Coins Flips, Fair Casinos and Martingales

An Introduction to the ABRACADABRA Theorem

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Probability Crash Course

Sample Space and Probability

The *sample space*, denoted S, is the set of all possible outcomes that can occur.

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Suppose I roll a six-sided dice. There are 6 possible outcomes: I roll a 1, I roll a 2, etc. This is my sample space. For convenience, write

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Let E denote the event "I roll a 1 or a 2". This corresponds to the subset $\{1, 2\}$.

The probability of E happening is $\mathbb{P}[E] = \frac{2}{6}$.

Random Variables

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The outcome of my dice roll is a random variable.

The outcome of a coin flip is a random variable.

Probability Distribution

A *probability distribution* describes all possible values of the random variable and their corresponding probabilities.

It assigns a probability value to each possible outcome in the sample space.

When writing probability distributions, we write particular values of a random variable using lower-case letters.

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When writing probability distributions, we write particular values of a random variable using lower-case letters.

The probability distribution of a dice roll is

x	1	2	3	4	5	6
$\mathbb{P}[X = x]$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

For example, $\mathbb{P}[X = 1] = \frac{1}{6}$, $\mathbb{P}[X = 1 \text{ or } 2] = \frac{2}{6}$.

Expectation of a Random Variable

We typically want to know the “average value” of a random variable X.

We call this the expectation of X, denoted $\mathbb{E}[X]$.

We define $\mathbb{E}[X]$ as

$$\mathbb{E}[X] = \sum_{x \in S} x \, \mathbb{P}[X = x] \, .$$

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$$\mathbb{E}[Y] = (1)\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) = 0.$$

Exercise

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$$\mathbb{E}[X] = (1)\left(\frac{1}{6}\right) + (2)\left(\frac{1}{6}\right) + (3)\left(\frac{1}{6}\right) + (4)\left(\frac{1}{6}\right) + (5)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{6}\right) = 3.5.$$

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Suppose the dice is now biased, so X has probability distribution

x	1	2	3	4	5	6
$\mathbb{P}[X = x]$	0.2	0.3	0	0	0.3	0.2

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Let X be the outcome of a dice roll. What is $\mathbb{E}[X]$?

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What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = 1(0.2) + 2(0.3) + 3(0) + 4(0) + 5(0.3) + 6(0.2) = 3.6.$$

Conditional Probability

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Given this information, the probability of an event changes. For instance, the probability that we roll a 3 is now much higher.

We write this as

$$\mathbb{P}[X = 3 \mid \underbrace{X \neq 1, X \neq 2}_{\text{given information}}].$$

Conditional Expectation

Similarly, given some information about the situation, the expectation of X also changes. The notation is identical:

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$$\mathbb{E}[X \mid A] = \sum_{x \in S} x \, \mathbb{P}[X = x \mid A] \, .$$

This is almost identical to what we saw previously:

$$\mathbb{E}[X] = \sum_{x \in S} x \, \mathbb{P}[X = x] \, .$$

Law of Total Expectation

The law of total expectation states that

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \, \mathbb{P}[A_i] \,,$$

where $A_1, A_2, \ldots, A_n$ partitions the sample space.

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Suppose you are picking a banknote from two bags, say Bag A and Bag B. Bag A has a \$2 note, a \$5 note and a \$10 note. Bag B has a \$50 note and a \$100 note. You have an 80% chance of taking a note from Bag A. What is your expected profit?

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Let X be my profit. By the law of total expectation,

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X \mid \text{Bag A}]\,\mathbb{P}[\text{Bag A}] + \mathbb{E}[X \mid \text{Bag B}]\,\mathbb{P}[\text{Bag B}] \\ &= \frac{2+5+10}{3}(0.8) + \frac{50+100}{2}(0.2) \\ &= 19.53.\end{aligned}$$

Hence, I expect to win \$19.53.

A Coin-Flip Problem

Our Problem

A fair coin is flipped repeatedly until a given sequence of Heads and Tails appears.

On average, how many times is the coin flipped?

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Ans: 2

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Suppose we flip a fair coin until we get Tails. On average, how many times is the coin flipped?

Ans: 2

Definition 2.

The **terminator** $\mathcal{T}$ is the sequence that terminates the coin-flipping.

Our Problem (Rephrased)

Let $W_{\mathcal{T}}$ be a word constructed by randomly concatenating the letters H and T until we reach a terminator $\mathcal{T}$. Let $L_{\mathcal{T}} = |W_{\mathcal{T}}|$ be the length of the resulting word.

