continuous Results

In the previous three sections, we derived two expressions for I_n. On one hand, when n is restricted to the non-negative natural numbers, we have

$$I_n = \frac{\pi(2n-1)!!}{n!\, 2^n a^{2n+1}}.$$

On the other hand, when n is allowed to take on any non-negative value, we have

$$I_n = \frac{1}{a^{2n+1}} B\left(\frac{1}{2}, n + \frac{1}{2}\right).$$

In this section, we aim to show that the latter expression is a natural extension of the former.

Proposition 1. *For all $n \in \mathbb{R}_0^+$ and $a \in \mathbb{R}^+$,*

$$I_n = \frac{1}{a^{2n+1}} B\left(\frac{1}{2}, n + \frac{1}{2}\right).$$

Proof. As derived earlier, we have

$$I_n = \frac{\pi(2n-1)!!}{n!\, 2^n a^{2n+1}}.$$

We now extend our result to $n \in \mathbb{R}_0^+$. The standard generalization for the odd double factorial is given by

$$z!! = \sqrt{\frac{2}{\pi}} 2^{\frac{z}{2}} \Gamma\Big(\frac{z}{2} + 1\Big).$$

Setting $z = 2n - 1$ yields

$$(2n-1)!! = \sqrt{\frac{2}{\pi}} 2^{\frac{2n-1}{2}} \Gamma\left(\frac{2n-1}{2} + 1\right) = \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right).$$

Altogether, we have

$$I_n = \frac{\pi}{n!\, 2^n a^{2n+1}} \cdot \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{a^{2n+1}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}.$$

Recall that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Hence,

$$I_n = \frac{1}{a^{2n+1}} \frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}.$$

Now recall that the Beta function has the following well-known relationship with the Gamma function:

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}.$$

Thus, I_n indeed extends to

$$\frac{1}{a^{2n+1}} B\left(\frac{1}{2}, n + \frac{1}{2}\right).$$

□

6. Special Identities

In this section, we derive a relationship between I_n and the series expansion of $\frac{1}{\sqrt{1-t}}$. We then deduce several identities stemming from this relationship, of which we separate into four classes: combinatorial identities, identities involving π, identities involving the Gaussian integral, and identities involving the Beta function. These identities are covered in Sections 6.1, 6.2, 6.3 and 6.4 respectively.

We begin by proving the following relationship between I_n and $\frac{1}{\sqrt{1-t}}$:

Proposition 2. *Let C_n be the coefficient of t^n in the series expansion of $\frac{1}{\sqrt{1-t}}$. Then*

$$C_n = \frac{a^{2n+1}}{\pi} I_n.$$

Proof. By the definition of C_n, we clearly have $\frac{1}{\sqrt{1-t}} = \sum_{n=0}^{\infty} C_n t^n$. It hence suffices to show that

$$\frac{1}{\sqrt{1-t}} = \sum_{n=0}^{\infty} \frac{a^{2n+1}}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^{n+1}} \, \mathrm{d}x \, t^n.$$

Let S denote the sum in consideration. Rearranging terms, we obtain

$$S = \frac{a}{\pi} \sum_{n=0}^{\infty} \frac{a^{2n}}{x^2+a^2} \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^n} \, \mathrm{d}x \, t^n.$$

Changing the order of summation and integration gives us our desired result:

$$\begin{aligned}
S &= \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\sum_{n=0}^{\infty}\left(\frac{a^2 t}{x^2+a^2}\right)^n \mathrm{d}x \\
&= \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\frac{1}{1-\frac{a^2 t}{x^2+a^2}} \mathrm{d}x \\
&= \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+a^2(1-t)} \mathrm{d}x \\
&= \frac{a}{\pi}\left[\frac{1}{\sqrt{a^2(1-t)}}\arctan\frac{t}{\sqrt{a^2(1-t)}}\right]_{-\infty}^{\infty} \\
&= \frac{1}{\sqrt{1-t}}.
\end{aligned}$$

□

We now look at several types of identities that stem from the above claim. Of the many identities derived below, only those of particular interest have been boxed.

