The Integral of Higher Powers of the Arctangent Derivative Function

Eytan Chong

1 Introduction

Define

$$f_n(x) = \frac{1}{(x^2+1)^{n+1}},$$

and let I_n and J_n be the definite and indefinite integrals of $f_n(x)$ as so:

$$I_n = \int_{-\infty}^{\infty} f_n(x)\,\mathrm{d}x \quad \text{and} \quad J_n(x) = \int f_n(x)\,\mathrm{d}x.$$

In this note, we prove using contour integration that for $n \in \mathbb{N}$[1],

$$I_n = \frac{\pi}{2^{2n}}\binom{2n}{n}. \tag{A.1}$$

We extend our result with the beta function and show that for $n \geq 0$, we have the formula

$$I_n = \mathrm{B}\left(\frac{1}{2}, n+\frac{1}{2}\right). \tag{A.2}$$

Lastly, we evaluate J_n to be

$$J_n = \frac{1}{2^{2n}}\binom{2n}{n}\left[\arctan x + \sum_{k=1}^{n} \frac{2^{2k-1}}{k}\binom{2k}{k}^{-1}\frac{x}{(x^2+1)^k}\right] + C,. \tag{B}$$

2 Evaluating I_n

2.1 The Discrete Case

Consider the case where $n \in \mathbb{N}$. We evaluate I_n using contour integration.

We look at a more general integral. Define

$$f(z) = \frac{\mathrm{e}^{\mathrm{i}\omega z}}{(z^2+1)^{n+1}}.$$

Let $\gamma_1 = [-R, R]$ and $\gamma_2 = \{z = R\mathrm{e}^{\mathrm{i}\theta}, \theta \in (0,\pi)\}$, where $R > 0$, as illustrated in Figure 1. We evaluate

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{(x^2+1)^{n+1}}\,\mathrm{d}x = \operatorname{Re} \lim_{R\to\infty} \int_{\gamma_1} f(z)\,\mathrm{d}z.$$

Date: July 16, 2025
Latest revision: https://asdia.dev/expository/arctan-int.pdf

[1] We take the convention that $0 \in \mathbb{N}$.

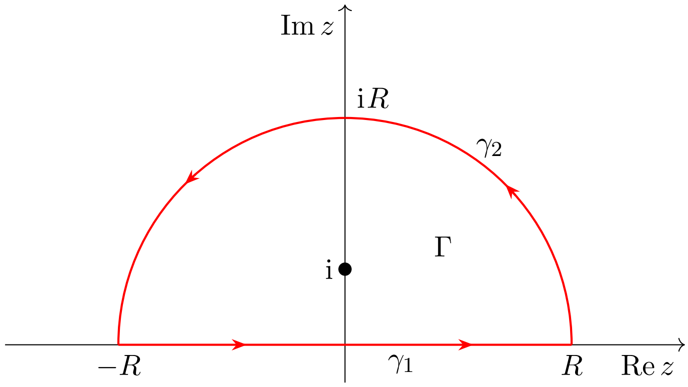

Figure 1: The semicircular, anti-clockwise contour Γ and the enclosed singularity i.

Proposition 1. *In the limit $R \to \infty$, we have $\oint_\Gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$.*

Proof. If $\omega > 0$, then a simple application of Jordan's lemma reveals

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \le \frac{\pi}{\omega} \sup_{\theta\in[0,\pi]} \left|\frac{1}{(R^2\mathrm{e}^{2\mathrm{i}\theta}+1)^{n+1}}\right| = O\left(\frac{1}{R^{2n+2}}\right) \to 0.$$

If $\omega = 0$, the ML inequality gives the same result:

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \le \pi R \sup_{\theta\in[0,\pi]} \left|\frac{1}{(R^2\mathrm{e}^{2\mathrm{i}\theta}+1)^{n+1}}\right| = O\left(\frac{R}{R^{2n+2}}\right) \to 0.$$

Thus,

$$\lim_{R\to\infty} \oint_\Gamma f(z)\,\mathrm{d}z = \lim_{R\to\infty}\left(\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z\right) = \lim_{R\to\infty}\int_{\gamma_1} f(z)\,\mathrm{d}z.$$

□

Definition 2. The *nth reverse Bessel polynomial* $\theta_n(x)$ is defined as

$$\theta_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!}\frac{x^{n-k}}{2^k}.$$

Proposition 3. *The residue of f at the singularity $z = \mathrm{i}$ is*

$$\operatorname*{Res}_{z=\mathrm{i}}(f) = \frac{\theta_n(\omega)}{n!\,\mathrm{e}^{\omega}2^{n+1}\mathrm{i}}.$$