What is $\mathbb{E}[L_{\mathcal{T}}]$?

A Coin-Flip Problem

A Naive Approach

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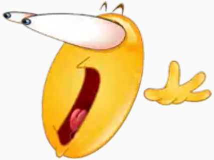
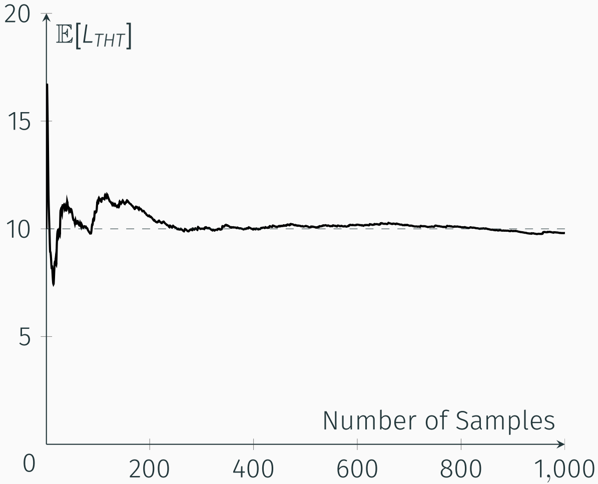


A Naive Approach



Martingales and the Optional Stopping Theorem

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Martingales

Informal Definition

A martingale is a random process which models a gambler's fortune in a *fair game*.

Is typically represented by a sequence of random variables $X_0, X_1, X_2, \ldots$.

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Is typically represented by a sequence of random variables $X_0, X_1, X_2, \ldots$.

Some motivating questions:

- What is a fair game?
- What properties do fair games possess?

A Prototypical Example

Flip a fair coin.

- If it comes up H, we win \$1.
- If it comes up T, we lose \$1.

Repeat this process forever.

A Prototypical Example

Let X_n be our wealth after the nth coin-flip.

Let Y_n represent the outcome of the nth coin-flip.

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- The coin is *fair*.

$$\mathbb{P}[Y_n = H] = \mathbb{P}[Y_n = T] = \frac{1}{2}.$$

A Prototypical Example

Let X_n be our wealth after the nth coin-flip.

Let Y_n represent the outcome of the nth coin-flip.

- The coin is *fair*.

$$\mathbb{P}[Y_n = H] = \mathbb{P}[Y_n = T] = \frac{1}{2}.$$

- The coin-flips are *independent*: The outcomes of past coin-flips do not influence the outcomes of future flips.

$$\mathbb{P}[Y_{n+1} \mid Y_1, \ldots, Y_n] = \mathbb{P}[Y_{n+1}]\,.$$

A Prototypical Example

$$\mathbb{E}[X_{n+1} \mid Y_1, Y_2, \ldots Y_n]$$

A Prototypical Example

$$\mathbb{E}[X_{n+1} \mid Y_1, Y_2, \ldots Y_n]$$
$$= (X_n + 1)\,\mathbb{P}[Y_{n+1} = H \mid Y_1, \ldots, Y_n] + (X_n - 1)\,\mathbb{P}[Y_{n+1} = T \mid Y_1, \ldots, Y_n]$$

A Prototypical Example

$$
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&\qquad = (X_n + 1)\,\mathbb{P}[Y_{n+1} = H] + (X_n - 1)\,\mathbb{P}[Y_{n+1} = T] \\
&\qquad = \frac{1}{2}\,(X_n + 1) + \frac{1}{2}\,(X_n - 1)
\end{aligned}
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&\mathbb{E}[X_{n+1} \mid Y_1, Y_2, \dots Y_n] \\
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&\qquad = (X_n + 1)\,\mathbb{P}[Y_{n+1} = H] + (X_n - 1)\,\mathbb{P}[Y_{n+1} = T] \\
&\qquad = \frac{1}{2}(X_n + 1) + \frac{1}{2}(X_n - 1) \\
&\qquad = X_n.
\end{aligned}$$

Our expected wealth after the next flip, given that we know all previous outcomes, is exactly our current wealth.

This is the defining property of a martingale.

Formal Definition

Definition 7.

A sequence of random variables $X_1, X_2, \dots$ is a **martingale** with respect to the sequence $Y_1, Y_2, \dots$ if

$$\mathbb{E}[X_{n+1} \mid Y_1, \dots, Y_n] = X_n.$$

Proposition 8.

Let X_n represent the price of a stock on day n. Then $\{X_n\}$ is a martingale with respect to itself.

The Stock Market

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Proof.