6.1. **Combinatorial Identities.** There have been some combinatorial identities involving I_n in the literature. For instance, Bailey et al. (2006) gives the following identity relating I_n at $a = 1$ and the central binomial coefficient:

$$\int_0^{\infty} \frac{1}{(x^2+1)^{n+1}} = \frac{\pi}{2^{2n+1}}\binom{2n}{n}. \tag{5}$$

Their proof involves the substitution $x \mapsto \tan\theta$ and relating it to Wallis' integrals, which can be solved using a recurrence relation similar to that of Method 1. We now give a more direct proof of (5) and generalize it over a.

Proposition 3. *For all $n \in \mathbb{R}_0^+$ and $a \in \mathbb{R}^+$,*

$$\boxed{I_n = \frac{\pi}{2^{2n}a^{2n+1}}\binom{2n}{n}.}$$

Proof. It is a well-known identity that

$$\binom{2n}{n} = \frac{2^n(2n-1)!!}{n!}.$$

Recalling our previous result for I_n, we have

$$\begin{aligned}
I_n &= \frac{\pi(2n-1)!!}{n!2^n a^{2n+1}} \\
&= \frac{\pi}{2^{2n}a^{2n+1}}\binom{2n}{n}
\end{aligned}$$

as desired. □

We now prove an identity related to the negative binomial coefficient.

Proposition 4. *For all $n \in \mathbb{Z}$,*

$$\boxed{\binom{n-\frac{1}{2}}{n} = (-1)^n\binom{-\frac{1}{2}}{n}.}$$

Proof. We first observe that when $n \in \mathbb{Z}^-$, we have $\Gamma(n+1) \to \infty$. Thus, both sides clearly evaluate to 0, making the equality trivially true. We hence focus only on the case $n \in \mathbb{Z}_0^+$ for the rest of the proof.

From Propositions 1 and 2, we have

$$\begin{aligned} C_n &= \frac{a^{2n+1}}{\pi} \cdot \frac{1}{a^{2n+1}} B\left(\frac{1}{2}, n + \frac{1}{2}\right) \\ &= \frac{1}{\pi} B\left(\frac{1}{2}, n + \frac{1}{2}\right). \end{aligned}$$

As previously discussed, we have

$$B\left(\frac{1}{2}, n + \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\,\Gamma(n + \frac{1}{2})}{\Gamma(n+1)}.$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we know $\frac{1}{\pi}\Gamma(\frac{1}{2}) = 1/\Gamma(\frac{1}{2})$, thus giving

$$C_n = \frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+1)}.$$

Abusing notation, we have

$$C_n = \frac{(n - \frac{1}{2})!}{(-\frac{1}{2})!\,n!},$$

which we recognize to be the binomial coefficient $\binom{n-\frac{1}{2}}{n}$.

However, by the binomial theorem, we have

$$\frac{1}{\sqrt{1-t}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-t)^n.$$

Comparing coefficients of t^n, we arrive at the conclusion that

$$\binom{n - \frac{1}{2}}{n} = (-1)^n \binom{-\frac{1}{2}}{n}.$$

□

Remark. As previously mentioned, this is related to the negative binomial coefficient identity, which states

$$\binom{-r}{k} = (-1)^k \binom{r + k - 1}{k},$$

where $r, k \in \mathbb{N}$. In our case however, we have $r = \frac{1}{2} - n$ and $k = n$. We have thus extended the negative binomial coefficient identity where r is a half-integer.

6.2. **Identities Involving π.** Using our previous results, we obtain an infinite family of expressions that evaluate to π:

Proposition 5. *For all $t \in [-1, 1)$,*

$$\boxed{\pi = \sqrt{1-t} \sum_{n=0}^{\infty} B\left(\frac{1}{2}, n + \frac{1}{2}\right) t^n.}$$

Proof. Recall that $C_n = \frac{1}{\pi} B(\frac{1}{2}, n + \frac{1}{2})$. We thus have for all $t \in [-1, 1)$,

$$\frac{1}{\sqrt{1-t}} = \sum_{n=0}^{\infty} \frac{1}{\pi} B\left(\frac{1}{2}, n + \frac{1}{2}\right) t^n, \tag{6}$$

which upon rearranging yields

$$\pi = \sqrt{1-t}\sum_{n=0}^{\infty} B\left(\frac{1}{2}, n+\frac{1}{2}\right)t^n$$

as desired. ☐

Corollary 3. *Using the relationship between the Gamma and Beta functions, we can rewrite (6) as*