Proof. The singularity at $z = \mathrm{i}$ has order $n+1$, so

$$\operatorname*{Res}_{z=\mathrm{i}}(f) = \frac{1}{n!}\lim_{z\to\mathrm{i}}\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z-\mathrm{i})^{n+1}f(z) = \frac{1}{n!}\lim_{z\to\mathrm{i}}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{\mathrm{e}^{\mathrm{i}\omega z}}{(z+\mathrm{i})^{n+1}}.$$

By the general Leibniz rule, we have

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{\mathrm{e}^{\mathrm{i}\omega z}}{(z+\mathrm{i})^{n+1}} &= \sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^k}{\mathrm{d}z^k}(z+\mathrm{i})^{-n-1}\frac{\mathrm{d}^{n-k}}{\mathrm{d}z^{n-k}}\mathrm{e}^{\mathrm{i}\omega z} \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{(-n-1)(-n-2)\dots(-n-k)}{(z+\mathrm{i})^{n+1+k}}\right]\left[(\mathrm{i}\omega)^{n-k}\mathrm{e}^{\mathrm{i}\omega z}\right] \\
&= \sum_{k=0}^{n}\left[\frac{n!}{k!(n-k)!}\right]\left[(-1)^k\frac{(n+k)!}{n!}\frac{1}{(z+\mathrm{i})^{n+1+k}}\right]\left[\mathrm{i}^{n-k}\omega^{n-k}\mathrm{e}^{\mathrm{i}\omega z}\right] \\
&= \mathrm{e}^{\mathrm{i}\omega z}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\frac{\mathrm{i}^{n+k}\omega^{n-k}}{(z+\mathrm{i})^{n+1+k}}.
\end{aligned}$$

As $z \to \mathrm{i}$, we have

$$\begin{aligned}\lim_{z\to\mathrm{i}} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \frac{\mathrm{e}^{\mathrm{i}z}}{(z+\mathrm{i})^{n+1}} &= \mathrm{e}^{-\omega} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} \frac{\mathrm{i}^{n+k}\omega^{n-k}}{(2\mathrm{i})^{n+1+k}} \\ &= \frac{1}{\mathrm{e}^{\omega}2^{n+1}\mathrm{i}} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} \frac{\omega^{n-k}}{2^k} \\ &= \frac{\theta_n(\omega)}{\mathrm{e}^{\omega}2^{n+1}\mathrm{i}}.\end{aligned}$$

We hence have

$$\operatorname*{Res}_{z=\mathrm{i}}(f) = \frac{\theta_n(\omega)}{n!\,\mathrm{e}^{\omega}2^{n+1}\mathrm{i}}$$

as desired. □

Lemma 4. *For $n \in \mathbb{N}$ and $\omega \geq 0$, we have*

$$\int_{-\infty}^{\infty} \frac{\cos\omega x}{(x^2+1)^{n+1}}\,\mathrm{d}x = \frac{\pi}{\mathrm{e}^{\omega}} \frac{\theta_n(\omega)}{n!\,2^n}.$$

Proof. From Proposition 1, we have

$$\int_{-\infty}^{\infty} \frac{\cos\omega x}{(x^2+1)^{n+1}}\,\mathrm{d}x = \operatorname{Re} \lim_{R\to\infty} \int_{\gamma_1} f(z)\,\mathrm{d}z = \operatorname{Re} \lim_{R\to\infty} \oint_{\Gamma} f(z)\,\mathrm{d}z.$$

Since the only singularity enclosed by Γ is $z = \mathrm{i}$, by Cauchy's residue theorem and Proposition 3, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos\omega x}{(x^2+1)^{n+1}}\,\mathrm{d}x = \operatorname{Re} \lim_{R\to\infty} \frac{2\pi\mathrm{i}\theta_n(\omega)}{n!\,\mathrm{e}^{\omega}2^{n+1}\mathrm{i}} = \frac{\pi}{\mathrm{e}^{\omega}} \frac{\theta_n(\omega)}{n!\,2^n}.$$

□

Remark. Taking $n = 0$, $a = 1$, and $\omega = 1$ recovers the celebrated result

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,\mathrm{d}x = \frac{\pi}{\mathrm{e}}.$$

Proposition 5. *For all $n \in \mathbb{N}$, we have $\theta_n(0) = (2n-1)!!$.*

Proof. From the definition of $\theta_n(x)$, we have

$$\theta_n(0) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} \frac{0^{n-k}}{2^k}.$$

Taking the combinatorial view that

$$0^m = \begin{cases} 1, & m = 1, \\ 0, & \text{otherwise}, \end{cases}$$

we see that everything apart from the $k = n$ term vanishes. Thus,

$$\theta_n(0) = \frac{(2n)!}{n!\,2^n} = \frac{(2n)!}{(2n)!!} = (2n-1)!!,$$

which was what we wanted. □

We are now ready to evaluate I_n.