Suppose $\mathbb{E}[X_{n+1} \mid X_1, \ldots, X_n] > X_n$.

- Buying the stock today and selling tomorrow yields a profit (in expectation).
- Those that own the stock today will not sell today, since its value is expected to increase tomorrow.

Demand increases, supply decrease $\implies$ today's stock price increases.

The Stock Market

Proposition 8.

Let X_n represent the price of a stock on day n. Then $\{X_n\}$ is a martingale with respect to itself.

Proof.

Suppose instead $\mathbb{E}[X_{n+1} \mid X_1, \ldots, X_n] < X_n$.

- Those who want to buy the stock would rather buy it tomorrow, since it will be cheaper.
- Those that own the stock today will want to sell today, since its value is expected to decrease tomorrow.

Demand decreases, supply increases $\implies$ today's stock price decreases.

Proposition 8.

Let X_n represent the price of a stock on day n. Then $\{X_n\}$ is a martingale with respect to itself.

Proof.

Today's stock price will eventually reach an equilibrium, where

$$\mathbb{E}[X_{n+1} \mid X_1, \ldots, X_n] = X_n.$$

This is exactly the condition for $\{X_n\}$ to be a martingale! $\square$

The Stock Market (In Real Life)

Efficient Market Hypothesis (EMH): Share prices reflect all available information and cannot consistently be beat.

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Efficient Market Hypothesis (EMH): Share prices reflect all available information and cannot consistently be beat.

The EMH is generally true. The market is not perfectly efficient

See: alpha generation in quantitative finance.

Exercise

Let $X_n = 10$ for all $n \geq 1$. Is $X_1, X_2, \ldots$ a martingale?

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Yes.

$$\mathbb{E}[X_{n+1} \mid Y_1, \ldots, Y_n] = \mathbb{E}[10 \mid Y_1, \ldots, Y_n] = 10 = X_n.$$

Exercise

Let $X_n = n$ for all $n \geq 1$. Is $X_1, X_2, \ldots$ a martingale?

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No.

$$\mathbb{E}[X_{n+1} \mid Y_1, \ldots, Y_n] = \mathbb{E}[n + 1 \mid Y_1, \ldots, Y_n] = n + 1 \neq n = X_n.$$

Exercise

Let $Y_1, Y_2, Y_3, \dots$ be a sequence of independent random variables, each equal to -1 with probability 1/2 and 1 with probability 1/2. Let $X_n = Y_1 + Y_2 + \cdots + Y_n$ for $n > 0$. Is X_n a martingale with respect to Y_n?

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Yes.

Observe that $X_{n+1} = X_n + Y_{n+1}$. So

$$\begin{aligned}\mathbb{E}[X_{n+1} \mid Y_1, \ldots, Y_n] &= \mathbb{E}[X_n + Y_{n+1} \mid Y_1, \ldots, Y_n] \\ &= \mathbb{E}[X_n \mid Y_1, \ldots, Y_n] + \mathbb{E}[Y_{n+1} \mid Y_1, \ldots, Y_n]\end{aligned}$$

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&= \mathbb{E}[X_n \mid Y_1, \ldots, Y_n] + \mathbb{E}[Y_{n+1} \mid Y_1, \ldots, Y_n] \\
&= \mathbb{E}[X_n] + \mathbb{E}[Y_{n+1}]
\end{aligned}$$

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&= \mathbb{E}[X_n] + \mathbb{E}[Y_{n+1}] \\
&= X_n + \left[\left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(-1)\right]
\end{aligned}$$

Exercise

Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of independent random variables, each equal to -1 with probability 1/2 and 1 with probability 1/2. Let $X_n = Y_1 + Y_2 + \cdots + Y_n$ for $n > 0$. Is X_n a martingale with respect to Y_n?

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&= \mathbb{E}[X_n] + \mathbb{E}[Y_{n+1}] \\
&= X_n + \left[\left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(-1)\right] \\
&= X_n.
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Martingales and the Optional Stopping Theorem

Stopping Times and Strategies

Informal Definition

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This is not very realistic, so we introduce the idea of a *stopping time*.

A stopping time τ is the round where a gambler quits playing a game.

He cannot see into the future (view the outcome of future rounds) to decide when to stop playing.

Formal Definition

Definition 9.

A **stopping time** τ with respect to a sequence $\{Y_n\}$ is a random variable taking values in $\mathbb{N} \cup \{\infty\}$ such that for all $n \in \mathbb{N}$, the event $\{\tau = n\}$ depends solely on $Y_1, \ldots, Y_n$. This event is called the gambler's **stopping strategy**.