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} t^n = \sqrt{\frac{\pi}{1-t}}. \tag{7}$$

When $t = -1$, we have the alternating sum

$$\sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} = \sqrt{\frac{\pi}{2}}.$$

When $t = \frac{1}{2}$, we can express the sum in terms of $\tau = 2\pi$:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{1}{2^n} = \sqrt{\tau}.$$

Corollary 4. *Substituting $t = \cos\theta$ into (7), we obtain*

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \cos^n\theta = \sqrt{\frac{\pi}{2}}\left|\csc\frac{\theta}{2}\right|,$$

which is valid for all $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.

When $\theta = \frac{2\pi}{3}$, we have the beautiful identity

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \cos^n\frac{2\pi}{3} = \sqrt{\frac{2\pi}{3}}.$$

6.3. **Identities Involving the Gaussian Integral.** Again rewriting the identity in (7), we have for all $t \in (0, 2]$,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} (1-t)^n = \sqrt{\frac{\pi}{t}}, \tag{8}$$

which we recognize to be the value of the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-tx^2}\,\mathrm{d}x\,.$$

We thus have a surprising connection between the Gamma function and the Gaussian integral:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} (1-t)^n = \int_{-\infty}^{\infty} e^{-tx^2}\,\mathrm{d}x\,. \tag{9}$$

We now generalize this result.

Proposition 6. *For all $t \in (0, 2]$ and $k \in \mathbb{N}_0$,*

$$\boxed{\sum_{n=0}^{\infty} \frac{\Gamma(n+k+\frac{1}{2})}{\Gamma(n+1)} (1-t)^n = \int_{-\infty}^{\infty} x^{2k} e^{-tx^2}\,\mathrm{d}x\,.}$$

Proof. Differentiating (9) k times with respect to t yields

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}(-1)^k \left(\prod_{i=0}^{k-1}(n-i)\right)(1-t)^{n-k} = \int_{-\infty}^{\infty} (-1)^k x^{2k} e^{-tx^2}\,\mathrm{d}x\,.$$

Using basic Gamma function properties, we see that the LHS becomes

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-k+1)}(1-t)^{n-k}$$

Now observe that for $k > n$, $\Gamma(n-k+1) \to \infty$, further simplifying the LHS down to

$$\sum_{n=k}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-k+1)}(1-t)^{n-k} = \sum_{n=0}^{\infty} \frac{\Gamma(n+k+\frac{1}{2})}{\Gamma(n+1)}(1-t)^{n}.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+k+\frac{1}{2})}{\Gamma(n+1)}(1-t)^{n} = \int_{-\infty}^{\infty} x^{2k} e^{-tx^2}\,\mathrm{d}x\,. \tag{10}$$

□

Remark. Taking $t = 1$ and $k = 0$, the LHS in the identity above becomes

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} 0^n.$$

For $n > 0$, the terms vanish. In the $n = 0$ case, taking 0^0 to be 1 gives

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} 0^n &= \frac{\Gamma(\frac{1}{2})}{\Gamma(1)} \\ &= \sqrt{\pi},\end{aligned}$$

recovering the famous result

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

6.4. **Identities Involving the Beta Function.** In this section, we prove that

$$\sum_{n=0}^{\infty} \frac{1}{n} \frac{B(n+1,k+1)}{B(n,k)} = 1.$$

To do so, we first find a closed form for (9) and generalize it for non-negative k.