Theorem A.1. *For $n \in \mathbb{N}$, we have*

$$I_n = \frac{\pi}{2^{2n}} \binom{2n}{n}.$$

Proof. Observe that our goal integral I_n corresponds to the case where $\omega = 0$. Applying Lemma 4 and Proposition 5 gives

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^{n+1}} \, \mathrm{d}x = \frac{\pi \theta_n(0)}{n! \, 2^n} = \frac{\pi (2n-1)!!}{n! \, 2^n} = \frac{\pi}{2^{2n}} \binom{2n}{n},$$

where we used the identity that

$$\frac{(2n-1)!!}{n! \, 2^n} = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{2^n} \binom{2n}{n}$$

in the last step. □

2.2 The Continuous Case

In this section, we use the beta function to extend the domain of our integral I_n from the natural numbers to the non-negative real numbers.

Definition 6. The *beta function* $\mathrm{B}(z_1, z_2)$ is defined as the definite integral

$$\mathrm{B}(z_1, z_2) = \int_0^{\infty} \frac{t^{z_1 - 1}}{(1+t)^{z_1 + z_2}} \, \mathrm{d}t,$$

where z_1 and z_2 are complex parameters with positive real parts.

Theorem A.2. *For $n \geq 0$, we have*

$$I_n = \mathrm{B}\left(\frac{1}{2}, n + \frac{1}{2}\right).$$

Proof. Under the substitution $t = x^2$, we obtain

$$I_n = 2 \int_0^{\infty} \frac{1}{(x^2+1)^{n+1}} \, \mathrm{d}x = \int_0^{\infty} \frac{t^{-1/2}}{(t+1)^{n+1}} \, \mathrm{d}t,$$

which we observe to be the beta function with parameters $z_1 = 1/2$ and $z_2 = n + 1/2$. □

2.3 Identities

Comparing our results for the discrete and continuous cases, we obtain a simple expression for $\mathrm{B}(1/2, n+1/2)$ when $n \in \mathbb{N}$.

Proposition 7. *For $n \in \mathbb{N}$, we have*

$$\mathrm{B}\left(\frac{1}{2}, n + \frac{1}{2}\right) = \frac{\pi}{2^{2n}} \binom{2n}{n}.$$

We can also derive the generating function for the central binomial coefficients easily.

Proposition 8. *The generating function for the central binomial coefficients is $1/\sqrt{1-4t}$.*

Proof. Consider the infinite sum

$$S = \sum_{n=0}^{\infty} \binom{2n}{n} t^n = \sum_{n=0}^{\infty} \left[\frac{4^n}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^{n+1}} \, \mathrm{d}x \right] t^n.$$

Switching the order of summation and integration gives us our desired result:

$$\begin{aligned} S &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2+1} \sum_{n=0}^{\infty} \left[\frac{4t}{x^2+1} \right]^n \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2+1} \left(\frac{1}{1 - \frac{4t}{x^2+1}} \right) \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2 + (1-4t)} \, \mathrm{d}x \\ &= \frac{1}{\pi} \left[\frac{1}{\sqrt{1-4t}} \arctan\left(\frac{t}{\sqrt{1-4t}} \right) \right]_{-\infty}^{\infty} \\ &= \frac{1}{\sqrt{1-4t}}. \end{aligned}$$

□

3 Evaluating J_n

In this section, we evaluate J_n by means of a recurrence relation.

Lemma 9. *For $k \geq 1$, we have*

$$J_k - \alpha_k J_{k-1} = \beta_k f_{k-1},$$

where

$$\alpha_k = \frac{2k-1}{2k} \quad \textit{and} \quad \beta_k = \frac{x}{2k}.$$

Proof. Consider

$$J_{k-1} = \int \frac{1}{(x^2+1)^k} \, \mathrm{d}x.$$

Integrating by parts, we obtain

$$\begin{aligned} J_{k-1} &= \frac{x}{(x^2+1)^k} + 2k \int \frac{x^2}{(x^2+1)^{k+1}} \, \mathrm{d}x \\ &= \frac{x}{(x^2+1)^k} + 2k \left(\int \frac{1}{(x^2+1)^k} \, \mathrm{d}x - \int \frac{1}{(x^2+1)^{k+1}} \, \mathrm{d}x \right) \\ &= \frac{x}{(x^2+1)^k} + 2k \left(J_{k-1} - 1 J_k \right), \end{aligned}$$

which rearranges to the first order recurrence relation

$$J_k - \frac{2k-1}{2k} J_{k-1} = \frac{x}{2k} f_{k-1}(x)$$

as desired. □

Proposition 10. *For $0 \leq k \leq n$, we have*

$$\alpha_{k+1} \alpha_{k+2} \dots \alpha_n = 2^{2k-2n} \binom{2n}{n} \binom{2k}{k}^{-1}.$$