An Example

Example 10.

Stopping strategy: quit after 10 games.

Stopping time: $\tau = 10$.

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Stopping strategy: quit after 10 games.

Stopping time: $\tau = 10$.

Stopping strategy: quit after 3 losses in a row.

Stopping time: $\{\tau = n\}$ is

$$\Bigg\{ \underbrace{Y_{n-2} = -1,\ Y_{n-1} = -1,\ Y_n = -1}_{\text{3 losses in a row}} \Bigg\}.$$

Exercise

Suppose a gambler plays a fair game with a \$1 stake per round. Determine if the following events are stopping strategies.

i. The third time the gambler loses in a row.

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ii. Two rounds before the gambler profits $50.

Exercise

Suppose a gambler plays a fair game with a \$1 stake per round. Determine if the following events are stopping strategies.

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Martingales and the Optional Stopping Theorem

The Optional Stopping Theorem

Motivating Question

Q: Is there a stopping strategy that returns a profit (on average)?

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A: Generally, no.

The OST

Theorem 11 (Doob's Optional Stopping Theorem).

Let $\{X_n\}$ be a martingale and let τ be a stopping time, both with respect to $\{Y_n\}$. Then $\mathbb{E}[X_\tau] = X_0$ if at least one of the following holds:

1. $|X_n|$ is bounded.
2. τ is bounded.
3. $\mathbb{E}[\tau]$ is finite, and all increments of X are bounded, i.e. there exists a constant C such that for all n,
$$|X_{n+1} - X_n| \leq C.$$

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$$|X_{n+1} - X_n| \leq C.$$

As long as our stopping strategy is *reasonable enough*, our expected payout must be equal to the amount we started with.

Breaking the OST

Suppose our strategy is profitable. Then it either

- breaks the validity of our stopping time, or
- breaks all three conditions of the OST.

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- $\tau = \infty$: We play the game forever (impossible).
- Same problems arise if $\mathbb{E}[\tau] = \infty$ and $|X_{n+1} - X_n|$ is unbounded.

Hence, practically all strategies obey the OST.

The Gambler's Ruin Problem

We start with $\$K$.

Each round, we flip a fair coin.

- If it lands H, we gain \$1.
- If it lands T, we lose \$1.

We keep playing until we go bankrupt, or have a total of $\$N$.

What is the probability of going bankrupt?

The Gambler's Ruin Problem

Let Y_n be the outcome of the nth flip.

Let X_n be our wealth after the nth flip.

Stopping time $\tau = \min\{n : X_n = 0 \text{ or } X_n = N\}$.

We wish to find $\mathbb{P}[X_\tau = 0]$.

We previously proved that $\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

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X_n is bounded: $0 \leq X_n \leq N$.

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X_n is bounded: $0 \leq X_n \leq N$.

By OST (scenario 1), $\mathbb{E}[X_\tau] = X_0 = K$.

The Gambler's Ruin Problem

We previously proved that $\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

X_n is bounded: $0 \leq X_n \leq N$.

By OST (scenario 1), $\mathbb{E}[X_\tau] = X_0 = K$.

We can also write $\mathbb{E}[X_\tau] = N\,\mathbb{P}[X_\tau = N] + 0\,\mathbb{P}[X_\tau = 0]$.

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So $\mathbb{P}[X_\tau = N] = \frac{K}{N}$ and $\mathbb{P}[X_\tau = 0] = 1 - \frac{K}{N}$.

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So $\mathbb{P}[X_\tau = N] = \frac{K}{N}$ and $\mathbb{P}[X_\tau = 0] = 1 - \frac{K}{N}$.

The greedier we are, the higher the probability of us going bankrupt.

The ABRACADABRA Theorem

The ABRACADABRA Theorem

The Fair Casino

Setting up the Scene

Imagine you work as a dealer at D'Casino.

There is only one game available for play at D'Casino.

The Game

Each round, you flip a fair-coin.

Gamblers go all-in, betting on the outcome of this coin-flip.

- If they win, they double their money, and they play again.
- If they lose, they lose everything and go home.

This repeats until a terminator (say *THT*) appears, at which point the casino closes.

Setting up the Scene

Let Y_n be the outcome of the nth coin flip.

Let τ be the stopping time.

Note that $\{\tau = n\}$ is the event that $Y_{n-2}Y_{n-1}Y_n$ is THT.

Setting up the Scene

A group of gamblers, obsessed with the sequence THT, frequents D'Casino.