Lemma 4. *For all* $t \in (0, 2]$ *and* $k \in \mathbb{R}^+$,

$$\boxed{\sum_{n=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+1)}(1-t)^n = \frac{\Gamma(k)}{t^k}.}$$

Proof. Observe that the integrand of the RHS in (10) is clearly even, giving us

$$\int_{-\infty}^{\infty} x^{2k} e^{-tx^2}\,\mathrm{d}x = 2\int_{0}^{\infty} x^{2k} e^{-tx^2}\,\mathrm{d}x\,.$$

Under the substitution $u = tx^2$, this integral simplifies as

$$\frac{1}{t^{k+\frac{1}{2}}} \int_{0}^{\infty} u^{k-\frac{1}{2}} e^{-u}\,\mathrm{d}u,$$

which we recognize to be of the form

$$\Gamma(z) = \int_0^\infty u^z e^{-u}\,\mathrm{d}u\,.$$

We thus have

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+k+\frac{1}{2})}{\Gamma(n+1)}(1-t)^n = \frac{1}{t^{k+\frac{1}{2}}}\Gamma\left(k+\frac{1}{2}\right).$$

Since the above identity is written in terms of the Gamma function, we can extend k to the non-negative real numbers. With the translation $k+\frac{1}{2} \mapsto k$, we have the desired identity:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+1)}(1-t)^n = \frac{\Gamma(k)}{t^k}.$$

□

Remark. Setting $k=\frac{1}{2}$, we recover (8):

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}(1-t)^n &= \frac{\Gamma(\frac{1}{2})}{t^{\frac{1}{2}}} \\ &= \sqrt{\frac{\pi}{t}}.\end{aligned}$$

Setting $k=1$ and $t \mapsto 1-t$ recovers the infinite geometric series formula:

$$\begin{aligned}\sum_{n=0}^{\infty} t^n &= \frac{\Gamma(1)}{(1-t)^1} \\ &= \frac{1}{1-t}\end{aligned}$$

We now prove the main result for this subsection.

Theorem 4. *For $t \in (0,2]$ and $k \in \mathbb{R}^+$,*

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{n}\frac{B(n+1,k+1)}{B(n,k)} = 1,}$$

Proof. After rearranging Lemma 4 and applying the property $\Gamma(n+1) = n\Gamma(n)$, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n}\frac{\Gamma(n+k)}{\Gamma(n)\,\Gamma(k)} t^k (1-t)^n = 1,$$

where the $n=0$ term should be taken as a limit. Invoking the relationship between the Gamma and Beta functions once more, we have

$$\sum_{n=0}^{\infty} \frac{1}{n}\frac{1}{B(n,k)} t^k (1-t)^n = 1.$$

Integrating both sides with respect to t over the interval $(0,1)$ yields

$$\sum_{n=0}^{\infty} \frac{1}{n}\frac{1}{B(n,k)} \int_0^1 t^k (1-t)^n\,\mathrm{d}t = 1. \tag{11}$$

Recall that the Beta function has the definition

$$B(z_1,z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,\mathrm{d}t\,.$$

Hence, the inner integral in (11) collapses to $B(k+1, n+1)$. Given the symmetry of the Beta function, this is equivalent to $B(n+1, k+1)$, thus giving

$$\sum_{n=0}^{\infty} \frac{1}{n} \frac{B(n+1, k+1)}{B(n, k)} = 1$$

as desired. □

Remark. It is not too hard to verify the above identity. Expanding the Beta function in terms of the Gamma function gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n} \frac{B(n+1, k+1)}{B(n, k)} &= \sum_{n=0}^{\infty} \frac{1}{n} \; \frac{\Gamma(n+1)\,\Gamma(k+1)}{\Gamma(n+k+2)} \Bigg/ \frac{\Gamma(n)\,\Gamma(k)}{\Gamma(n+k)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n} \; \frac{n\Gamma(n)\,k\Gamma(k)}{(n+k+1)(n+k)\Gamma(n+k)} \Bigg/ \frac{\Gamma(n)\,\Gamma(k)}{\Gamma(n+k)} \\
&= \sum_{n=0}^{\infty} \frac{k}{(n+k+1)(n+k)}.
\end{aligned}$$

Performing partial fraction decomposition gives us the telescoping sum

$$\sum_{n=0}^{\infty} \left(\frac{k}{n+k} - \frac{k}{n+k+1} \right),$$

which takes on the value of its 0th term, i.e. $\frac{k}{0+k} = 1$.

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