Proof. We have

$$\begin{aligned}
\alpha_{k+1}\alpha_{k+2}\ldots\alpha_n &= \prod_{i=k+1}^{n}\frac{2i-1}{2i} \\
&= \frac{(2n-1)!!}{(2n)!!}\frac{(2k)!!}{(2k-1)!!} \\
&= \left[\frac{1}{2^{2n}}\binom{2n}{n}\right]\left[\frac{1}{2^{2k}}\binom{2k}{k}\right]^{-1} \\
&= 2^{2k-2n}\binom{2n}{n}\binom{2k}{k}^{-1}.
\end{aligned}$$

Note that in the case where $k = 0$, the result still holds with the standard convention that $0!! = (-1)!! = 1$. □

Theorem B. *For $n \in \mathbb{N}$, we have*

$$J_n = \frac{1}{2^{2n}}\binom{2n}{n}\left[\arctan x + \sum_{k=1}^{n}\frac{2^{2k-1}}{k}\binom{2k}{k}^{-1}\frac{x}{(x^2+1)^k}\right] + C.$$

Proof. Using the recurrence relation derived in Lemma 9, we easily obtain the following n equations:

$$\begin{array}{rlcrlcr}
 & J_n & - & \alpha_n & J_{n-1} & = & f_{n-1}\beta_n, \\
\alpha_n & J_{n-1} & - & \alpha_{n-1}\alpha_n & J_{n-2} & = & f_{n-2}\beta_{n-1}\alpha_n \\
\alpha_{n-1}\alpha_n & J_{n-2} & - & \alpha_{n-2}\alpha_{n-1}\alpha_n & J_{n-3} & = & f_{n-3}\beta_{n-2}\alpha_{n-1}\alpha_n \\
 & \vdots & & & \vdots & & \vdots \\
\alpha_2\alpha_3\alpha_4\ldots\alpha_n & J_1 & - & \alpha_1\alpha_2\alpha_3\ldots\alpha_n & J_0 & = & f_0\beta_1\alpha_2\alpha_3\ldots\alpha_n.
\end{array}$$

Summing each column yields

$$J_n - \alpha_1\ldots\alpha_n J_0 = \sum_{k=1}^{n} f_{k-1}\beta_k\alpha_{k+1}\ldots\alpha_n.$$

Substituting $J_0 = \arctan x + C$ and invoking Proposition 10, we finally obtain

$$\begin{aligned}
J_n &= 2^{-2n}\binom{2n}{n}\arctan x + \sum_{k=1}^{n}\left[\frac{1}{(x^2+1)^k}\right]\left[\frac{x}{2k}\right]\left[2^{2k-2n}\binom{2n}{n}\binom{2k}{k}^{-1}\right] + C \\
&= \frac{1}{2^{2n}}\binom{2n}{n}\left[\arctan x + \sum_{k=1}^{n}\frac{2^{2k-1}}{k}\binom{2k}{k}^{-1}\frac{x}{(x^2+1)^k}\right] + C.
\end{aligned}$$

□

Remark. Given this formula for J_n, we can easily verify Theorem A.1:

$$\begin{aligned}
I_n &= \frac{1}{2^{2n}}\binom{2n}{n}\left[\arctan x + \sum_{k=1}^{n}\frac{2^{2k-1}}{k}\binom{2k}{k}^{-1}\frac{x}{(x^2+1)^k}\right]_{-\infty}^{\infty} \\
&= \frac{1}{2^{2n}}\binom{2n}{n}\left[\arctan x\right]_{-\infty}^{\infty} \\
&= \frac{\pi}{2^{2n}}\binom{2n}{n}.
\end{aligned}$$

References

[1] D. H. Bailey, J. M. Borwein, N. J. Calkin, R. Girgensohn, D. R. Luke, and V. H. Moll. *Experimental Mathematics In Action.* A K Peters, 2006.

[2] L. Carlitz. A note on the Bessel polynomials. *Duke Mathematical Journal*, 24(2):151 – 162, 1957.

[3] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Elsevier, 2007.

[4] S. Hassani. *Mathematical Methods: For Students of Physics and Related Fields.* Springer, 2000.