Every flip, a new gambler from this group arrives with \$1 and plays the game, hoping that the subsequent flips appear T, H, T in that order.

A Simple Example

$$Y_1 = T$$

Gambler #1

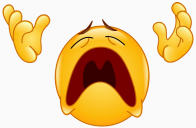
- Bets \$1 that $Y_1 = T$.
- Wins \$2 and plays again.

A Simple Example

$$Y_2 = H$$

Gambler #1

- Bets \$2 that $Y_2 = H$.
- Wins \$4 and plays again.



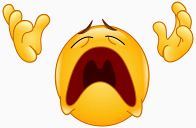
Gambler #2

- Bets \$1 that $Y_2 = T$.
- Loses \$1 and stops playing.

$$Y_3 = T$$

Gambler #1

- Bets \$4 that $Y_3 = T$.
- Wins \$8.



Gambler #2

Gambler #3

- Bets \$1 that $Y_3 = T$.
- Wins \$2.

Since terminator *THT* appears, the game stops.

A Simple Example

	Coin Flip #1	Coin Flip #2	Coin Flip #3
Total won	\$2	\$4	\$8 + \$2
Total lost	\$1	\$2 + \$1	\$4 + \$1

A Simple Example

	Coin Flip #1	Coin Flip #2	Coin Flip #3
Total won	\$2	\$4	\$8 + \$2
Total lost	\$1	\$2 + \$1	\$4 + \$1

What if we tabulate by gambler instead of by coin-flips?

	Gambler #1	Gambler #2	Gambler #3
Total won	\$2 + \$4 + \$8	-	\$2
Total lost	\$1 + \$2 + \$4	\$1	\$1

Notation

Key Idea: We want to track the total money earned by the first n gamblers.

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Let R_n and C_n be the total revenue earned and total cost incurred by the first n gamblers, respectively.

Let $X_n = R_n - C_n$ be the combined profit earned by the first n gamblers.

Walkthrough: Gambler #1

Coin-flips: *HTTHT*

Walkthrough: Gambler #1

Coin-flips: $HTTHT$

Gambler #1 bets \$1 that $Y_1 = T$. He loses.

Walkthrough: Gambler #1

Coin-flips: *HTTHT*

Gambler #1 bets \$1 that $Y_1 = T$. He loses.

We record this as a loss of \$1.

Walkthrough: Gambler #1

Coin-flips: *HTTHT*

Gambler #1 bets \$1 that $Y_1 = T$. He loses.

We record this as a loss of \$1.

$R_1 = 0$, $C_1 = 1$.

Walkthrough: Gambler #2

Coin-flips: *HTTHT*

Walkthrough: Gambler #2

Coin-flips: *HTTHT*

Gambler #2 bets \$1 that $Y_2 = T$. He wins. He now has \$2.

Walkthrough: Gambler #2

Coin-flips: $HTTHT$

Gambler #2 bets \$1 that $Y_2 = T$. He wins. He now has \$2.

He bets \$2 that the $Y_3 = H$. He loses.

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Walkthrough: Gambler #2

Coin-flips: $HTTHT$

Gambler #2 bets \$1 that $Y_2 = T$. He wins. He now has \$2.

He bets \$2 that the $Y_3 = H$. He loses.

We record this as a loss of \$1.

$R_2 = 0$, $C_2 = 2$.

Walkthrough: Gambler #3

Coin-flips: *HTTHT*

Walkthrough: Gambler #3

Coin-flips: *HTTHT*

Gambler #3 bets \$1 that $Y_3 = T$. He wins. He now has \$2.

Walkthrough: Gambler #3

Coin-flips: *HTTHT*

Gambler #3 bets \$1 that $Y_3 = T$. He wins. He now has \$2.

He bets \$2 that $Y_4 = H$. He wins. He now has \$4.

Walkthrough: Gambler #3

Coin-flips: *HTTHT*

Gambler #3 bets \$1 that $Y_3 = T$. He wins. He now has \$2.

He bets \$2 that $Y_4 = H$. He wins. He now has \$4.

He bets \$4 that $Y_5 = T$. He wins. He now has \$8.

Walkthrough: Gambler #3

Coin-flips: *HTTHT*

Gambler #3 bets \$1 that $Y_3 = T$. He wins. He now has \$2.

He bets \$2 that $Y_4 = H$. He wins. He now has \$4.

He bets \$4 that $Y_5 = T$. He wins. He now has \$8.

We record this as a gain of \$8 and a loss of \$1.

$R_3 = 8$, $C_3 = 3$.

Walkthrough: Gambler #4

Coin-flips: *HTTHT*

Walkthrough: Gambler #4

Coin-flips: $HTTHT$

Gambler #4 bets \$1 that $Y_4 = T$. He loses.

Walkthrough: Gambler #4

Coin-flips: *HTTHT*

Gambler #4 bets \$1 that $Y_4 = T$. He loses.

We record this as a loss of \$1.

Walkthrough: Gambler #4

Coin-flips: *HTTHT*

Gambler #4 bets \$1 that $Y_4 = T$. He loses.

We record this as a loss of \$1.

$R_4 = 8$, $C_4 = 4$.

Walkthrough: Gambler #4

Coin-flips: *HTTHT*

Walkthrough: Gambler #4

Coin-flips: *HTTHT*

Gambler #5 bets \$1 that $Y_5 = T$. He wins. He now has \$2.

Walkthrough: Gambler #4

Coin-flips: *HTTHT*

Gambler #5 bets \$1 that $Y_5 = T$. He wins. He now has \$2.

After the fifth coin-flip, the casino closes. Hence, he cannot continue betting.

Walkthrough: Gambler #4

Coin-flips: *HTTHT*

Gambler #5 bets \$1 that $Y_5 = T$. He wins. He now has \$2.

After the fifth coin-flip, the casino closes. Hence, he cannot continue betting.

We record this as a gain of \$2 and a loss of \$1.

Walkthrough: Gambler #4

Coin-flips: $HTTHT$

Gambler #5 bets \$1 that $Y_5 = T$. He wins. He now has \$2.

After the fifth coin-flip, the casino closes. Hence, he cannot continue betting.

We record this as a gain of \$2 and a loss of \$1.

$R_5 = 10$, $C_5 = 5$.

Walkthrough

Total revenue: $R\tau = \$10$.

Total cost: $C\tau = \$5$.

Total profit: $X\tau = R\tau - C\tau = \$5$.

Observation #1

Proposition 12.

$\mathbb{E}[L_{THT}] = \mathbb{E}[C_\tau]$.

Proof.

The number of coin-flips, is equal to the number of gamblers.

Because of the way we recorded losses, each gambler incurs a loss of exactly $1.

Hence, the number of coin-flips made, L_{THT}, is equal to the total cost incurred, C_τ.

Taking expectations, $\mathbb{E}[L_{THT}] = \mathbb{E}[C_\tau]$. □

Observation #2

Proposition 13.
$\mathbb{E}[C_\tau] = \mathbb{E}[R_\tau]$.

Because the coin flips are fair, X_n is a martingale. We can show that the stopping time τ is finite, so by the Optional Stopping Theorem,

$$\mathbb{E}[X_\tau] = X_0 = 0,$$

but $X_\tau = R_\tau - C_\tau$, so

$$\mathbb{E}[R_\tau] = \mathbb{E}[C_\tau]\,.$$

Observation #2

Proposition 13.

$\mathbb{E}[C_\tau] = \mathbb{E}[R_\tau]$.

Because the coin flips are fair, X_n is a martingale. We can show that the stopping time τ is finite, so by the Optional Stopping Theorem,

$$\mathbb{E}[X_\tau] = X_0 = 0,$$

but $X_\tau = R_\tau - C_\tau$, so

$$\mathbb{E}[R_\tau] = \mathbb{E}[C_\tau] .$$

(For a more rigorous proof, see the next section)

Observation #3

Proposition 14.

$\mathbb{E}[R_\tau] = R_\tau = 10$.

Proof.

By the rules of the game, only the last three gamblers can earn money. (why?)

So R_τ depends solely on the last three coin-flips.

But the last three coin-flips are always *THT*.

Hence, R_τ is a constant, thus $\mathbb{E}[R_\tau] = R_\tau = 10$. □

Observations

We observed that

$$\mathbb{E}[L_{THT}] = \mathbb{E}[C_\tau]$$

$$\mathbb{E}[C_\tau] = \mathbb{E}[R_\tau]$$

$$\mathbb{E}[R_\tau] = R_\tau = 10.$$

Observations

We observed that

$$\mathbb{E}[L_{THT}] = \mathbb{E}[C_\tau]$$

$$\mathbb{E}[C_\tau] = \mathbb{E}[R_\tau]$$

$$\mathbb{E}[R_\tau] = R_\tau = 10.$$

Therefore,

$$\mathbb{E}[L_{THT}] = \mathbb{E}[C] = \mathbb{E}[R] = R = 10.$$

The ABRACADABRA Theorem

Proof of Proposition 13

Outline of Proof

Proposition 13.

$\mathbb{E}[C_\tau] = \mathbb{E}[R_\tau]$.

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Proposition 13.
$\mathbb{E}[C_\tau] = \mathbb{E}[R_\tau]$.

Outline:

- $\{X_n\}$ is a martingale.
- $\{X_n\}$ and τ obeys the OST.
- Invoke OST.

Proof of Proposition 13

Lemma 15.

$\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

Proof.

It suffices to show that (1) $\mathbb{E}[X_n]$ is finite, and (2) $\mathbb{E}[X_{n+1} \mid Y_1, \ldots, Y_n] = X_n$.

Proof of Proposition 13

Lemma 15.

$\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

Proof.

(1) $\mathbb{E}[X_n]$ is finite.

X_n attains a maximum when all n gamblers win.

X_n attains a minimum when all n gamblers lose.

Hence, $|X_n| \leq n \cdot 2^n$, so $\mathbb{E}[X_n]$ must also be bounded and thus finite.

Proof of Proposition 13

Lemma 15.

$\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

Proof.

(2) $\underline{\mathbb{E}[X_{n+1} \mid Y_1, \ldots, Y_n] = X_n.}$

Let A_n be the total wealth of gamblers that have lost before the nth flip.

Since A_n is constant, we have

$$\mathbb{E}[A_{n+1} \mid Y_1, \ldots, Y_n] = A_{n+1} = A_n.$$

Proof of Proposition 13

Lemma 15.

$\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

Proof.

Let B_n be the total wealth of gamblers that are still betting at the nth flip.

Since the coin is fair and independent, and the gamblers bet double-or-nothing, we have

$$\mathbb{E}[B_{n+1} \mid Y_1, \ldots, Y_n] = \frac{1}{2}(2B_n) + \frac{1}{2}(0) = B_n.$$

Proof of Proposition 13

Lemma 15.

$\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

Proof.

Because $X_n = A_n + B_n$,

$$\begin{aligned}\mathbb{E}[X_{n+1} \mid Y_1, \ldots, Y_n] &= \mathbb{E}[A_{n+1} \mid Y_1, \ldots, Y_n] + \mathbb{E}[B_{n+1} \mid Y_1, \ldots, Y_n] \\ &= A_n + B_n \\ &= X_n.\end{aligned}$$

Hence, $\{X_n\}$ is a martingale with respect to $\{Y_n\}$. □

Proof of Proposition 13

Proof of Proposition 13.

We show that scenario 3 of OST is satisfied: (1) $\mathbb{E}[\tau]$ is finite and (2) increments of X_n are bounded.

We will prove (2) first.

Proof of Proposition 13.

(2) Increments of X_n are bounded.

Maximum increase in X_n occurs when last three gamblers bet on Y_τ and win:
$X_{n+1} - X_n \leq 3 \cdot 2^3$.

Maximum decrease in X_n occurs when last three gamblers bet on Y_τ and lose:
$X_{n+1} - X_n \geq -3$.

Hence, $|X_{n+1} - X_n|$ is bounded.

Proof of Proposition 13.

(1) $\mathbb{E}[\tau]$ is finite.

Consider the following modified game:

> *Suppose the terminator has length n. Each round, n coins are flipped. If these n coins matches the terminator (i.e. come up THT), we stop flipping. If not, we continue on with another round.*

Let the stopping time for this game be τ'.

Clearly, this game takes longer to finish: $\mathbb{E}[\tau] \leq \mathbb{E}[\tau']$.

Proof of Proposition 13

Proof of Proposition 13.

Let M be the number of rounds played under this game.

Each round, coin-flips have a 1/8 chance of matching the terminator.

So M follows a geometric distribution with probability of success $p = 1/8$. Hence, $\mathbb{E}[M] = 1/p = 8$.

Since a total of $3M$ coin-flips are made in this game, $0 < \mathbb{E}[\tau] \leq \mathbb{E}[\tau'] = 3 \cdot 8$.

Thus, $\mathbb{E}[\tau]$ is bounded and thus finite.

Proof of Proposition 13

Proof of Proposition 13.

Thus, $\{X_n\}$ and τ satisfy scenario 2 of the OST.

Invoking OST, $\mathbb{E}[X_\tau] = X_0 = 0$.

But $X_\tau = R_\tau - C_\tau$, so $\mathbb{E}[C_\tau] = \mathbb{E}[R_\tau]$ as desired. $\square$

The ABRACADABRA Theorem

Correlations and the ABRACADABRA Theorem

Since R_τ depends solely on the last few gamblers, we have an easy way of calculating $\mathbb{E}[L_{\mathcal{T}}]$.

Example 16.

Imagine the terminator *THT* has already been flipped. Working backwards,

- The third-last gambler wins \$$2^3$, since he sees *THT*.
- The second-last gambler wins \$0, since he sees *H* and loses.
- The last gambler wins \$$2^1$, since he sees *T* before the casino closes.

Hence, $\mathbb{E}[L_{THT}] = R_\tau = 2^3 + 2^1 = 10$.

Correlations

We can abstract this process of calculating R_τ using the *correlation* of two strings.

Definition 17.

Let X and Y be two words. The **correlation polynomial** of X and Y, denoted $\rho_z(X, Y)$, is a polynomial in z of maximum degree $|X|$.

The coefficients of $\rho_z(X, Y)$ are determined as follows: place Y under X so that its leftmost character is under the ith character of X (from the right). Then, if all pairs of characters in the overlapping segment are identical, the coefficient of z^i is 1, else it is 0.

Example 18.

Let $X = HTHTTH$ and $Y = HTTHT$. Then $\rho_z(X, Y) = z^4 + z^1$.

X:	H	T	H	T	T	H					
Y:	H	T	T	H	T						0
		H	T	T	H	T					0
			H	T	T	H	T				z^4
				H	T	T	H	T			0
					H	T	T	H	T		0
						H	T	T	H	T	z^1

Correlation and R_τ

We see that $R_\tau = \rho_2(\mathcal{T}, \mathcal{T})$!

Correlation and R_τ

We see that $R_\tau = \rho_2(\mathcal{T}, \mathcal{T})$!

Example 19.

Suppose $\mathcal{T} = THT$. Then $R_\tau = \rho_2(THT, THT) = 2^3 + 2^1 = 10$.

X:	T	H	T			
Y:	T	H	T			2^3
		T	H	T		0
			T	H	T	2^1

We can write this more concisely:

	T	H	T
THT	2^3		2^1

Example 20.

Let $\mathcal{T} = THHTHHTHH$.

	T	H	H	T	H	H	T	H	H
$THHTHHTHH$	2^9			2^6			2^3		

So $\mathbb{E}[L_{\mathcal{T}}] = 2^9 + 2^6 + 2^3 = 584$.

Suppose we change the final character to a T: $\mathcal{T} = THHTHHTHT$.

	T	H	H	T	H	H	T	H	T
$THHTHHTHT$	2^9								2^1

So $\mathbb{E}[L_{\mathcal{T}}] = 2^9 + 2^1 = 514$.

Correlation and R_τ

The "self-repetition" of a terminator determines how big $\mathbb{E}[L_\mathcal{T}]$ is.

THHTHHTHH self-repeats many times (at the sixth-last and third-last characters), while *THHTHHTHT* only repeats itself at the last character.

The ABRACADABRA Problem

Example 21.

A monkey types one random character on a typewriter every second. On average, how long would it take the monkey to type the word "ABRACADABRA"?

The ABRACADABRA Problem

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A monkey types one random character on a typewriter every second. On average, how long would it take the monkey to type the word "ABRACADABRA"?

We now have 26 letters to construct our sequence from: $\{A, B, C, \ldots, X, Y, Z\}$. We hence evaluate the correlation polynomial at $z = 26$ instead.

	A	B	R	A	C	A	D	A	B	R	A
$ABRACADABRA$	26^{11}							26^4			26^1

Hence, the expected time taken is $26^{11} + 26^4 + 26$ seconds, or 116.4 million years.

The ABRACADABRA Theorem

Theorem 22 (ABRACADABRA Theorem).

Suppose we have *n* possible letters. Then

$$\mathbb{E}[L_{\mathcal{T}}] = \rho_n(\mathcal{T}, \mathcal{T}).$$

Exercise

How long would it take the monkey to type “ENTANGLEMENT”?

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	E	N	T	A	N	G	L	E	M	E	N	T
$ENTANGLEMENT$	26^{12}									26^3		

So the expected time taken is $26^{12} + 26^3$ seconds, or 3.026 billion years.

Further Questions

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What if we have more than one terminator, say $\mathcal{T} = \{THT, HTT\}$?

- Can we find $\mathbb{E}[L_{\mathcal{T}}]$?
- What is the probability that $L_{\mathcal{T}}$ ends with THT?
- What is the expected length of $L_{\mathcal{T}}$, given that $L_{\mathcal{T}}$ ends with THT?

Any questions?